Relation between scattering amplitude and Bethe-Salpeter wave function in quantum field theory

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Purpose

(re)introduce a simple relation between scattering amplitude and Bethe-Salpeter (BS) wave function inside interaction range $R$ rather than HALQCD method

Outline

- Wave function in finite volume method
- BS wave function outside $R$
- BS wave function inside $R$
- Fundamental relation in quantum mechanics
- Expansion of reduced BS wave function
- Summary
Lüscher’s finite volume method

spinless two-particle elastic S-wave scattering in center of mass frame

Quantum mechanics

Important assumption

1. Two-particle interaction is localized. → Interaction range $R$ exists.

\[ V(r) \begin{cases} 
  \neq 0 & (r \leq R) \\
  = 0 & \sim e^{-cr} (r > R)
\end{cases} \]

2. $V(r)$ is not affected by boundary. → $R < L/2$

Two-particle wave function $\phi(\vec{r}; k)$ in $r > R$ satisfies Helmholtz equation.

\[ (\Delta + k^2) \phi(\vec{r}; k) = 0 \text{ in } r > R, \quad E_k = 2\sqrt{m^2 + k^2} \]
Lüscher’s finite volume method

[Bruderer, NPB354:531(1991)]

Helmholtz equation on $L^3$

1. Solution of $(\Delta + k^2)\phi(\vec{r}; k) = 0$ in $r > R$

$$\phi(\vec{r}; k) = G(\vec{r}; k) = C \cdot \sum_{\vec{n} \in \mathbb{Z}^3} \frac{e^{i\vec{r} \cdot \vec{n} (2\pi/L)}}{\vec{n}^2 - q^2}, \quad q^2 = (Lk/2\pi)^2 \neq \text{integer}$$

2. Expansion by spherical Bessel $j_l(kr)$ and Neumann $n_l(kr)$ functions

$$\phi(\vec{r}; k) = \beta_0(k)n_0(kr) + \alpha_0(k)j_0(kr) + (l \geq 4)$$

$$= e^{i\delta(k)} \sin(kr + \delta(k))/kx + (l \geq 4)$$

3. $S$-wave scattering phase shift $\delta(k)$ in infinite volume

$$\frac{\beta_0(k)}{\alpha_0(k)} = \tan \delta(k) = \frac{\pi^{3/2}q}{Z_{00}(1;q^2)}$$

$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{(\vec{n}^2 - q^2)^s}$$

Relation between $\delta(k)$ and $k$

$$E_k = 2\sqrt{m^2 + k^2}$$

$\phi(\vec{r}; k)$ disappears in final formula.
BS wave function through LSZ reduction formula

Quantum field theory

BS wave function of two pions in infinite volume (Only S-wave)

\[ \phi(x; k) = \langle 0|\pi_1(x/2)\pi_2(-x/2)|\pi_1(k)\pi_2(-k); \text{in} \rangle \]

Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

\phi(x; k) from 4-point correlation function \( C(\vec{x}, t) \) on lattice

\[ C(\vec{x}, t - t_s) = \langle 0|\pi_1(\vec{x}/2, t)\pi_2(-\vec{x}/2, t)\Omega_{\pi\pi}(t_s)|0\rangle \]

\[ = \sum_k C_k \phi(x; k)e^{-E_k(t - t_s)} \]

where \( \Omega_{\pi\pi}(t_s) = \text{two-pion operator} \), \( C_k = \langle 0|\Omega_{\pi\pi}(0)|E_k\rangle \)
BS wave function through LSZ reduction formula

Quantum field theory

[Lin et al., NPB619:467(2001)]

BS wave function of two pions in infinite volume (Only S-wave)

\[
\phi(x; k) = \langle 0|\pi_1(\bar{x}/2)\pi_2(-\bar{x}/2)|\pi_1(\bar{k})\pi_2(-\bar{k}); \text{in}\rangle
\]

\[
= e^{ik\cdot x} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\epsilon} e^{ip\cdot x}
\]

Inelastic scattering contribution and unnecessary overall factors are neglected.

\[\text{NOT exactly same as one in BS equation}\]

Half off-shell amplitude \( H(p; k) \)

\[
H(p; k) = \frac{E_p + E_k}{8E_pE_k} M(p; k)
\]

\( M(p; k) \) defined by LSZ reduction formula

\[
e^{-iq\cdot x} \frac{-i\sqrt{Z} M(p; k)}{-q^2 + m^2 - i\epsilon} = \int d^4x d^4y_1 d^4y_2 K(p, z) K(-k_1, y_1) K(-k_2, y_2) G_4(z, x, y_1, y_2)
\]

\[
K(p, z) = \frac{i}{\sqrt{Z}} e^{ip\cdot z}(-p^2 + m^2), \quad G_4(z, x, y_1, y_2) = \langle 0|T[\pi_1(z)\pi_2(x)\pi_1(y_1)\pi_2(y_2)]|0\rangle
\]

\[
p = (E_p, \vec{p}), \quad k_1 = (E_k, \vec{k}), \quad k_2 = (E_k, -\vec{k}), \quad q = (2E_k - E_p, -\vec{p})
\]

off-shell momentum
BS wave function through LSZ reduction formula

[Lin et al., NPB619:467(2001)]

Quantum field theory

BS wave function of two pions in infinite volume (Only S-wave)

\[
\phi(x; k) = \langle 0| \pi_1(x/2) \pi_2(-x/2) | \pi_1(k) \pi_2(-k); \text{in} \rangle
\]

\[
= e^{ik \cdot x} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\epsilon} e^{ip \cdot x}
\]

Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

Half off-shell amplitude \(H(p; k)\)

\[
H(p; k) = \frac{E_p + E_k}{8E_pE_k} M(p; k)
\]

\(M(p; k)\) at on-shell \(p = k\)

\[
M(k; k) = \frac{16\pi E_k}{k} e^{i\delta(k)} \sin \delta(k) \Rightarrow H(k; k) = \frac{4\pi}{k} e^{i\delta(k)} \sin \delta(k)
\]
BS wave function outside $R$ [CP-PACS, PRD71:094504(2005)]

Quantum field theory

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k)$$

Assumption: $h(x; k) = 0$ outside interaction range ($x > R$)

c.f. $(\Delta + k^2)\phi(x; k) = mV(x)\phi(x; k)$ in quantum mechanics

Using the assumption and $\phi(x; k) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p}\cdot\vec{x}}$ in $x > R$

$$\phi(x; k) = e^{i\delta(k)} \frac{\sin(kx + \delta(k))}{kx}$$

agrees with wave function in quantum mechanics

Following derivation in quantum mechanics,

finte volume formula can be derived from BS wave function.
**BS wave function outside** $R$  

Quantum field theory

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k)$$

Assumption: $h(x; k) = 0$ outside interaction range $(x > R)$

C.f. $(\Delta + k^2)\phi(x; k) = mV(x)\phi(x; k)$ in quantum mechanics

$I = 2$ S-wave two-pion BS wave function

- assumption is valid in lattice QCD
- $k^2$ from $\phi(r; k)$ in $r > R$

Using $G(\vec{r}; k)$: Solution of Helmholtz equation on $L^3$

$$\phi(x; k) \ r > R \text{ can be used for calculation of } \delta(k).$$
Our results
BS wave function inside $R$

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x; k) = (\Delta + k^2) \phi(x; k)$$
BS wave function inside $R$

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k) = -\int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} H(p; k)$$

$$\therefore \phi(x; k) = e^{i\vec{k} \cdot \vec{x}} + \int \frac{d^3 p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p} \cdot \vec{x}}$$

$h(x; k) = 0$

$h(x; k) \neq 0$
**BS wave function inside** $R$

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k) = -\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} H(p; k)$$

\[\downarrow\] Fourier transformation
BS wave function inside $R$

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k) = -\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} H(p; k)$$

\[\downarrow\] Fourier transformation

Fundamental relation in this talk

$$H(p; k) = -\int d^3x \ e^{-i\mathbf{p} \cdot \mathbf{x}} h(x; k)$$

Relation between $H(p; k)$ and $h(x; k)$ i.e. $\phi(x; k)$ inside $R$
**BS wave function inside** \( R \)

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

\[
h(x; k) = (\Delta + k^2)\phi(x; k) = -\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} H(p; k)
\]

\[\Downarrow\] Fourier transformation

**Fundamental relation in this talk**

\[
H(p; k) = -\int d^3x \, e^{-i\vec{p}\cdot\vec{x}} h(x; k)
\]

Relation between \( H(p; k) \) and \( h(x; k) \) i.e. \( \phi(x; k) \) inside \( R \)

At on-shell \( p = k \)

\[
H(k; k) = \frac{4\pi}{k} e^{i\delta(k)} \sin \delta(k) = -\int d^3x \, e^{-i\vec{k}\cdot\vec{x}} h(x; k)
\]

\( h(x; k) \) is essential to calculate \( H(p; k) \).
BS wave function inside $R$

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x; k) = (\Delta + k^2)\phi(x; k) = -\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} H(p; k)$$

$\downarrow$ Fourier transformation

Fundamental relation in this talk

$$H(p; k) = -\int d^3x \ e^{-i\vec{p}\cdot\vec{x}} h(x; k)$$

Relation between $H(p; k)$ and $h(x; k)$ i.e. $\phi(x; k)$ inside $R$

can be used for calculation on finite volume

Exploratory study with fundamental relation

$\rightarrow$ Next talk by Namekawa
Fundamental relation in quantum mechanics
[TY and Kuramashi, PRD96:114511,11(2017)]

Interpretation of HALQCD method in this frame work

\(V(x; k)\) is defined by \(h(x; k)\) as

\[
V(x; k) = \begin{cases} 
\frac{1}{m} \frac{h(x; k)}{\phi(x; k)} & (x \leq R) \\
0 & (x > R)
\end{cases}
\]

corresponding to LO HALQCD method

\(V(x; k)\) is regarded as potential in Shrödinger equation.

\[(\Delta + p^2)\overline{\phi}(x; p) = mV(x; k)\overline{\phi}(x; p)\]

\(\overline{\phi}(x; p)\) is a solution of the equation with given \(p\).

Scattering phase shift \(\overline{\delta}(p)\) from Shrödinger equation

[textbook of quantum mechanics]

\[
\frac{e^{i\overline{\delta}(p)} \sin \overline{\delta}(p)}{p} = -\frac{m}{4\pi} \int d^3x \ e^{-ip\cdot x} V(x; k) \overline{\phi}(x; p)
\]
Fundamental relation in quantum mechanics

[TY and Kuramashi, PRD96:114511,11(2017)]

Scattering phase shift $\bar{\delta}(p)$ from Shrödinger equation

$$\frac{e^{i\bar{\delta}(p)} \sin \bar{\delta}(p)}{p} = -\frac{1}{4\pi} \int d^3 x \ e^{-ip \cdot x} \frac{h(x; k)}{\phi(x; k)} \bar{\phi}(x; p)$$

At $p = k$, $\bar{\phi}(x; k) = \phi(x; k) \quad \therefore (\Delta + k^2)\phi(x; k) = h(x; k)$

$$\frac{e^{i\bar{\delta}(k)} \sin \bar{\delta}(k)}{k} = -\frac{1}{4\pi} \int d^3 x \ e^{-i\vec{k} \cdot \vec{x}} h(x; k) = \frac{H(k; k)}{4\pi} = \frac{e^{i\bar{\delta}(k)} \sin \delta(k)}{k} \quad \bar{\delta}(k) = \delta(k)$$

At $p \neq k$, $\bar{\phi}(x; p) \neq \phi(x; k)$ in general

$$\frac{e^{i\bar{\delta}(p)} \sin \bar{\delta}(p)}{p} = -\frac{1}{4\pi} \int d^3 x \ e^{-ip \cdot x} \frac{h(x; k)}{\phi(x; k)} \bar{\phi}(x; p) \neq \frac{e^{i\bar{\delta}(p)} \sin \delta(p)}{p}$$

Same $\delta(k)$ is obtained at only $p = k$, where $V(x; k)$ is defined.

Above discussion corresponding to LO HALQCD method
Expansion of reduced BS wave function

[TY and Kuramashi, PRD96:114511,11(2017)]

Derivative expansion in HALQCD method

\[ h(x; k) = (\Delta + k^2)\phi(x; k) = \sum_{n=0}^{\infty} V_n(x) \Delta^n \phi(x; k), \quad V_n(x) \text{ independent of } k \]

[HALQCD, PTEP2018:043B04(2018)]

Convergence of expansion is unclear.

→ Large number of terms would be necessary in general.

A few terms are not enough for convergence test.

c.f.) convergent test with two terms [HALQCD, arXiv:1805.02365]
Expansion of reduced BS wave function

[TY and Kuramashi, PRD96:114511,11(2017)]

Derivative expansion in HALQCD method

\[ h(x; k) = \sum_{n=0}^{\infty} V_n(x) \Delta^n \phi(x; k), \quad V_n(x) \text{ independent of } k \]

[HALQCD, PTEP2018:043B04(2018)]

Truncated in practical determination of \( V_n(x) \)

\[ \rightarrow V_n(x) \text{ depends on input } k. \]

Approximation \( h(x; k) = V_0(x) + V_1(x) \Delta \phi(x; k) \) with inputs \( h(x; k_1), h(x; k_2) \)

\[ V_0(x) = \frac{k_1^2 \phi(x; k_1) h(x; k_2) - k_2^2 \phi(x; k_2) h(x; k_2)}{\phi(x; k_1) h(x; k_2) - \phi(x; k_2) h(x; k_1) + \phi(x; k_1) \phi(x; k_2) (k_1^2 - k_2^2)} \]

\[ V_1(x) = \frac{\phi(x; k_1) h(x; k_2) - \phi(x; k_2) h(x; k_1)}{\phi(x; k_1) h(x; k_2) - \phi(x; k_2) h(x; k_1) + \phi(x; k_1) \phi(x; k_2) (k_1^2 - k_2^2)} \]

\( V_0(x), V_1(x) \) change with \( k_1, k_2 \).
Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator $n = 1, \cdots, NO$

$$C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)\phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha}e^{-E_\alpha t}, \quad \phi_\alpha(x) = \phi(x; k_\alpha)$$

Truncated approximation $h_\alpha(x) = h(x; k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x)\Delta^n\phi_\alpha(x)$

Common $V_n(x)$ in all $\alpha$

Simultaneous equations

$$M(x, t)V(x) = (\Delta + f(\partial_t))C(x, t)$$

$$(\Delta + f(\partial_t))C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)h_\alpha(x), \quad f(\partial_t)A_{i\alpha}(t) = k_\alpha^2 A_{i\alpha}(t)$$

$$M_{in}(x, t) \equiv \Delta^n C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)\Delta^n\phi_\alpha(x)$$
Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator $n = 1, \cdots , NO$

$$C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t) \phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha} e^{-E_\alpha t}, \phi_\alpha(x) = \phi(x; k_\alpha)$$

Truncated approximation $h_\alpha(x) = h(x; k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x) \Delta^n \phi_\alpha(x)$

Common $V_n(x)$ in all $\alpha$

Simultaneous equations

$$\left( \Delta + f(\partial_t) \right) C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t) h_\alpha(x)$$

$$M(x, t)V(x) = A(t)h(x)$$

$$M_{in}(x, t) \equiv \Delta^n C_i(x, t)$$

$$\left[ NO \times NV \right] \left[ NV \right] = \left[ NO \times N_\alpha \right] \left[ N_\alpha \right]$$

sizes for matrices and vectors
Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

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4-point function with different operator $n = 1, \cdots, NO$

$$C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t) \phi_{\alpha}(x), \quad A_{i\alpha}(t) = B_{i\alpha} e^{-E_\alpha t}, \phi_{\alpha}(x) = \phi(x; k_\alpha)$$

Truncated approximation $h_\alpha(x) = h(x; k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x) \Delta^n \phi_{\alpha}(x)$

Common $V_n(x)$ in all $\alpha$

Simultaneous equations

$$M(x,t)V(x) = A(t)h(x)$$

$$[NO \times NV][NV] = [NO \times N_\alpha][N_\alpha]$$

Necessary condition to determine $V(x)$

$$NO = NV \text{ for } (M(x,t))^{-1}$$
Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator \( n = 1, \ldots, N_O \)

\[
C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t) \phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha} e^{-E_\alpha t}, \quad \phi_\alpha(x) = \phi(x; k_\alpha)
\]

Truncated approximation \( h_\alpha(x) = h(x; k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x) \Delta^n \phi_\alpha(x) \) Common \( V_n(x) \) in all \( \alpha \)

Simultaneous equations

\[
M(x, t)V(x) = A(t)h(x) \quad M_{in}(x, t) \equiv \Delta^n C_i(x, t)
\]

\[
[N_O \times N_V][N_V] = [N_O \times N_\alpha][N_\alpha]
\]

sizes for matrices and vectors

Necessary condition to determine \( V(x) \)

\( N_O = N_V \) for \( (M(x, t))^{-1} \) and \( N_O = N_\alpha \) for \( (A(t))^{-1} \)

otherwise operator dependence remains in \( V(x) \).
Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator \( n = 1, \cdots, N_O \)

\[
C_i(x, t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t) \phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha} e^{-E_\alpha t}, \phi_\alpha(x) = \phi(x; k_\alpha)
\]

Truncated approximation \( h_\alpha(x) = h(x; k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x) \Delta^n \phi_\alpha(x) \) Common \( V_n(x) \) in all \( \alpha \)

Simultaneous equations

\[
M(x, t)V(x) = A(t)h(x)
\]

\[
[N_O \times N_V][N_V] = [N_O \times N_\alpha][N_\alpha]
\]

sizes for matrices and vectors

Necessary condition to determine \( V(x) \)

\[
N_O = N_V \text{ for } (M(x, t))^{-1} \text{ and } N_O = N_\alpha \text{ for } (A(t))^{-1}
\]

otherwise operator dependence remains in \( V(x) \).

the same condition for generalized eigenvalue problem

need data in large \( t \) region to satisfy \( N_O = N_\alpha \)
Summary

Simple relation between BS wave function inside $R$ and half off-shell scattering amplitude $H(p; k)$

$$H(p; k) = -\int d^3 x \ e^{-i \vec{p} \cdot \vec{x}} h(x; k), \quad H(k; k) = \frac{4\pi}{k} e^{i\delta(k)} \sin \delta(k)$$

Reduced BS wave function $h(x; k) = (\Delta + k^2)\phi(x; k)$

exploratory study: next talk by Namekawa might be possible to derive similar relations in more than two particles

$\bar{\delta}(p)$ from Shrödinger equation with $V(x; k) = h(x; k)/\phi(x; k)$

At $p = k$, $\bar{\delta}(k) = \delta(k)$, but at $p \neq k$, $\bar{\delta}(p) \neq \delta(p)$.

Derivative expansion of $h(x; k)$

- convergence is unclear.
- $V_n(x)$ depends on $k$ if truncated in finite $\Delta^n$ terms
- time-dependent HALQCD method: same necessary condition to GEVP
  $\rightarrow$ large $t$ data necessary as in calculation of energy