Motivation

- Perturbative expansion for observable O in QFTs
  \[ \langle O \rangle = \sum_{n=0}^{\infty} a_n g^n, \quad g \ldots \text{coupling} \]

- In general divergent (asymptotic series) with \( a_n \sim n! \)

- Assign a unique value to \( \langle O \rangle \)?

- Borel summation
  - If \( a_n \) are alternating in sign (\( a_n \sim (-1)^n n! \))
  - Otherwise: Pole in the Borel transformation \( \rightarrow \) Renormalon

- Renormalon gives rise to ambiguity \( \sim e^{-\frac{1}{g}} \)
  (Resurgence, OPE, \cdots)

- Interesting physics in the behaviour of expansion coefficients
Sigma Models in 2D

▶ The $\mathbb{CP}(N - 1)$ Model

▶ Lattice action:

$$S = -2\beta N \sum_{x, \nu} \text{Re} \left( n^\dagger(x) U_\nu(x) n(x + \nu) \right)$$

▶ Fields $n \in \mathbb{C}^N$, constraint: $n^\dagger n = 1$, links $U_\nu \in U(1)$ (auxiliary)

▶ The Principal Chiral Model (PCM($N$))

▶ Lattice action

$$S = -2\beta N \sum_{x, \nu} \text{Re} \text{Tr} \left( U(x) U(x + \nu)^\dagger \right)$$

▶ Fields $U \in \mathbb{C}^{N \times N}$, constraint: $U \in SU(N)$
Sigma Models in 2D

- Features
  - Asymptotic freedom: $\text{CP}(N - 1)$, $\text{PCM}(N)$
  - Non-perturbative mass gap: $\text{CP}(N - 1)$, $\text{PCM}(N)$
  - Confinement: $\text{CP}(N - 1)$, $\text{PCM}(N)$
  - Instantons: $\text{CP}(N - 1)$
  - …

- Study interesting physics in a (relatively) simple setting
Numerical Stochastic Perturbation Theory (NSPT)

- Ansatz: Expand fields in coupling $g$ up to order $M$
  
  $\phi = \phi_0 + \phi_1 g + \phi_2 g^2 + \cdots + \phi_M g^M = \sum_{n=0}^{M} \phi_n g^n$

- Define sum and product for finite series

  $\phi + \psi = \sum_{n=0}^{M} (\phi_n + \psi_n) g^n,$
  
  $\phi \cdot \psi = \sum_{n=0}^{M} \left( \sum_{l=0}^{n} \phi_l \psi_{l-n} \right) g^n$

- Define functions via (finite) Taylor series

- Numerical cost $\sim M^2$
Numerical Stochastic Perturbation Theory (NSPT)

- MC with accept/reject step not expandable in $g$
- Introduce additional dimension: “stochastic time” $\tau$
- Langevin equation describes $\tau$-dependence

\[
\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta S[\phi(x, \tau)]}{\delta \phi(x, \tau)} + \eta(x, \tau)
\]

- Gaussian noise $\eta(x, \tau)$ with

\[
\langle \eta(x, \tau) \rangle_{\eta} = 0, \quad \langle \eta(x, \tau) \eta(x', \tau') \rangle_{\eta} = 2\delta(x - x')\delta(\tau - \tau')
\]
Numerical Stochastic Perturbation Theory (NSPT)

- Expectation values $\langle \cdots \rangle_{\eta}$ with respect to the noise

\[ \langle \cdots \rangle_{\eta} = \frac{\int D[\eta](\cdots) e^{-\frac{1}{4} \int dx d\tau \, \eta^2(x,\tau)}}{\int D[\eta] e^{-\frac{1}{4} \int dx d\tau \, \eta^2(x,\tau)}} \]

- Asymptotic ($\tau \to \infty$) distribution $P(\phi) \propto e^{-S[\phi]}$

- For expectation values of operators:

\[ \lim_{\tau \to \infty} \langle O(\phi(x,\tau)) \rangle_{\eta} = \langle O(\phi) \rangle = \frac{1}{Z} \int D[\phi] e^{-S[\phi]} O(\phi) \]

- Stochastic Quantisation
  - no accept/reject step $\Rightarrow$ compatible with expansion in $g$
Numerical Stochastic Perturbation Theory (NSPT)

- Discretise stochastic time for numerical treatment:
  \[ \tau \rightarrow \tau_k = k\epsilon \]

- Euler approximation \((O(\epsilon))\) of the Langevin equation:
  \[
  \phi(x, \tau_k + \epsilon) = \phi(x, \tau_k) - \epsilon \frac{\delta S[\phi(x, \tau_k)]}{\delta \phi(x, \tau_k)} - \sqrt{\epsilon} \eta(x, \tau_k)
  \]

- Asymptotic \(P(\phi) \propto e^{-\overline{S}[\phi]}\) with \(\overline{S}[\phi] = S[\phi] + O(\epsilon)\)
  - Systematic error \(\rightarrow\) extrapolation to \(\epsilon = 0\)
  - Runs for several \(\epsilon\) values necessary
  - Use Runge-Kutta \((O(\epsilon^2))\) approximation
    (Bali et al. *PRD* 87, 094517 [1303.3279])
Langevin for constrained fields

- \( \phi \) subject to constraint \( C \) such that \( C(\phi) = 0 \)
- In general: \( C(\phi(x, \tau_k)) = 0 \Rightarrow C(\phi(x, \tau_k + \epsilon)) = 0 \)
- Large constraint violations for long Langevin trajectories
- Modify Langevin update
  (E.g., Batrouni et al. PRD 32, 2736)
- Langevin update with modifications respects constraints
  (up to numerical errors)
Numerical Results

- Expand around vacuum
  - $\text{CP}(N-1)$: \( n_0(x) \equiv \frac{1}{\sqrt{N}} (1, \cdots, 1)^\dagger \), \( U_0(x, \nu) \equiv 1 \)
  - $\text{PCM}(N)$: \( U_0(x) \equiv 1 \)

- Compute coefficients of the energy density $E$

\[
E = \sum_{n=0}^{\infty} E_{2n} g^{2n} \quad \text{(Odd terms vanish)}
\]

- Analytic expressions known for leading coefficients
Energy Density Coefficients \textit{PCM(3)}

\textbf{PCM(3)} \quad V = 16 \times 16

- $\epsilon = 0.0100$
- $\epsilon = 0.0250$
- $\epsilon = 0.0500$

Analytic
# Energy Density Coefficients \( \text{PCM}(3) \)

<table>
<thead>
<tr>
<th>Coefficient ( E_2 )</th>
<th>Stochastic Time ( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>20</td>
</tr>
<tr>
<td>0.102</td>
<td>40</td>
</tr>
<tr>
<td>0.104</td>
<td>60</td>
</tr>
<tr>
<td>0.106</td>
<td>80</td>
</tr>
<tr>
<td>0.108</td>
<td>100</td>
</tr>
</tbody>
</table>

\[ \text{PCM}(3) \quad V = 16 \times 16 \]

- \( \epsilon = 0.0100 \)
- \( \epsilon = 0.0250 \)
- \( \epsilon = 0.0500 \)
- Analytic
Energy Density Coefficients $\text{PCM}(3)$

$\epsilon = 0.0100$
$\epsilon = 0.0250$
$\epsilon = 0.0500$
analytic
Energy Density Coefficients $\text{PCM}(3)$

$\text{PCM}(3) \quad V = 16 \times 16$

- $\epsilon = 0.0100$
- $\epsilon = 0.0250$
- $\epsilon = 0.0500$
- analytic

Coefficient $E_6$

Stochastic time $\tau$
Energy Density Coefficients $\text{PCM}(3)$

$\text{PCM}(3) \quad V = 16 \times 16$

- $\epsilon = 0.0100$
- $\epsilon = 0.0250$
- $\epsilon = 0.0500$
Energy Density Coefficients \( CP(1) \)

\[ CP(1) \quad V = 16 \times 16 \]

\[ \epsilon = 0.0100 \quad \epsilon = 0.0250 \quad \epsilon = 0.0500 \]

analytic

stochastic time \( \tau \)
### Energy Density Coefficients \( CP(1) \)

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>Coefficient ( E^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0100</td>
<td>-2.25</td>
</tr>
<tr>
<td>0.0250</td>
<td>-2.2</td>
</tr>
<tr>
<td>0.0500</td>
<td>-2.15</td>
</tr>
</tbody>
</table>

\( CP(1) \quad V = 16 \times 16 \)

- \( \epsilon = 0.0100 \)
- \( \epsilon = 0.0500 \)
- \( \epsilon = 0.0250 \)
- analytic

![Graph](image-url)
Energy Density Coefficients $\text{CP}(1)$

$\text{CP}(1) \quad V = 16 \times 16$

$\epsilon = 0.0100$  
$\epsilon = 0.0250$  
$\epsilon = 0.0500$  
analytic
Energy Density Coefficients \( CP(1) \)

\[
\begin{align*}
\text{Coefficient } E_{16} & = 16 \times 16 \\
\epsilon &= 0.0100 \\
\epsilon &= 0.0250 \\
\epsilon &= 0.0500
\end{align*}
\]
Energy Density Coefficients \( CP(1) \)

\[
CP(1) \quad V = 16 \times 16
\]

\[
\epsilon = 0.0100, \quad \epsilon = 0.0250, \quad \epsilon = 0.0500
\]
Constraint Violation $\text{CP}(1)$

$\text{CP}(1) \quad V = 16 \times 16$

$\langle |n^\dagger n - 1| \rangle_{\text{order } 16}$

stochastic time $\tau$

$\epsilon = 0.0100$
$\epsilon = 0.0250$
$\epsilon = 0.0500$
Constraint Violation \( CP(1) \)

\[ CP(1) \quad V = 16 \times 16 \]

\[ \langle |n^\dagger n - 1| \rangle \text{ order 18} \]

Stochastic time \( \tau \)

\( \epsilon = 0.0100 \)

\( \epsilon = 0.0250 \)

\( \epsilon = 0.0500 \)
Good agreement with analytic (low order) coefficients (even for finite $\epsilon$)

Numerical issues for $\text{CP}(N-1)$
  - Field constraint violated in higher orders
  - Results reliable?

Focus on $\text{PCM}(N)$ for further calculations
  - Larger lattice $V = 32 \times 32$
  - Smaller $\epsilon$
  - Higher orders
  - $N$ dependence

Expected large $n$ behavior: $E_n \sim a^n n!$ ($a \propto \beta_0$)
PCM(N) coefficients

N=5, L=32

ε = 0.0100
ε = 0.0075
analytic
PCM($N$) coefficients

N=6, L=32

ε = 0.0100
ε = 0.0075
analytic
PCM(N) comparison

All N, \( \epsilon = 0.0075 \)
Summary

- NSPT for $\text{CP}(N-1)$ and $\text{PCM}(N)$
- First results for expansion of energy density
- Good agreement with analytic coefficients (where available)
- Numerical issues spoil high order calculations in $\text{CP}(N-1)$
- No renormalon observed in $\text{PCM}(N)$ up to order 40 (20) for $N \leq 6$ ($N = 12$)
- Expansion coefficients $E_n(N)$ decrease slower for larger $N$

Outlook

- Better control of numerical errors in $\text{CP}(N-1)$
- Twisted boundary conditions in $\text{PCM}(N)$
Backup Slides
Langevin for constrained fields

- Find a Lie Group $G$ such that

$$\forall g \in G : C(\phi) = 0 \implies C(g\phi) = 0$$

- Define Lie derivative ($\Lambda^\alpha$ generators of $G$)

$$f(e^{i\epsilon \Lambda^\alpha} \phi) = \left(1 + \epsilon \nabla^\alpha + O(\epsilon^2)\right) f(\phi)$$

- New Langevin equation ($\nabla_x = \Lambda^\alpha x^\alpha$, $\eta(x, \tau) = \Lambda^\alpha \eta^\alpha(x, \tau)$)

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = -i \left(\nabla_x S[\phi(x, \tau)] + \eta(x, \tau)\right) \phi(x, \tau)$$
Discretised constrained Langevin can be written as

\[ \phi(x, \tau_i + \epsilon) = e^{-if(x, \tau_i)\alpha \wedge \alpha} \phi(x, \tau_i) \]

To order \( \mathcal{O}(\epsilon) \) ("Euler")

\[ f(x, \tau_i)^\alpha = \epsilon \nabla_x^\alpha \mathcal{S}[\phi(x, \tau_i)] + \sqrt{\epsilon} \eta^\alpha(x, \tau) \]

By construction \( e^{-if(x, \tau_i)\alpha \wedge \alpha} \in G \) and

\[ C(\phi(x, \tau_i)) = 0 \quad \Rightarrow \quad C(\phi(x, \tau_i + \epsilon)) = 0 \]
Constraint Violation in NSPT

- Constraints have to hold order by order

\[
n^\dagger n = \sum_{l=0}^{M} \left( \sum_{k=0}^{l} n_k^\dagger n_{l-k} \right) \beta^{-\frac{1}{2}} = 1
\]

- Since \( n_0^\dagger n_0 = 1 \) for our expansion

\[
\sum_{k=0}^{l} n_k^\dagger n_{l-k} = 0 \quad \forall \ l > 0
\]

- Analogous formula for gauge field \( U_\nu \) and matrices \( U \)
Constraint Violation in NSPT

- Numerical errors accumulate and lead to constraint violation
- “Normalise” the fields during Langevin runs?

\[
\sum_{k=0}^{l} n_k^\dagger n_{l-k} = n_0^\dagger n_l + n_l^\dagger n_0 + \sum_{k=1}^{l-1} n_k^\dagger n_{l-k} = 0
\]

\[
\Rightarrow n_l \rightarrow \frac{1}{\alpha} n_l \quad \text{with} \quad \alpha = -\frac{\sum_{k=1}^{l-1} n_k^\dagger n_{l-k}}{2 \text{Re}(n_0^\dagger n_l)}
\]

- Numerically unstable!
Constraint Violation in NSPT

- “Normalisation” is a lot easier for $U$ and $U_\nu$

$$A = \log(U) \Rightarrow \exp(A) \in SU(N)$$

- This gives constraints for $A$ ...

$$A \doteq -A^\dagger \quad \text{and} \quad \text{Tr} A \doteq 0$$

- ... which are simple to implement order by order

$$A_l \leftrightarrow \frac{A_l - A_l^\dagger}{2} \quad \text{and} \quad A_l \leftrightarrow A_l - \frac{\text{Tr} A_l}{N} \mathbb{I}$$