Fermions on Simplicial Lattices and their Dual Lattices

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2018
What Am I Talking About?

Background

Naive and Staggered Fermions on an $A_4$ lattice

Naive and Staggere Fermions on an $A_4^*$ lattice

Final Remarks and Sales Pitch
The isotropic lattices in every dimension

The notation comes from the book by Conway and Sloane.

- $\mathbb{Z}^n$; The hypercubic lattices. Automorphism group has $2^n \cdot n!$ elements ($=384$ in 4-d).

- $A_n$; Also called "simplicial." Group order $= 2 \cdot n!$ ($=240$ in 4-d). In 2-d, triangular lattice. FCC in 3-d. Pure gauge models were simulated on an $A_4$ lattice.

- $A^*_n$; The lattice dual to $A_n$. In 3-d $A^*_3$ is the BCC lattice.

- $D_n$; Also known as the "checkerboard" lattice. $D_3 = A_3$ is FCC. $D_4 = F_4$ is self-dual. Automorphism group of $D_4$ has 1152 elements. $D_3$, $D_4$, and $D_5$ are the densest possible lattice packings in 3, 4 and 5 dimensions.

- Hyperdiamond lattice is not a Bravais lattice. Union of 2 $A_n$ lattices.
Extremely Abridged History

Noticed a long time ago [Celmaster and Krausz, (1983)] that fermions on non-cubic lattices are problematic:

\[ \sum \bar{\psi}_n e_i \cdot \gamma (\psi_{n+e_i} - \psi_{n-e_i}) \]

Equations for doublers break rotational symmetry. There must be a symmetry connecting doublers to have rotational invariance and a reduction to staggered fermions.

Could add Wilson term. On \( D_4 \) you have rotational symmetry broken only at \( O(a^4) \).

In 4-d, staggered fermions have only been satisfactorily formulated on hypercubic lattices.

Drouffe and Moriarty (1983) did simulations of pure SU(2) and SU(3) gauge theories on the \( A_4 \) lattice.
A Lattice Fermion Popularity Contest

Counting papers on hep-lat since 2017 using lattice fermions:

- 155 Wilson/clover,
- 86 domain wall
- 62 staggered
- 57 overlap
- 0 on non-cubic lattices
The $A_4$ lattice

Coordinate vector of $A_d$ lattice:

$$(n_1, n_2, \ldots, n_{d+1}) \text{ where } \sum n_i = 0 \quad \text{Surface in } Z_{d+1} \text{ lattice.}$$

Nearest neighbor vectors:

$$\epsilon_{12} = (1, -1, 0, 0, 0), \epsilon_{13} = (1, 0, -1, 0, 0), \ldots, \epsilon_{45} = (0, 0, 0, 1, -1)$$

and negatives of these.

So 20 neighbors in 4-d, compared to 8 for hc.

Take primitive lattice vectors $\tau_\mu = \epsilon_{\mu 5}$:

$$\tau_1 = (1, 0, 0, 0, -1), \ldots, \tau_4 = (0, 0, 0, 1, -1)$$

Reciprocal lattice vectors, $b_\mu$, defined by $b_\mu \cdot \tau_\nu = 2\pi \delta_{\mu\nu}$ are

$$b_1 = \kappa(4, -1, -1, -1, -1), \ldots, b_4 = \kappa(-1, -1, -1, 4, -1)$$

with $\kappa = 2\pi/5$, generate the lattice $A_4^\ast$. 
Also need a set of orthonormal vectors on $A_4$:

\[ e_1 = \left(1, -1, 0, 0, 0\right)/\sqrt{2}, \quad e_2 = \left(1, 1, -2, 0, 0\right)/\sqrt{6}, \]
\[ e_3 = \left(1, 1, 1, -3, 0\right)/\sqrt{12}, \quad e_4 = \left(1, 1, 1, 1, -4\right)/\sqrt{20}. \]
The action:

\[ S_A = \frac{\sqrt{2}}{8} i \sum_n \sum_{j>i} \bar{\psi}_n \gamma_i \gamma_j (\psi_{n+\epsilon_{ij}} - \psi_{n-\epsilon_{ij}}) \]

\[ \{\gamma_i, \gamma_j\} = 2\delta_{\mu \nu} \]

The inverse free propagator in momentum space:

\[ D(k) \propto \sum_{j>i}^{5} \gamma_i \gamma_j \sin(k \cdot \epsilon_{ij}) \]

which leads to the propagator

\[ S(k) \propto \sum_{j>i} \gamma_i \gamma_j \sin(k \cdot \epsilon_{ij}) / \sum_{j>i} \sin^2(k \cdot \epsilon_{ij}) \]
The modes

Poles at \( k = 0 \) and at

\[
k = \frac{b_\mu}{2}
\]

and sums of 2, 3 and all 4 of these, 16 in total.

5 modes at \(|k| = \sqrt{\frac{4}{5} \pi} \Leftrightarrow \frac{\pi}{5} (-4, 1, 1, 1, 1), \ldots \frac{\pi}{5} (1, 1, 1, 1, -4)\)

10 modes at \(|k| = \sqrt{\frac{6}{5} \pi} \Leftrightarrow \frac{\pi}{5} (3, 3, -2, -2, -2), \ldots \)
Symmetries connecting modes

The action is invariant under

\[ \psi_n \rightarrow T(n) \psi_n, \quad \bar{\psi}_n \rightarrow \bar{\psi}_n T(n) \]

where

\[ T(n) = (-1)^{n\mu} \gamma_\mu \]

and products of these.

Since all modes are equivalent need only examine the one at \( k \approx 0 \)
For $k \approx 0$

$$D(k) \approx -\frac{1}{\sqrt{5}} \sum_{j>i} \gamma_i \gamma_j \mathbf{k} \cdot \mathbf{e}_{ij} \equiv i \sum_{\mu=1}^{4} \Gamma_\mu \mathbf{k} \cdot \mathbf{e}_\mu$$

Solving for $\Gamma_\mu$:

$$\Gamma_\mu = i \sum_{i=1}^{5} e^i_{\mu} \gamma_i A$$

where

$$A = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$

The $\Gamma_\mu$ comprise a set of Euclidean Dirac matrices:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$

Thus the action describes 16 Dirac fermions. We also have

$$\Gamma_5 = A = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$
Short paws
Symmetry group of the $A_4$ lattice

Permutations of $(n_1, n_2, n_3, n_4, n_5)$, the "symmetric" group $S_5$.

Negation of all the coordinates is also a symmetry.

So $2 \times 5! = 240$ elements.

$S_5$ is generated by single exchanges: e.g. $(21345)$

The action is invariant provided

$$\psi_n \to \frac{1}{\sqrt{2}} (\gamma_1 - \gamma_2) \psi_{n'}$$

$$\bar{\psi}_n \to \bar{\psi}_{n'} \frac{1}{\sqrt{2}} (\gamma_1 - \gamma_2).$$
Representations of some lattice objects

\[ \epsilon_{ij}, \gamma_i \gamma_j, U_{ij} = e^{i A_{ij}} \text{ transform as 10-d rep. of } S_5. \]

Orthogonality of characters \( \rightarrow 10 = 4 \oplus 6 \)

\[ i \gamma_i \gamma_j = \sqrt{\frac{2}{5}} \epsilon^\mu_{ij} \Gamma_\mu + i \sum_{\nu > \mu} (e^i_\mu e^j_\nu - e^i_\nu e^j_\mu) \Gamma_\mu \Gamma_\nu \]

showing reduction to vector and antisymmetric tensor.
Likewise:

\[ A_{ij} = \epsilon_{ij}^\mu B_\mu + \sum_{\nu > \mu} (e_i^\mu e_j^\nu - e_i^\nu e_j^\mu) Y_{\mu\nu} \]

the naive continuum limit:

\[ \int d^4x \bar{\psi} \left\{ \Gamma_\mu (\partial_\mu - igB_\mu) + g\sigma_{\mu\nu} Y_{\mu\nu} \right\} \psi + m\bar{\psi}\psi \]

\( Y_{\mu\nu} \) is short range \( \rightarrow \) four-fermion interaction with coupling of order \( a^2 g^2 \).
The Action for the Link Variables

\[ (\alpha F_{\mu\nu}(B) + \kappa Y_{\mu\nu})^2 \]

\[ (\alpha F_{\mu\nu}(B) - \kappa Y_{\mu\nu})^2 \]

\[ \Rightarrow F^2 + Y^2 + F\Delta Y \]
Absence of additive mass renormalization

Additive mass renormalization is forbidden, even though there is no exact axial symmetry. The action

\[ S_A = \frac{\sqrt{2}}{8} i \sum_n \sum_{j>i} \bar{\psi}_n \gamma_i \gamma_j U_{n,ij} \psi_{n+\epsilon_{ij}} + h.c. \]

is invariant under negation of all the coordinates provided

\[ U_{ij} \rightarrow U_{ij}^\dagger; \quad \psi_n \rightarrow \psi_{-n}; \quad \bar{\psi}_n \rightarrow -\bar{\psi}_{-n} \]

This implies for the full propagator:

\[ S(-p) = -S(p) \]

which forbids a mass term.

Mass or Wilson terms are not invariant.
No exact chiral symmetry $\rightarrow$ fermion determinant is not real (except for free fermions).

- In a simulation, the pseudo-fermion action

$$\phi (D^\dagger D + m^2)^{-1} \phi$$

is real and $\approx det (D + m)$.

- Or to get to reality you can double the fermions $\psi \rightarrow (\psi_1, \psi_2)$ with a mass term $m \psi \sigma_3 \psi$.

- Or go to a hyperdiamond lattice ($A_4 \cup A_4$) with $\psi_1$ on one $A_4$ with mass $m$ and $\psi_2$ on the other with mass $-m$. The coupling $\rightarrow$ axial-vector interaction mixing 1 and 2.
Axial Vector Interaction

Using

\[ \gamma_i = -i \sum_{\mu} e_{\mu}^i \Gamma_{\mu} \Gamma_5 + \frac{1}{\sqrt{5}} \Gamma_5 \]

a rotationally invariant, axial vector interaction is

\[ \sum_{n} \sum_{i} (\bar{\psi}_n \gamma_i \psi_{n+r_i} + \bar{\psi}_{n+r_i} \gamma_i \psi_n) Z_i(n) \]

the same for all doublers, where

\[ r_1 = (4, -1, -1, -1, -1), \ldots, r_5 = (-1, -1, -1, -1, 4) \]

generate an \( A_4^* \) sublattice. So axial currents live on a dual sublattice.

Naive continuum limit \( \Rightarrow \bar{\psi} \Gamma_{\mu} \Gamma_5 \psi A_5^{\mu} + \bar{\psi} \Gamma_5 \psi \phi \)
Reduction to Staggered Fermions

Naive action is diagonalized by:

$$\psi_n \rightarrow \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4} \gamma_5^{(n_1+n_2+n_3+n_4)} \psi_n$$

leading to the staggered fermion action

$$S_{st} = \sum \bar{\chi}_n \eta_i(n) \eta_j(n) (\chi_n + \epsilon_{ij} - \chi_n - \epsilon_{ij}) + m \bar{\chi}_n \chi_n$$

where $\chi_n$ is a single anticommuting variable and the phases are

$$\eta_1 = 1, \quad \eta_2 = (-1)^{n_1}, \quad \eta_3 = (-1)^{n_1+n_2}, \quad \eta_4 = (-1)^{n_1+n_2+n_3},$$

$$\eta_5 = (-1)^{n_1+n_2+n_3+n_4}$$
Can make blocks of 16 points as on hypercubic lattice.

Degrees of freedom in a block couple to degrees of freedom in 20 neighboring blocks.

All the symmetries of the naive fermions carry through to the staggered case. There is no additive mass renormalization.
Staggered Blocks on Triangular Lattice
Fermions on an $A^*_4$ lattice

The action:

$$S = \frac{5}{16} \sum_n \sum_j \bar{\psi}_n \gamma_i \left( \psi_{n+f_j} - \psi_{n-f_j} \right)$$

where

$$f_1 = \kappa(4, -1, -1, -1, -1), \ldots, f_5 = \kappa(-1, -1, -1, -1, 4)$$

with $\kappa = 1/\sqrt{20}$.

Take the first 4 to be primitive vectors. The doubling symmetry is then

$$\psi_n \rightarrow (-1)^{n \mu} \gamma_\mu \psi_n$$
The propagator

\[ S(k) \propto \sum_i \gamma_i \sin(k \cdot f_i) / \sum_i \sin^2(k \cdot f_i) \]

has a mode at \( k = 0 \), and 10 modes at

\[ \alpha(1, -1, 0, 0, 0), \ldots, \alpha(0, 0, 0, 1, -1); \quad \alpha = 2\pi/\sqrt{5} \]

and 5 modes at

\[ \alpha(0, 1, 1, -1, -1), \ldots, \alpha(1, 1, -1, -1, 0) \]
For $\mathbf{k} \approx 0$ the inverse propagator

$$\Rightarrow \frac{2}{\sqrt{5}} \sum_i \gamma_i \mathbf{k} \cdot \mathbf{f}_i \equiv \sum_{\mu=1}^{4} \Gamma_{\mu} \mathbf{k} \cdot \mathbf{e}_\mu$$

$$\Rightarrow \Gamma_{\mu} = \frac{2}{\sqrt{5}} \sum_{i=1}^{5} \mathbf{f}_i \cdot \mathbf{e}_\mu \gamma_i$$

which obey

$$\{\Gamma_{\mu}, \Gamma_{\nu}\} = 2\delta_{\mu\nu}$$

and as for $A_4$

$$\Gamma_5 = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$
The naive continuum limit is

\[ \int d^4x \bar{\psi} \{ \Gamma_\mu (\partial_\mu - igB_\mu) + g\Gamma_5 \phi \} \psi + m\bar{\psi}\psi \]

Absence of additive mass renormalization works the same.

The staggered action is

\[ S_{st} = \sum \bar{\chi}_n \ \eta_i(n) \ (\chi_{n+f_i} - \chi_{n-f_i}) + m\bar{\chi}_n\chi_n \]

where

\[ \eta_1 = 1, \ \eta_2 = (-1)^{n_1}, \ \eta_3 = (-1)^{n_1+n_2}, \ \eta_4 = (-1)^{n_1+n_2+n_3}, \ \eta_5 = (-1)^{n_1+n_3} \]
Axial Interactions on the $A_4^*$ lattice

An axial interaction with the same charge for all the doublers is

$$\sum_{n} \sum_{j>i}^{5} \left( \bar{\psi}_n \gamma_i \gamma_j \psi_{n+f_i-f_j} + \bar{\psi}_{n+f_i-f_j} \gamma_i \gamma_j \psi_n \right) A_{ij}$$

The vectors $f_i - f_j$ generate an $A_4$ sublattice.

So, again, axial interactions live on a dual sublattice.
The Last Slide

Fermions on $A_4$ and $A_4^*$ lattices are interesting (at least to one person), and might be useful in simulations. Drouffe and Moriarty claimed that (quenched) simulations on $A_4$ are faster than on hypercubic.

Mean field calculations, including $1/d$ corrections, are better. The corrections are smaller because you’re really expanding in $1/(kissing\ number)$.

The duality between vector and axial vector currents paralleling the duality between $A_4$ and $A_4^*$ lattices is interesting.

Would be interesting to find a fermion formulation on $D_n$ ($D_4 = F_4$) lattices, as they have more rotational symmetry (broken at $O(a^4)$). At least someone could try Wilson fermions.
That's all Folks!
Odd numbers of exchanges, e.g. (23145) or (21435) are rotations. Subgroup of $S_5$ called $A_5$, the alternating group.

In even dimensions, negation of all the coordinates has $\det = 1$, a 180 deg rotation.

$S_5$ has representations of dimensions 1, 1, 4, 4, 5, 5 and 6.
Chiral Symmetry

Recall

\[ \Gamma_5 = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i \]

Can’t do:

\[ \psi_n \rightarrow e^{i\phi \Gamma_5} \psi_n \]

No doubling symmetry.

Chiral transformation same for all modes:

\[ \psi_n \rightarrow \psi_n + \frac{i}{\sqrt{5}} \phi \sum_j \gamma_j \sum_{\sigma_j} \psi_{n+\sigma_j} \]

e.g.

\[ \sigma_1 = (0, 1, 1, -1, -1), (0, 1, -1, 1, -1), \ldots (0, -1, -1, 1, 1) \]
The Anomaly

\[ \delta S \Rightarrow i \varphi \]

\[ \langle \bar{\psi}_n\, \chi_i\, \psi_{n+e_{ij}+\delta_j} - \ldots \rangle = c \tilde{F} \]

\[ \Rightarrow SS = c \varphi F \tilde{F} \]
Hexagonal Lattice

\[(\sigma_1 \Delta_1 + \sigma_2 \Delta_2 + m)(x_1, x_2) = 0\]
More nearest-neighbors ⇒
- longer correlation length for given bare coupling constant.
- Faster thermalization times ⇒ Shorter auto-correlation times? At least in a disordered phase.
- More rotational symmetry.

The Bad: more nearest-neighbors ⇒
- More computation per simulation step.
- More link degrees of freedom per site.