Gauss’s Law, Duality, and the Hamiltonian Framework of U(1) Lattice Gauge Theory

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Work based on arXiv:1806.08797
(submitted to PRL)
Outline

1. Context of project

2. Recap: Conventional Hamiltonian LGT

3. The emergence of duality
   - Original theory set-up
   - Reconstruction begets duality
Roadmap

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Unitary evolution on a quantum computer

Digital quantum computers (QC):

- Unitary gates $\sim e^{-it\hat{H}}$ of some $\hat{H}$.
- Want to simulate a lattice gauge theory (LGT).
- How to map its $\hat{H}$ and its Hilbert space $\mathcal{H}$ on to QC?
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Near-term QC architectures will have very limited capabilities

- How to most wisely spend those qubits?
Previous work

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  - For modern discussion in context of QC see, e.g., Byrnes and Yamamoto 2006; Wiese 2014; Zohar et al. 2017; P. Dreher’s talk
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- **Taking pure U(1) LGT, we seek **most economical construction**
  - Leads directly to duality transformation
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- Taking pure U(1) LGT, we seek most economical construction
  - Leads directly to duality transformation
- Dualities also extensively studied in LGTs and many other areas
  - See, e.g., Anishetty and Sharatchandra 1990; Mathur 2006; Anishetty and Sreeraj 2018
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Recap: Conventional Hamiltonian LGT
Conventional construction

Link operators raise or lower electric field:

\[ \hat{U} | \psi \rangle = | \phi \rangle \]
Recap: Conventional Hamiltonian LGT

Conventional construction

Link operators raise or lower electric field:

\[ \hat{U} | \psi \rangle = | \psi' \rangle \]

Kogut-Susskind Hamiltonian:

\[
H_E = \frac{1}{2a_s} \sum_\ell \tilde{g}_t^2 \hat{E}_\ell^2, \quad H_B = \frac{1}{2a_s} \left[ \frac{1}{\tilde{g}_s^2} \sum_p \left( 2 - \hat{P}_p - \hat{P}_p^\dagger \right) \right]
\]

\[
H_E + H_B \xrightarrow{a_s \to 0} H = \frac{1}{2} \int d^Dx (E^2 + B^2)
\]
Recap: Conventional Hamiltonian LGT

Issues with standard formulation

1. Must impose **Gauss’s law** on kets [Kogut and Susskind 1975; Zohar et al. 2017]
   - Most directions in $\mathcal{H}$ unphysical.
   - Danger of leaving $\mathcal{H}_{\text{phys}}$ due to errors, noise
   - If truncating states (by e.g. $|\mathcal{E}_\ell| \leq \Lambda$ in $U(1)$), makes awkward constraints around cutoff.
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2. Electric fluctuations large at weak coupling
   - Expect large $\mathbf{E}$ fluctuations as $a_s \to 0$ in $D = 2$ gauge theories and in asymptotically-free theories in $D = 3$
   - Rate of convergence as $a_s \to 0$ unclear when truncating on $\mathbf{E}$
The emergence of duality

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Starting point for original theory

We start with a symmetric Hamiltonian,\(^1\)

\[
\begin{align*}
\hat{H} &= \hat{H}_E + \hat{H}_B, \\
\hat{H}_B &= \frac{1}{2a_s} \left[ \frac{1}{\tilde{g}_s^2} \sum_p \left( 2 - \hat{P}_p - \hat{P}_p^\dagger \right) \right], \\
\hat{H}_E &= \frac{1}{2a_s} \left[ \frac{\tilde{g}_t^2}{\xi^2} \sum_\ell \left( 2 - \hat{Q}_\ell - \hat{Q}_\ell^\dagger \right) \right].
\end{align*}
\]

\(\times\) Hilbert space \(\mathcal{H}\) and \(\hat{H}_B\) are conventional

\(^1\)Different, but similar to [Horn, Weinstein, and Yankielowicz 1979].
The emergence of duality

Original theory set-up

Starting point for original theory

We start with a symmetric Hamiltonian,

\[ \hat{H} = \hat{H}_E + \hat{H}_B , \]

\[ \hat{H}_B = \frac{1}{2a_s} \left[ \frac{1}{\tilde{g}_s^2} \sum_p \left( 2 - \hat{P}_p - \hat{P}_p^\dagger \right) \right] , \]

\[ \hat{H}_E = \frac{1}{2a_s} \left[ \frac{\tilde{g}_t^2}{\xi^2} \sum_\ell \left( 2 - \hat{Q}_\ell - \hat{Q}_\ell^\dagger \right) \right] . \]

\[ \text{Hilbert space } \mathcal{H} \text{ and } \hat{H}_B \text{ are conventional} \]

\[ \text{We exponentiated } E: \]

\[ \hat{Q}_\ell \equiv e^{i\xi \hat{E}_\ell} . \]

Think of \( \xi \ll 1 \) as \( a_t/a_s \).

\[ ^1 \text{Different, but similar to [Horn, Weinstein, and Yankielowicz 1979].} \]
The emergence of duality

Reconstruction begets duality

Hilbert space generation

A basis for $\mathcal{H}_{\text{phys}}$ is generated by acting with plaquettes on trivial state.

$$|\Omega\rangle \equiv \bigotimes_{\ell} |0\rangle_{\ell} ,$$

$$|\mathcal{A}_L\rangle \equiv \prod_{p} \left( \hat{P}_p \right)^{\mathcal{A}_p} |\Omega\rangle .$$
Hilbert space generation

A basis for $\mathcal{H}_{\text{phys}}$ is generated by acting with plaquettes on trivial state.

$$\left| \Omega \right\rangle \equiv \bigotimes_{\ell} \left| 0 \right\rangle_{\ell},$$

$$\left| \mathcal{A}_L \right\rangle \equiv \prod_{p} \left( \hat{P}_p \right)^{\mathcal{A}_p} \left| \Omega \right\rangle .$$

$D = 2$ for this talk.
Take $A$’s further: Use as *quantum numbers*.

Notice:

- Plaquettes $\mathbf{p} \sim$ dual sites $\mathbf{n}^*$.  
  $\Rightarrow \mathcal{A}_\mathbf{p}$ is scalar field $\mathcal{A}_{\mathbf{n}^*}$ on $L^*$.

- $E_\ell$ on a link $\sim$ difference $\Delta \mathcal{A}_{\mathbf{n}^*}$ along a dual link.
The emergence of duality

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Hilbert space transcription

Take $\mathcal{A}$’s further: Use as *quantum numbers*

Notice:

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$\blacktriangleright \blacktriangleleft E_\ell$ on a link $\sim$ difference $\Delta \mathcal{A}_{n^*}$ along a dual link

1.) Identify

$$
\prod_p \left( \hat{P}_p \right)^{\mathcal{A}_p} \left| \Omega \right> \longleftrightarrow \bigotimes_{n^*} \left| \mathcal{A}_{n^*} \right>
$$
2. Define identical local orthonormal bases, \( \{ |\mathcal{A}_n^* \rangle \} \), which diagonalize

\[
\hat{U}_n^* \equiv \sum_{\mathcal{A}_n^* = -\infty}^{\infty} |\mathcal{A}_n^* \rangle e^{i \xi \mathcal{A}_n^*} \langle \mathcal{A}_n^* | .
\]

3. Global basis states:

\[ |\mathcal{A}_L^* \rangle \equiv \bigotimes n^* |\mathcal{A}_n^* \rangle \]

4. (Local) raising operators:

\[
\hat{Q}_n^* \equiv \sum_{\mathcal{A}_n^* = -\infty}^{\infty} |\mathcal{A}_n^* + 1 \rangle \langle \mathcal{A}_n^* | .
\]
2. Define identical local orthonormal bases, \( \{ |\mathcal{A}_{n^*}\rangle \} \), which diagonalize

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\hat{U}_{n^*} \equiv \sum_{n^* = -\infty}^{\infty} |\mathcal{A}_{n^*}\rangle e^{i \xi \mathcal{A}_{n^*}} \langle \mathcal{A}_{n^*}| .
\]

3. Global basis states:

\[
|\mathcal{A}_{L^*}\rangle \equiv \bigotimes_{n^*} |\mathcal{A}_{n^*}\rangle
\]

4. (Local) raising operators:

\[
\hat{V}_{n^*} \equiv \sum_{n^* = -\infty}^{\infty} |\mathcal{A}_{n^*} + 1\rangle \langle \mathcal{A}_{n^*}|
\]

Redundancy:

Since \( \prod_p \left( \hat{P}_p \right) = \hat{1} \), must impose

\[
\prod_{n^*} \hat{V}_{n^*} |\mathcal{A}_{L^*}\rangle = |\mathcal{A}_{L^*}\rangle
\]

on \( \mathcal{H}^* \). This is magnetic Gauss law.
## The dual formulation

<table>
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<th>Dual</th>
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We have $\langle A'_L | \hat{H} | A_L \rangle = \langle A'_L^* | \hat{H} | A_L^* \rangle$ for the dual Hamiltonian

$$\hat{H} = \frac{1}{2a_s} \sum_{n^*} \left[ \frac{1}{\tilde{g}_s^2} \left( 2 - \hat{Q}_{n^*} - \hat{Q}_{n^*}^\dagger \right) \right.$$

$$\left. - \frac{\tilde{g}_t^2}{\xi^2} a_s^2 \hat{U}_{n^*}^\dagger \partial_i^+ \partial_i^- \hat{U}_{n^*} \right], \quad (D = 2)$$

(subject to magnetic Gauss).
Solving the dual Gauss law:

1. Fix one $\mathcal{A}_n = 0$.

   ▶ Break translational symmetries

   ▶ $\hat{\mathcal{H}}$ becomes nonlocal

   ■ Truncation can be done as

   $|\mathcal{A}_n| \leq \Lambda$
Solving the dual Gauss law:

1. Fix one $A_n^* = 0$.
   - Break translational symmetries
   - $\hat{\mathcal{H}}$ becomes nonlocal
   - Truncation can be done as $|A_n^*| \leq \Lambda$

2. Restrict states to subspace on which $\prod_n^\ast \hat{Q}_n^\ast = 1$
   - Truncation can be done on argument of $Q_n^\ast$ phases (equivalent to regulating $B$ in original theory)
Summary

1. Duality transformation naturally emerges from building $\nabla \cdot \mathbf{E} = 0$ into $\mathcal{H}$

2. Formulating and truncating dual theory preferable for weak coupling
Summary

1. Duality transformation naturally emerges from building $\nabla \cdot E = 0$ into $\mathcal{H}$

2. Formulating and truncating dual theory preferable for weak coupling

Current/future work

- Putting in matter
  - Want: Local Hilbert spaces, $\hat{\mathcal{H}}$ built from local operators
  - How much redundancy?

- Extend to non-Abelian
  - Local field description possible with non-Abelian lattice duality? (prepotential formalism)
I thank Natalie Klco and Martin Savage at the Institute for Nuclear Theory for helpful conversations.

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E fluctuations at weak coupling

Analogy to SHO: (electric field is momentum, gauge field is coordinate)

\[ H_E = \frac{1}{2a_s} \sum_{\ell} \tilde{g}_t^2 \hat{E}_\ell^2 \sim \frac{1}{2m} \hat{p}^2 \]

\[ H_B = \frac{1}{2a_s} \left[ \frac{1}{\tilde{g}_s^2} \sum_p \left( 2 - \hat{P}_p - \hat{P}_p^\dagger \right) \right] \sim \frac{k}{2} \hat{x}^2 \]

Read off

\[ m \sim 1/\tilde{g}_t^2, \quad k \sim 1/\tilde{g}_s^2 \]

By dimensional analysis,

\[ \langle \hat{p}^2 \rangle \propto \sqrt{mk} \sim \frac{1}{\tilde{g}_t \tilde{g}_s}, \quad \langle \hat{x}^2 \rangle \propto \frac{1}{\sqrt{mk}} \sim \tilde{g}_t \tilde{g}_s \]
Topological sectors

Original formulation (on periodic lattice) has many gauge-invariant states decoupled from $|\Omega\rangle$

- Topological Polyakov loops are gauge-invariant
- Define class representatives,

$$|\nu\rangle \equiv \prod_{i=1}^{d} \left( \hat{W}(C_i) \right)^{\nu_i} |0\rangle, \quad \nu_i \in \mathbb{Z}.$$ 

with $\hat{W}(C_i)$ the product of oriented $\hat{U}_i$'s along a closed loop $C_i$ wrapping direction $i$.

- An $\hat{H}$ containing only elementary Wilson loops cannot cause transitions

Fully general state:

$$|\mathcal{A}\rangle_{\nu} = \prod_{p} \left( \hat{P}_p \right)^{\mathcal{A}_p} |\nu\rangle, \quad \mathcal{A}_p \in \mathbb{Z}.$$
Dual Hamiltonian with topology

Since ν’s don’t talk to each other, we fix ν. We must adapt \( \mathcal{H} \) to get the right matrix elements:

\[
\mathcal{H} \rightarrow \mathcal{H}^\nu = \mathcal{H}_B + \mathcal{H}_E^\nu, \quad (\mathcal{H}_B \text{ unchanged})
\]

\[
\mathcal{H}_E^\nu = \frac{1}{2a_s} \sum_{n^*} \left[ -\frac{g_t^2}{\xi_2} a_s^2 \hat{\mathcal{U}}_{n^*}^\dagger \Delta \hat{\mathcal{U}}_{n^*} \right], \quad (D = 2)
\]

Here we have generalized to a **covariant Laplacian** \( \Delta = \sum_{i=1}^{2} D_i^+ D_i^- \),

\[
D_1^+ F_{n^*} = (\mathcal{W}\{n^*,n^*-e_1\} F_{n^*-e_1} - F_{n^*})/a_s ,
\]

\[
D_2^+ F_{n^*} = (\mathcal{W}\{n^*,n^*+e_2\} F_{n^*+e_2} - F_{n^*})/a_s ,
\]

involving the (dual lattice) **connection**

\[
\mathcal{W}_\mathcal{L}^* = \begin{cases} 
    e^{i\xi \nu_i} , & \text{if } \mathcal{L} \in C_i; \\
    1 , & \text{otherwise}
\end{cases}
\]
Appendix

Further details

Dual Hamiltonian in $d = 3 + 1$

For $D = 3$ spatial dimensions, $p \leftrightarrow \ell^*$ (rather than $p \leftrightarrow n^*$).

We define $\hat{D}_{\ell^*}$’s and $\hat{U}_{\ell^*}$’s on local dual link Hilbert spaces by direct analogy.

Then

$$\hat{H}_\nu = \frac{1}{2a_s} \left[ \sum_{\ell^*} \frac{1}{\tilde{g}_s^2} \left( 2 - \hat{D}_{\ell^*} - \hat{D}_{\ell^*}^\dagger \right) \right. $$

$$+ \left. \frac{\tilde{g}_t^2}{\xi^2} \sum_{p^*} \left( 2 - \left( \hat{W}_{p^*} \hat{P}_{p^*} + \text{h.c.} \right) \right) \right] ~ (D = 3).$$

Dual plaquettes $\hat{P}_{p^*}$ are usual products of $\hat{U}_{\ell^*}$’s, and

$$\hat{W}_{p^*} = \begin{cases} e^{i \xi \nu_i}, & \text{if } \ell \in C_i; \\ 1, & \text{otherwise}. \end{cases}$$


