Review on Lattice Muon g-2 HVP Calculation

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Hadron Vacuum Polarization (HVP) Contribution to Muon $g - 2$
### SM contribution $a_\mu^{\text{contrib.}} \times 10^{10}$ Ref.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value</th>
<th>Reference</th>
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<tbody>
<tr>
<td>QED [5 loops]</td>
<td>$11658471.8951 \pm 0.0080$</td>
<td>[Aoyama et al ’12]</td>
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<tr>
<td>HVP-LO (pheno.)</td>
<td>$692.6 \pm 3.3$</td>
<td>[Davier et al ’16]</td>
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<td></td>
<td>$694.9 \pm 4.3$</td>
<td>[Hagiwara et al ’11]</td>
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<tr>
<td></td>
<td>$681.5 \pm 4.2$</td>
<td>[Benayoun et al ’16]</td>
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<td></td>
<td>$688.8 \pm 3.4$</td>
<td>[Jegerlehner ’17]</td>
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<td>HVP-NLO (pheno.)</td>
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<td>HVP-NNLO</td>
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<td>[Kurz et al ’11]</td>
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<td>HLbyL</td>
<td>$10.5 \pm 2.6$</td>
<td>[Prades et al ’09]</td>
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<td>Weak (2 loops)</td>
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<td>[Gnendiger et al ’13]</td>
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<td>SM tot [0.42 ppm]</td>
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<td>[Davier et al ’11]</td>
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<td>[0.43 ppm]</td>
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<td>$26.1 \pm 7.8$</td>
<td>[Hagiwara et al ’11]</td>
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<td></td>
<td>$24.9 \pm 8.7$</td>
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</tr>
</tbody>
</table>

\[ a_{\mu}^{\text{LO-HVP}|_{\text{NoNewPhys}}} \times 10^{10} \approx 720 \pm 7, \]

FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm
Really $a_{\mu}^{\text{exp.}} \neq a_{\mu}^{\text{SM}}$?
Motivation

HVP in Phenomenology

- The HVP in Phen. is:
  \[ \hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\text{Im}\Pi(s)}{\pi} = (Q^2/(12\pi^2)) \int_0^\infty ds \frac{R_{\text{had}}(s)}{s(s+Q^2)}, \]

- with R-ratio [right fig. Jegerlehner EPJ-Web2016] given by
  \[ R_{\text{had}}(s) \equiv \frac{\sigma(e^+e^-\rightarrow\text{had.})}{4\pi\alpha^2(s)/(3s)}, \]

- where the systematics is challenging to control (next talk). Some tension among experiments in \( \sigma(e^+e^- \rightarrow \pi^+\pi^-) \).

Requirement for Lattice QCD:

- Independent cross-check of Hadronic Vacuum Polarization Contribution to muon g-2 \( a_{\mu}^{\text{HVP}} \),
- Permil-Level determination of \( a_{\mu}^{\text{HVP}} \) w.r.t. FNAL/J-PARC expr.
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  \[ \hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\text{Im}\Pi(s)}{\pi} \]
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Requirement for Lattice QCD:

- Independent cross-check of Hadronic Vacuum Polarization Contribution to muon g-2 (\( a^{\text{HVP}}_\mu \)),

- Permil-Level determination of \( a^{\text{HVP}}_\mu \) w.r.t. FNAL/J-PARC expr.
Objective in This Work

- Hadron Vacuum Polarization (HVP):
  \[ \Pi_{\mu\nu}(Q) = \int d^4 x \ e^{iQx} \langle j_\mu(x)j_\nu(0) \rangle \]
  \[ = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(Q^2) \]
  \[ j_\mu = \frac{2}{3} \bar{u}\gamma_\mu u - \frac{1}{3} \bar{d}\gamma_\mu d - \frac{1}{3} \bar{s}\gamma_\mu s + \frac{2}{3} \bar{c}\gamma_\mu c + \cdots . \]

- Leading-Order (LO) HVP Contr. to Muon g-2:
  \[ a_{\mu}^{\text{LO-HVP}} = (\alpha/\pi)^2 \int_0^\infty dQ^2 \ \omega(Q^2/m_{\mu}^2)\hat{\Pi}(Q^2) \]
  \[ \hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0) . \]

- HVP Time-Moments:
  \[ \hat{\Pi}(Q^2) = \sum_{n=1} Q^{2n} \Pi_n , \]
  \[ \Pi_n = \frac{1}{n!} \frac{d^n \hat{\Pi}(Q^2)}{(dQ^2)^n} \bigg|_{Q^2 \to 0} = \sum_x \frac{(-\bar{x}^2)^{n+1}}{(2n+2)!} \langle j_\mu(x)j_\mu(0) \rangle . \]

Model Independent Approximants

Pade Approximant

- For $Q^2 < Q^2_{cut}$, lattice HVP data are fitted to

\[ \hat{\Pi}(Q^2) = \frac{A_2 Q^2 + \cdots}{1 + B_2 Q^2 + \cdots}. \]  

(1)

- The dispersion relation $\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\text{Im} \Pi(s)}{\pi}$ is seen as so-called 

\textit{Stieltjes Integral} [Aubin et.al., PRD2012], which guarantees a finite conversion radius.

Time-Momentum Representation (TMR)

- For $Q^2 < Q^2_{uv-cut}$, define [Bernecker and Meyera, EPJA2011],

\[ \hat{\Pi}(Q^2) = \sum_i t^2 \left[ 1 - \left( \frac{\sin[Qt/2]}{Qt/2} \right)^2 \right] \frac{1}{3} \sum_{i=1}^3 \langle j_i(t)j_i(0) \rangle. \]  

(2)

- The momentum $Q$ is \textit{Continuous}. The Sine-Cardinal $\sin[Qt/2]/(Qt/2)$ accounts for a periodic feature of lattice correlators $\langle j_i(t)j_i(0) \rangle$. 

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Pade Approximant

- For $Q^2 < Q^2_{\text{cut}}$, lattice HVP data are fitted to

$$\hat{\Pi}(Q^2) = \frac{A_2 Q^2 + \cdots}{1 + B_2 Q^2 + \cdots}.$$  \hspace{1cm} (1)

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Example of TMR

Figure: From BMW Ensemble ($a = 0.064 \text{ fm}$) used in PRD2017 and PRL2018.
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Multi-Exponential Fits [HPQCD PRD2017]

**Left:** HPQCD PRD2017, vector-current correlator with ud-quarks and a fit line $t > t^*$: $G^{ud}(t, t^*) = G^{ud}_{\text{data}}(t < t^*)$ or $(G_{\text{fit}}(t > t^*) + G_{\pi\pi}(t > t^*))$, where $t^* \in [0.5, 1.5] \text{fm}$. Multi($N = 5$)-Exponential Ansatz are adopted and $\rho$-meson dominates.

**Right:** From a slide of Van de Water at Mainz Workshop 2018. Diagrams in effective theory to correct missing effects in the fits. Taste-spliting and finite volume corrections are also taken account.
**Multi-Exponential Fits [FNAL/HPQCD/MILC Preliminary]**

- **Left:** The $t^*$ dependence of

  \[ a_{\mu, ud}^{\text{LO-HVP}}(t^*) = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dQ^2 \, \omega(Q^2/m_{\mu}^2) \, \mathcal{F} \mathcal{T} [G^{ud}(t, t^*), Q^2] \text{ with Pade}. \]  

  With high-statistics, $a_{\mu, ud}^{\text{LO-HVP}}$ get stable at larger $t^*$. For $t^* \lesssim 2 \text{ fm}$, low-(used in PRD2017) and high-statistics are consistent.

- **Right:** The high-statistics in the left-panel is compared with *Bounding Method* (next page).
Bounding [BMW PRD2017 and PRL2018]

The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3\text{fm}$. To control statistical error, consider $C^{ud}(t > t_c) \to C^{ud}_{\text{up/low}}(t, t_c)$, where

$$C^{ud}_{\text{up}}(t, t_c) = C^{ud}(t_c) \varphi(t)/\varphi(t_c),$$
$$C^{ud}_{\text{low}}(t, t_c) = 0.0,$$

with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$, and $E_{2\pi} = 2(M^2_{\pi} + (2\pi/L)^2)^{1/2}$.

Similarly, $C^{\text{disc}}(t) \to C^{\text{disc}}_{\text{up/low}}(t, t_c)$,

$$-C^{\text{disc}}_{\text{up}}(t > t_c) = 0.1C^{ud}(t_c) \varphi(t)/\varphi(t_c),$$
$$-C^{\text{disc}}_{\text{low}}(t > t_c) = 0.0.$$

By construction,

$$C^{ud,\text{disc}}_{\text{low}}(t, t_c) \leq C^{ud,\text{disc}}(t) \leq C^{ud,\text{disc}}_{\text{up}}(t, t_c).$$

Figure shows

$$C^{ud}(t) = \frac{5}{9} \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^{3} \langle j_i^{ud}(\vec{x}, t)j_i^{ud}(0) \rangle,$$

by BMW Ensemble with $a = 0.078 \text{[fm]}$ used in PRD2017/PRL2018.
Corresponding to $C^{ud, disc\, up/low}_{\mu}(t_c)$, we obtain upper/lower bounds for muon g-2: $a^{ud, disc\, up/low\, (t_c)}_{\mu}$. Two bounds meet around $t_c = 3fm$. Consider the average of bounds: $\bar{a}^{ud, disc\, (t_c)}_{\mu} = 0.5(a^{ud, disc\, up,(t_c)}_{\mu} + a^{ud, disc\, low,(t_c)}_{\mu})$, which is stable around $t_c = 3fm$.

We pick up such averages $\bar{a}^{ud, disc\, (t_c)}_{\mu}$ with 4 – 6 kinds of $t_c$ around 3fm. The average of average is adopted as $a^{LO-HVP\, up/disc\, (t_c)}_{\mu}$ to be analysed, and a fluctuation over selected $t_c$ gives systematic error.

A similar method is proposed by C. Lehner in Lattice2016 and used in RBC/UKQCD-PRL2018. Improved bounding method with GEVP: [A. Meyer/C. Lehner, 27 Fri Hadron Structure].

Figure: BMW, PRL2018.
Large Distance Control Using $F_\pi$, [Mainz CLS JHEP2017]

- **Isospin Decom. of Vector-Current Correlator:**

$$G(t, L) = G^{l=1}(t, L) + G^{l=0}(t, L), \quad G^{l=1}(t, L) = \sum_{n=1} |A_n|^2 e^{-\omega_n t},$$

where $\omega_n = 2\sqrt{M_\pi^2 + k_n^2}$. Investigate the large distance behavior of $G^{l=1}(t)$.

- **L"uscher’s Formula [NPB1991]:** The p-wave phase shift determines $k_n$,

$$\delta_{l=1}(k_n) + \phi(k_n L/(2\pi)) = n\pi,$$

where $\phi$ is a known kinematical function.

- **Meyer’s Formula [PRL2011]:**

$$|F_\pi(\omega_n)|^2 = \frac{3\pi\omega_n^2}{2k_n^5} \left( k_n \frac{\partial \delta_1(k_n)}{\partial k_n} + q_n \frac{\partial \phi(q_n)}{\partial q_n} \right) |A_n|^2, \quad q_n = \frac{k_n L}{2\pi},$$

which is analogous to Lellouch-L"uscher Formula [CMP2012].

- **Gounaris-Sakurai(GS) [PRL1968]** (c.f. Fransis et.al. [PRD2013]):

$$(k^3/\omega) \cot \delta^{\text{GS}}_1(k) = k^2 h(\omega) - k_\rho^2 h(M_\rho) + b[k_\rho, M_\rho, \Gamma_\rho](k^2 - k_\rho^2),$$

$$F^{\text{GS}}_\pi(\omega) = f_0[M_\pi, M_\rho, \Gamma_\rho]/((k^3/\omega)(\cot[\delta^{\text{GS}}_1(k)] - i)), \quad k_\rho^2 = (M_\rho^2/4) - M_\pi^2.$$

- **Construct $G^{l=1}(t)$:** For given lattice data $(M_{\pi, \rho})$, using GS formulae with Eqs. (5) and (6), $G^{l=1}_{\text{lat}}(t)$ is fitted to Eq. (4) to determine $(A_n, k_n, \Gamma_\rho)$. 

Large Distance Control Using $F_{\pi}$, [Mainz CLS JHEP2017]

- **Isospin Decomp. of Vector-Current Correlator:**

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  \[ F_\pi^{GS}(\omega) = f_0[M_\pi, M_\rho, \Gamma_\rho]/((k^3/\omega)(\cot[\delta_1^{GS}(k)] - i)) , \quad k_\rho^2 = (M_\rho^2/4) - M_\pi^2 . \]

- **Construct $G^{l=1}(t)$:** For given lattice data $(M_{\pi, \rho})$, using GS formulae with Eqs. (5) and (6), $G^{l=1}_{lat}(t)$ is fitted to Eq. (4) to determine $(A_n, k_n, \Gamma_\rho)$. 
Large Distance Control Using $F_\pi$

\[ G_n(t, L) = \sum_{j=1}^{n} |A_j|^2 e^{-\sqrt{M_{\pi}^2 + k_j^2} t}, \]

\[ G(t > t^*, L \to \infty) = \frac{1}{48\pi^2} \int_{2M_{\pi}}^{\infty} d\omega \omega^2 \left(1 - \frac{4M_{\pi}^2}{\omega^2}\right)^{3/2} |F_{\pi}(\omega)|^2 e^{-\omega |t|}. \]

- Figure: [Mainz Prelim], update of [Mainz Lat2017]. $(\tilde{K}(t)/m_{\mu})G_n(t, L)$ vs $x_0 = t$ for $N_f = 2 + 1, M_{\pi} = 200$ MeV. $G_n$ is given by Eq. (A). c.f. Talk by H. Wittig (27 Fri, Hadron Structure).

- The lowest mode ($n = 1$) becomes dominant at around 3 [fm]. A single exponential-fit provides a good approximation at long-distance.

- Using $F_{\pi}^{GS}(\omega)$, the infinite-volume correlator $G^{l=1}(t, L \to \infty)$ is given by Eq. (B). Comparing $a_{\mu, ud}^{l0-HVP}$ obtained with $G^{l=1}(t > t^*, L \to \infty)$ or $G^{l=1}_{lat}(t > t^*, L)$, a finite volume effect can be estimated.
Large Distance Control Using $F_\pi$

Finite Volume Effects

Consider $a_{\mu,ud}^{\text{LO-HVP}}(L_2) - a_{\mu,ud}^{\text{LO-HVP}}(L_1)$.

- $(L_1, L_2) = (4.66, 6.22)\,\text{[fm]}$, physical $M_\pi$
  [RBC/UKQCD Prelim., talk by C. Lehner (27 Fri, Hadron Structure)]
- XPT: $12.2 \times 10^{-10}$,
- LQCD: $21.6(6.3) \times 10^{-10}$,
- GSL: $20(3) \times 10^{-10}$.

- $(L_1, L_2) = (5.4, 10.8)\,\text{[fm]}$, $M_\pi = 135\,\text{[MeV]}$
  [talk by E. Shintani (24 Tue, Hadron Spectroscopy), update of PACS 1805.04250]
- LQCD: $40(18) \times 10^{-10}$, 2.5 times larger than XPT estimates.

- $L_2 = \text{large}$, $M_\pi L_1 \sim 4$
  - XPT/RBCUK-PRL18: $16(4) \times 10^{-10}$,
  - GSL/RBCUK-Prelim: $22(1) \times 10^{-10}$,
  - XPT/BMW-PRL18: $15(15) \times 10^{-10}$,
  - GSL/Mainz-Prelim: $20.4(4.2) \times 10^{-10}$,
  - GSL+dual/ETM-prelim: $31(6) \times 10^{-10}$.

Figure: RBC/UKQCD Preliminary.

- $E_\rho = 0.766(21)\,\text{[GeV]}$
  (c.f. PDG: $0.77549(34)\,\text{[GeV]}$).
- $\Gamma_\rho = 0.139(18)\,\text{[GeV]}$
  (c.f. PDG: $0.1462(7)\,\text{[GeV]}$).
Continuum Extrapolation
Controlled Continuum Extrap. [BMW PRL2018]

BMW Ensemble PRD2017 and PRL2018

- 6-\(\beta\), 15 simulation with all physical masses.
- Nf=(2+1+1) staggered quarks.
- Large Volume: \((L, T) \sim (6, 9−12)\) fm.

- Get systematic uncertainty from various cuttings: no-cut, or cutting \(a \geq 0.134, 0.111, \) or 0.095.
- Strong \(a^2\) deps. for \(a^{\text{LO-HVP}}_{\mu, ud/\text{disc}}\) due to taste violations, and for \(a^{\text{LO-HVP}}_{\mu, c}\) due to large \(m_c\).
- Get good \(\chi^2/dof\) with extrapolation linear in \(a^2\), and interpolation linear in \(M_K^2\) (strange) or \(M_{\pi}^2\) and \(M_{\eta c}\) (charm).
Red open-circles are raw lattice data and continuum-extrapolated (red filled-circle). Then finite-volume correction using XPT is added to get the green-square point.

Similarly to HPQCD-PRD2017, raw data (red-circles) are first corrected with finite-volume and taste-partner effects to get blue open-triangles, which are continuum-extrapolated to get blue filled-triangle.
Continuum Extrapolation, Comparison

Figure: BMW-PRL2018 vs HPQCD-PRD2017 and FNAL/HPQCD/MILC-Prelim.
QED and Strong-Isospin Breaking Corrections

\[ \mathcal{O}(\alpha) \sim \mathcal{O}\left( \frac{m_d - m_u}{\Lambda_{QCD}} \right) \sim 1\% \text{ Correction} \]
Strong Isospin Breaking (SIB)

Strong isospin breaking: $m_d - m_u = 2.41(6)(4)(9)$ [BMW PRL2016] in $\overline{\text{MS}}$-2[GeV].

- Direct Simulations with $m_u \neq m_d$ [FNAL/HPQCD/MILC-PRL2018].
- Perturbative Method [RM123-JHEP2012, RBC/UKQCD-JHEP17]:

$$\langle O \rangle = \langle O \rangle_{mu/d=\hat{m}} + (m_u/d - \hat{m}) \left. \frac{\partial \langle O \rangle}{\partial m_{u/d}} \right|_{m_{u/d}=m_d} + \mathcal{O}((m_{u/d} - \hat{m})^2),$$

$$= \langle O \rangle_{mu/d=\hat{m}} - (m_u/d - \hat{m}) \langle OS \rangle_{m_{u/d} = \hat{m}},$$

where $\hat{m} = (m_u + m_d)/2$, and $S = \sum_x \bar{q}_{u/d} q_{u/d}(x)$.


Right: FNAL/HPQCD/MILC-PRL2018 (Van de Water, Mainz g-2 workshop).

Valence-quark dep. of $a_{\mu}^{\text{LO-HVP}}$ for (2+1+1) and (1+1+1+1) ensemble. Two ensemble results agree at $m_l = (m_u + m_d)/2$; sea-quark SIB are negligible. To quantify SIB, define, $\Delta a_{\mu}^{\text{LO-HVP}} = (4a_{\mu}^{\text{LO-HVP}}|m_u + a_{\mu}^{\text{LO-HVP}}|m_d)/5 - a_{\mu}^{\text{LO-HVP}}|m_l$. SIB corr. = $\Delta a_{\mu}^{\text{LO-HVP}}/a_{\mu}^{\text{LO-HVP}}|m_l = 1.5(7)%$.
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\[
\langle O \rangle = \langle O \rangle_{m_u/d=\hat{m}} + (m_u/d - \hat{m}) \left. \frac{\partial \langle O \rangle}{\partial m_u/d} \right|_{m_u=m_d} + \mathcal{O}((m_u/d - \hat{m})^2), \\
= \langle O \rangle_{m_u/d=\hat{m}} - (m_u/d - \hat{m}) \langle OS \rangle_{m_u/d=\hat{m}},
\]

where \( \hat{m} = (m_u + m_d)/2 \), and \( S = \sum_x \bar{q}_{u/d} q_{u/d}(x) \).

**Up:** Strong Isospin Breaking Diagrams.

**Right:** FNAL/HPQCD/MILC-PRL2018 (Van de Water, Mainz g-2 workshop).

Valence-quark dep. of \( a_{\mu}^{\text{LO-HVP}} \) for (2+1+1) and (1+1+1+1) ensemble. Two ensemble results agree at \( m_l = (m_u + m_d)/2 \); sea-quark SIB are negligible. To quantify SIB, define, \( \Delta a_{\mu}^{\text{LO-HVP}} = (4a_{\mu}^{\text{LO-HVP}}|_{m_u} + a_{\mu}^{\text{LO-HVP}}|_{m_d})/5 - a_{\mu}^{\text{LO-HVP}}|_{m_l} \).

SIB corr. = \( \Delta a_{\mu}^{\text{LO-HVP}} / a_{\mu}^{\text{LO-HVP}}|_{m_l} = 1.5(7)\% \)

\( \Delta a_{\mu}^{\text{HVP}} = +1.5(7)\% \)
QED Correction

- Consider QCD + QED Euclidean partition function:

\[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}[q, \bar{q}, U] \mathcal{D}[A] \ O \ e^{-S_F[q, \bar{q}, U, A] - S_G[U]} e^{-S_\gamma[A]} . \quad (7) \]

- **Full QCD + QED:** First Come Out! [QCDSF-Prelim, talk by J. Zanotti (27 Fri, Hadron Structure)].

- **Stochastic Method:** Stochastic photon fields \( A_\mu \) are generated with weight \( e^{-S_\gamma} \) independently of gluon fields \( U_\mu \) (electro-quenched), and multiplied, \( U_\mu(x) \to e^{-ieq_f A_\mu(x)} U_\mu(x) \) [Duncan et.al. PRL1996].

- **Perturbative Method:** QED can be treated in a perturbative way in \( \alpha = e^2/(4\pi^2) \) [RM123-PRD2013]:

\[ \langle O \rangle = \langle O \rangle_0 + \frac{e^2}{2} \frac{\partial^2 \langle O \rangle}{\partial e^2} \bigg|_{e=0} + O(\alpha^2) . \quad (8) \]

The stochastic and perturbative methods gave consistent corrections [RBC/UKQCD-Lat2017].

- To control QED FV effects, QED\(_L\) prescription [Hayakawa PTP2008] is used; spatial zero-modes and the universal \( 1/L^{n=1,2} \) corrections to mass are removed [BMW Science2015], while a reflection positivity is preserved.
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QED Correction Diagrams in Perturbative Approach

- **Left:** ■ = vector-current, ▲ = tadpole, ⊕ = (pseudo-)scalar insertions.
- **Right:** [ETM JHEP2017, talk by D. Giusti, (27 Fri, Hadron structure)] with corrections [1],[2],[3],[8] (mass retuning) and [9] (keeping maximal twist) for strange component.
- RBCUKQCD (Domain-Wall) considered [1],[2],[3],[4]; the others \( \sim 1/N_c \) or irrelevant. One must take are a double counting problem in [4] w.r.t. single-photon and additional glues [talks by RBC/UKQCD (27 Fri, Hadron Structure)].
- For diagram details, see [talk by A. Risch (24 Tue, Hadron Spectroscopy)].
SIB + QED Corrections, Short Summary

- **ETMc Preliminary**
  \[ \delta a_{\mu}^{\text{LO-HVP}} \times 10^{10} = 7(2) \] (quark connected and qQED).

- **BMW PRL2018**
  \[ \delta a_{\mu}^{\text{LO-HVP}} \times 10^{10} = 7.8(5.1) \] (pheno. \( \pi^0 \gamma, \eta \gamma, \rho - \omega \text{ mix, } M_{\pi \pm} \)).

- **RBC/UKQCD PRL2018**
  \[ \delta a_{\mu}^{\text{LO-HVP}} \times 10^{10} = 9.5(10.2) \] (quark connected + one disconnected and qQED. Also relevant to use tau decay input for HVP, [M. Bruno, 27 Fri Hadron Structure].)

- **FNAL/HPQCD/MILC PRL2018**
  \[ \delta a_{\mu}^{\text{LO-HVP}} \times 10^{10} = 9.5(4.5) \] (Strong Isospin Breaking only).

- **QCDSF Prelim:**
  \[ \frac{\delta a_{\mu}^{\text{LO-HVP}}}{a_{\mu}^{\text{LO-HVP}}} \lesssim 1\% \] (Dynamical QED, \( M_{\pi} \sim 400\text{[MeV]} \)).
The obvious: $a_{\mu}^{\text{LO-HVP}}$

- Lattice errors $\sim 2\%$ vs phenomenology errors $\sim 0.4\%$.
- Some lattice results suggest new physics others not but all compatible with phenomenology.
\( a_{LO-HVP}^{\mu} \): flavor by flavor comparison

- \( a_{\mu ud}^{LO-HVP} \cdot 10^{10} \)
  - Mainz 17 (TMR+FV)
  - N\(_f=2\)
  - N\(_f=2+1\)
  - N\(_f=2+1+1\)
  - Mainz (prelim)
  - RBC/UKQCD 18
  - HPQCD 16
  - BMWc 17
  - ETM (prelim)
  - FHM (prelim)

- \( a_{\mu, u}^{LO-HVP} \cdot 10^{10} \)
  - Mainz 17 (TMR)
  - N\(_f=2\)
  - N\(_f=2+1\)
  - N\(_f=2+1+1\)
  - Mainz (prelim)
  - RBC/UKQCD 18
  - HPQCD 14
  - ETM 17
  - BMWc 17

- \( a_{\mu, s}^{LO-HVP} \cdot 10^{10} \)
  - Mainz 17 (TMR)
  - RBC/UKQCD 18
  - N\(_f=2\)
  - N\(_f=2+1\)
  - N\(_f=2+1+1\)
  - HECQD 14
  - ETM 17
  - BMWc 17

- \( a_{\mu, disc}^{LO-HVP} \cdot 10^{10} \)
  - Mainz 17 (TMR)
  - RBC/UKQCD 18
  - N\(_f=2\)
  - N\(_f=2+1\)
  - N\(_f=2+1+1\)
  - BMWc 17

- \( a_{LO-HVP}^{\mu, s, c, disc} \) already known with high enough precision for FNAL E989

- “Disagreement” is on \( a_{\mu, ud}^{LO-HVP} \)
Derivatives of $\Pi(Q^2)$ at $Q^2 = 0$: $ud$ contribution

\[ \Pi_n = \frac{1}{n!} \left. \frac{d^n \hat{\Pi}(Q^2)}{dQ^2^n} \right|_{Q^2 \to 0} = \sum x \frac{(-x^2)^{n+1}}{(2n+2)!} \langle j_\mu(x) j_\mu(0) \rangle. \]

- In Pad picture, larger $\Pi_1(\Pi_2)$ $\to$ larger (smaller) $a_\mu$.

- HPQCD 16 has slightly smaller $\Pi_{1ud}$ and larger $-\Pi_{2ud}$ than BMWc 16 and RBC/UKQCD 18 $\to$ combine to give smaller $a^{LO-HVP}_{\mu, ud}$.

- Suggests that HPQCD 16 has smaller $C(t)$ for $t \sim 1$ fm but larger for $t \gtrsim 2$ fm

- Difference comes from HPQCD 16’s large corrections
Time window: lattice + phenomenology

- **Figure:** [RBC/UKQCD-PRL2018, talk by C. Lehner and Colleagues (27 Fri, Hadron Structure)]. In $a_{\mu}^{\text{LO-HVP}} = (\alpha/\pi)^2 \sum_t W(t, Q^2/m_{\mu}^2) C(t)$, consider lattice/pheno correlators;

$$C_{\text{lat}}(t) = \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^{3} \langle j_i(\vec{x}, t) j_i^{ud}(0) \rangle, \quad C_{\text{pheno}}(t) = \frac{1}{2} \int_0^\infty ds \sqrt{s} R(s) e^{-\sqrt{s}|t|}.$$

$C_{\text{lat}}(t)$ may be more precise in intermediate $t \sim 1$ [fm].

- Consider the decomposition $C(t) = (C^{\text{SD}} + C^{\text{W}} + C^{\text{LD}})(t)$, where $(C^{\text{SD}}, C^{\text{W}}, C^{\text{LD}})(t) = C(t)(1 - \Theta(t, t_0, \Delta), \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta), \Theta(t, t_1, \Delta))$ with the smeared step function, $\Theta(t, t', \Delta) = (1 + \tanh[(t - t' / \Delta)])/2$.

- For $C^{\text{W}}(t)$, use lattice data $C^{\text{W}}_{\text{lat}}$. For the others, use phenomenological data $C^{\text{SD/LD}}_{\text{pheno}}(t_0, t_1, \Delta) = (0.4, 1.0, 0.15)[\text{fm}]$, $a_{\mu}^{\text{LO-HVP}} = 692.5(2.7) \cdot 10^{-10}$ [RBC/UKQCD-PRL2018].
Window Method: DWF vs HISQ vs Pheno.

**Fig.:** T. Blum (27 Fri). Continuum extrapolation of $a^W_{\mu} = \sum_t C^W_{\text{lat}}(t) W(t, m_{\mu})$, where $C^W_{\text{lat}}(t) = C_{\text{lat}}(t)((\Theta(t, t_0, \Delta) – \Theta(t, t_1, \Delta))$ with $t_0 = 4.0$, $t_1 = 1.0$, $\Delta = 0.15$[fm].

*(2+1+1)* HISQ(MILC ensemble) and DWF all physical points in 5.5 [fm] boxes. HISQ and DWF shows 2-3 $\sigma$ tension; lattice spacing, statistics may be responsible. The DWF result is consistent with phenomenology.
Other Important Subjects

- Lattice ($Q^2 < Q^2_{cut}$) - Perturbation ($Q^2 \geq Q^2_{cut}$) Matching [BMW-PRL2018].
- Lattice results of Higher-Order HVP [FNAL/HPQCD/MILC, 1806.08190].
- Dual Propagator + Gounaris-Sakurai-Lüscher Propagator [ETMc-Prelim, Mainz g-2 Workshop].
- Omnès Formula for time-like pion form factor [Mainz Preliminary, talk by H. Wittig (27 Fri, Hadron Structure)].
- HVP for $\sin^2 \theta_W$ [talk by Cè Marco, (27 Fri, Hadron Structure)].
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4 Summary and Conclusions
Lattice computation of $a_{\mu}^{\text{LO-HVP}}$ has total error $\sim 2\% \gg \sim 0.4\%$ from phenomenology. Some results are consistent with no new physics and phenomenology, others with phenomenology and new physics.

Difference comes from $ud$ contribution and most probably from treatment of long-distance physics, for which many progress have been done but need more understandings.

Comparison of $ud$ time moments suggests:
- larger intermediate-distance contribution in [BMWc-PRL2018 and RBC/UKQCD-PRL2018]
- larger long-distance contribution in [HPQCD-PRD2017], associated with model description

With current lattice results, too early to make detailed comparisons with dispersive approach. However, combination of lattice and phenomenology [RBC/UKQCD PRL18, T. Blum Preliminary] may deliver a reliable $0.2\% a_{\mu}^{\text{LO-HVP}}$.

Lattice combined with Experimental Data: Next Talk by Marina.
Summary and Conclusions

- Lattice computation of $a_{\mu}^{\text{LO-HVP}}$ has total error $\sim 2\% \gg \sim 0.4\%$ from phenomenology. Some results are consistent with no new physics and phenomenology, others with phenomenology and new physics.

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- Lattice combined with Experimental Data: Next Talk by Marina.
Top panel [ETMc JHEP2017]: Vector-current correlator data are well described by 1-loop QCD up to $1 \text{fm} > \frac{\hbar c}{\Lambda_{\text{QCD}}}$. This was interpreted as the onset of SVZ Quark-Hadron Duality [NPB1979].

Motivated by the duality, consider the following expression for the vector-current correlator,

$$ V_{\text{dual}}(t) = \frac{5R_{\text{dual}}}{72\pi^2} \int_{s_{\text{dual}}}^{\infty} ds \sqrt{s} e^{-\sqrt{s}t} R_{1l-\text{QCD}}^{1l}(s), $$

where,

$$ R_{1l-\text{QCD}}^{1l}(s) = \left(1 - \frac{4m_{ud}^2}{s}\right)^{1/2} \left(1 + \frac{2m_{ud}^2}{s}\right). $$

This expression differs from 1-loop QCD by two fit params $(R_{\text{dual}}, s_{\text{dual}})$, and combined with 2-pion correlator $V_{\pi\pi}$ constructed via Gounaris-Sakurai $F_{\pi}^{\text{GS}}$.

Bottom panel [ETMc Preliminary]: $(V_{\text{dual}} + V_{\pi\pi})$ describes well lattice data whole range. FV effects and other systematics can be studied with this.
Omnès Formula (Nuovo Cimento (1958))

- Figs: Mainz Preliminary, Thanks to F. Erben (GSI-HIM).
- \[ F_{\pi}(\omega) = \exp \left[ \omega^2 P_{n-1}(\omega^2) + \frac{\omega^{2n}}{\pi} \int_{4M_{\pi}^2}^{\infty} ds \frac{\delta_1(s)}{s^n(s-\omega^2-i\epsilon)} \right] \]

  - Lattice data are used for \( F_{\pi} \) and \( \delta_1 \) and fit parameters are in the Polynomial \( P_{n-1} \).
  - Omnès gives a better description than GS in the middle range.
**Left:** ETM Preliminary. From slide by S.Simula in Mainz g-2 workshop 2018. The continuum limit line (black-solid) becomes sensitive to $m_{ud}$ at physical point.

**Right:** Mainz Preliminary. From slide by H.Meyer in Mainz g-2 workshop 2018. $\tilde{y} = (M_\pi/(4\pi f_\pi))^2$. 

\[ m_\mu = m_\mu^{\text{phys}} \]

\[ \tilde{y} = \left( \frac{M_\pi}{4\pi f_\pi} \right)^2 \]
**Left:** ETMc Preliminary, (SIB + QED) corrections for light components. The chiral/continuum-extrapolation is investigated with FV effects taken account.

**Right:** ETMc JHEP2017, (SIB + QED) corrections for strange component integrand for each diagrams shown previous pages. The charm is also investigated. In both, partial cancellations among the various diagrams.
Comparison of derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

$$\Pi_n = \frac{1}{n!} \frac{d^n \hat{\Pi}(Q^2)}{(dQ^2)^n} \bigg|_{Q^2 \to 0} = \sum_x \left( \frac{-\hat{x}^2}{2n+2} \right)^{n+1} \frac{1}{(2n+2)!} \langle j_\mu(x) j_\mu(0) \rangle.$$