Comparing Different Parameterizations of the z-expansion

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Background: Decay Process: $B \rightarrow \pi \ell \nu_\ell$

- Decay Rate Expression

**Differential Decay Rate (Massless Lepton Limit)**

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{192\pi^3 m_B^3} \lambda (q^2)^{3/2} |f_+(q^2)|^2$$

-$\lambda$(mass of W)
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- $\lambda(q^2) = \left((m_B^2 + m_\pi^2 - q^2)^2 - 4m_B^2 m_\pi^2\right)$
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- $\lambda(q^2) = ((m_B^2 + m_\pi^2 - q^2)^2 - 4m_B^2 m_\pi^2)$

- Exclusive and inclusive decays have determinations of $V_{ub}$ which differ by $2.4\sigma$ [1]
Conformal Mapping

- Transform $q^2 \rightarrow z(q^2, t_0) = \frac{\sqrt{t_+-q^2}-\sqrt{t_+-t_0}}{\sqrt{t_+-q^2}+\sqrt{t_+-t_0}}$ [5]
Conformal Mapping

- Transform $q^2 \rightarrow z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$ [5]
- Visually what is happening:

**Figure**: Image is borrowed from upcoming Fermilab $B \rightarrow K$ paper, Image Credit: Yuzhi Liu
BGL expansion

Parameterization of vector form factor

\[ f_+(q^2; t_0) = \frac{1}{B(q^2)\phi(q^2)} \sum_{n=0}^{N} a_n z^n \] [4]

- \( B(q^2) \) is a function which characterizes the pole in the \( q^2 \) plane
- \( \phi(q^2) \) is a function which arises from unitarity requirements and imposes a simple constraint on the coefficients
Parameterization of the vector form factor

\[
f_+(q^2; t_0) = \frac{1}{1-q^2/m_B^2} \sum_{n=0}^{N-1} b_n \left( z^n - (-1)^{N-n} \frac{n}{N} z^N \right) [3]
\]

- The complicated function of \( z \) comes from the conservation of angular momentum requirement that: \( \frac{df_+(q^2)}{dz} \bigg|_{z=-1} = 0 \).
- \( z = -1 \) corresponds to the threshold for \( B^* \).
- Fixes issue with BGL parameterization by having the appropriate \( 1/q^2 \) falloff behavior.
Outline of methodology

1.) Fit the parameterization of the form factor over different regions of experimental data.
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4.) Test stability of fit coefficients

5.) We do not use any lattice data
Efficacy of predictions: BGL parameterization

\[ X_p^2 = \frac{1}{N_{\text{data points}}} \sum_i \frac{(\Delta B_{\text{exp}} - \Delta B_{\text{fit}})_i}{(\sigma_i^2)} \]

- \( X_p^2 \) is not minimized.

<table>
<thead>
<tr>
<th>fit region</th>
<th>3 params</th>
<th>4 params</th>
<th>5 params</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 26.4 GeV^2</td>
<td>1.02</td>
<td>0.88</td>
<td>1.00</td>
</tr>
<tr>
<td>10 – 26.4 GeV^2</td>
<td>2.12</td>
<td>3.23</td>
<td>5.15</td>
</tr>
<tr>
<td>15 – 26.4 GeV^2</td>
<td>3.42</td>
<td>1.90</td>
<td>7.79</td>
</tr>
<tr>
<td>17 – 26.4 GeV^2</td>
<td>17.56</td>
<td>897</td>
<td>809</td>
</tr>
</tbody>
</table>
Figure: Traditional BGL fits with number of parameters ranging from 3 to 5 (left to right) and fit ranges decreasing (largest: top to smallest: bottom)
stability of fits: coefficients
Efficacy of predictions: BCL parameterization

\[ X_p^2 = \frac{1}{N_{data\ points}} \sum_{i} \frac{(\Delta B_{exp} - \Delta B_{fit})_i}{(\sigma_i^2)} \]

- \( X_p^2 \) is not minimized.

<table>
<thead>
<tr>
<th>fit region</th>
<th>2 params.</th>
<th>3 params.</th>
<th>4 params.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 – 26.4 GeV(^2)</td>
<td>1.04</td>
<td>1.05</td>
<td>0.95</td>
</tr>
<tr>
<td>10 – 26.4 GeV(^2)</td>
<td>1.793</td>
<td>2.073</td>
<td>3.77</td>
</tr>
<tr>
<td>15 – 26.4 GeV(^2)</td>
<td>2.62</td>
<td>3.34</td>
<td>4.33</td>
</tr>
<tr>
<td>17 – 26.4 GeV(^2)</td>
<td>7.97</td>
<td>48.5</td>
<td>156</td>
</tr>
</tbody>
</table>
Figure: Traditional BCL fits with number of parameters ranging from 2 to 4 (left to right) and fit ranges decreasing (largest: top to smallest: bottom)
stability of fits: Coefficients $b_i$

- stable coefficients: $b_0$, $b_1$, and $b_2$
- coefficient $b_3$ is less well distributed.
The BCL parameterizations is stable up to order $z^3$ (3 parameters)
BCL takeaway

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- The overestimation of the partial branching fractions is likely caused by overfitting due to the large statistical uncertainties in the large $q^2$ regime.
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- The overestimation of the partial branching fractions is likely caused by overfitting due to the large statistical uncertainties in the large $q^2$ regime.

- Predictions become far more accurate when extended to the $15 \text{ GeV}^2 < q^2 < 26.4 \text{ GeV}^2$ region, slightly outside the region where we have lattice determinations of the form factors.
Comparison of BGL and BCL near lattice range (15 – 26.4 GeV$^2$) at maximal order $z^2$

**BGL fit:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>0.0245(21)</td>
</tr>
<tr>
<td>$a_1$</td>
<td>-0.013(20)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.13(19)</td>
</tr>
<tr>
<td>$\chi^2$/d.o.f.</td>
<td>0.91</td>
</tr>
<tr>
<td>$X_p^2$</td>
<td>3.23</td>
</tr>
</tbody>
</table>

**BCL fit:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0$</td>
<td>0.406(11)</td>
</tr>
<tr>
<td>$b_1$</td>
<td>-0.42(10)</td>
</tr>
<tr>
<td>$b_2$</td>
<td>[0.70(67)]</td>
</tr>
<tr>
<td>$\chi^2$/d.o.f.</td>
<td>0.97</td>
</tr>
<tr>
<td>$X_p^2$</td>
<td>2.62</td>
</tr>
</tbody>
</table>
Comparison of BGL and BCL in lattice range (17 – 26.4 GeV$^2$) at order $z^2$

**BGL Data**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>0.0240(20)</td>
</tr>
<tr>
<td>$a_1$</td>
<td>-0.009(32)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.03(41)</td>
</tr>
<tr>
<td>$\chi^2$/d.o.f.</td>
<td>0.96</td>
</tr>
<tr>
<td>$X_p^2$</td>
<td>17.59</td>
</tr>
</tbody>
</table>

**BCL Data**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0$</td>
<td>0.405(11)</td>
</tr>
<tr>
<td>$b_1$</td>
<td>-0.30(16)</td>
</tr>
<tr>
<td>$b_2$</td>
<td>[-0.6(1.5)]</td>
</tr>
<tr>
<td>$\chi^2$/d.o.f.</td>
<td>0.96</td>
</tr>
<tr>
<td>$X_p^2$</td>
<td>7.97</td>
</tr>
</tbody>
</table>
Examination

• for 15 – 26.4 GeV$^2$ fit region predictions are nearly identical. BCL errorbands are smaller.
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- Comparing $\chi^2$/d.o.f. values for fit are nearly identical: $\chi^2$/d.o.f. = 0.91 (BGL) and $\chi^2$/d.o.f. = 0.97
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Comparing $\chi^2$/d.o.f. values for fit are nearly identical: $\chi^2$/d.o.f. = 0.91 (BGL) and $\chi^2$/d.o.f. = 0.97

Considering only the lattice region (17 – 26.4 GeV^2) BCL parameterization overestimates partial branching fractions less than BGL parameterization.
Examination

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- Comparing $\chi^2$/d.o.f. values for fit are nearly identical: $\chi^2$/d.o.f. $= 0.91$ (BGL) and $\chi^2$/d.o.f. $= 0.97$

- Considering only the lattice region ($17 - 26.4$ GeV$^2$) BCL parameterization overestimates partial branching fractions less than BGL parameterization.

- Comparing $\chi^2$/d.o.f. are nearly equivalent.
Comparisons between BCL and BGL

Take Away

What is the take away?

- the BCL parameterization provides a better estimate of the low $q^2$ regime than the BGL parameterization does.
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- order $z^2$ and $z^3$ fits provide determinations determinations of the decay spectrum than $z^4$ parameter fits.
What is the take away?

- the BCL parameterization provides a better estimate of the low $q^2$ regime than the BGL parameterization does.
- order $z^2$ and $z^3$ fits provide determinations of the decay spectrum than $z^4$ parameter fits.
- Efficacy of this tool when examining $B \rightarrow \pi \ell \nu$ is limited by the statistical uncertainty associated with partial branching fractions measured in the high $q^2$ region due to phase space suppression.
Why should the lattice community care?

- this procedure can help us identify which parameterizations of the form factors provide better extrapolation of our lattice calculations.
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- this procedure can help us identify which parameterizations of the form factors provide better extrapolation of our lattice calculations.
- this procedure can identify possible energy regions of interest to examine using lattice calculations that have not been currently unexamined due to noise in signal extraction.
Where to go?

- Examine other semileptonic decay: e.g. $B_s \rightarrow K\ell\nu$, $B \rightarrow D\ell\nu$
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- Examine FCNC decays: e.g. $B \rightarrow \pi\ell\ell$, $\Lambda_b \rightarrow \Lambda\ell\ell$
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- Examine other semileptonic decay: e.g. $B_s \rightarrow K \ell \nu$, $B \rightarrow D \ell \nu$
- Examine FCNC decays: e.g. $B \rightarrow \pi \ell \ell$, $\Lambda_b \rightarrow \Lambda \ell \ell$
- Re-examine $B \rightarrow \pi \ell \nu$ when LHCb releases the results.
We would like to thank A. Schwartz for discussions regarding $B \rightarrow D$ decays. This research was supported in part by the Department of Energy under Award Numbers DOE grant DE-SC0010113.
Further Reading I


Further Reading II


Appendix: BGL functions

- \[ B(q^2) = \frac{z(q^2, t_0) - z(m_{B^*}^2, t_0)}{1 - z(q^2, t_0)z(m_{B^*}^2, t_0)} \]

- \[ \phi(q^2, t_0) = \sqrt{\frac{1}{32\pi \chi_1(0)}} \left( \sqrt{t_+ - q^2} + \sqrt{t_+ - t_0} \right) \]
  \[ \times \frac{t_+ - q^2}{(t_+ - t_0)^{1/4}} \left( \sqrt{t_+ - q^2} + \sqrt{t_+} \right)^{-5} \]
  \[ \times \left( \sqrt{t_+ - q^2} + \sqrt{t_+ - t_-} \right)^{3/2} \]