Pion Distribution Amplitude from lattice QCD: towards the continuum limit

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Pion distribution amplitude

**Definition**

Pion DA is the quantum amplitude that the pion moving with momentum $p$ is built of a pair of quark and antiquark moving with momentum $xp$ and $(1-x)p$ respectively.

**Relevance**

Pion photoproduction: two off-shell photons provide the hard scale necessary for the factorization into the perturbative and non-perturbative parts. Transition form factor measured most recently experimentally by BaBar '09 and Belle '12.

**Implementation**

2nd moment of the pion DA, $\langle \xi^2 \rangle$, can be obtained numerically from two-point correlation functions.
Pion distribution amplitude

**Definition**

\[
\langle 0| \bar{d}(z_2 n) \gamma_5 [z_2 n, z_1 n] u(z_1 n) |\pi(p)\rangle = \]

\[
= i f_\pi(p \cdot n) \int_0^1 dx e^{-i(z_1 x + z_2 (1-x))p \cdot n} \phi_\pi(x, \mu^2)
\]

Neglecting isospin breaking effects, \( \phi_\pi(x) \) is symmetric under the interchange of momentum fraction \( x \to (1-x) \)

\[
\phi_\pi(x, \mu^2) = \phi_\pi(1-x, \mu^2)
\]

Moments of the momentum fraction difference \( \xi = x - (1-x) \) can be estimated on the lattice

\[
\langle \xi^{2n} \rangle = \int_0^1 dx (2x - 1)^{2n} \phi_\pi(x, \mu^2)
\]

\[
\phi_\pi(x, \mu^2) = 6x(1-x) \left[ 1 + \sum_n a_{2n}^{\pi}(\mu) C_{2n}^{3/2} (2x - 1) \right]
\]
Local operators

The nonlocal operator can be Taylor expanded and expressed in terms of local operators with derivatives

\[
\bar{d}(z_2 n) \gamma_5 [z_2 n, z_1 n] u(z_1 n) = \sum_{k,l=0}^{\infty} \frac{z_2^k z_1^l}{k!l!} n^\rho n^{\mu_1} \ldots n^{\mu_{k+l}} M^{(k,l)}_{\rho,\mu_1,\ldots,\mu_{k+l}}
\]

where

\[
M^{(k,l)}_{\rho,\mu_1,\ldots,\mu_{k+l}} = \bar{d}(0) \bar{D}_{\mu_1} \ldots \bar{D}_{\mu_k} \bar{D}_{\mu_{k+1}} \ldots \bar{D}_{\mu_{k+l}} \gamma_5 u(0)
\]

Consequently,

\[
i^{k+l} \langle 0 | M^{(k,l)}_{\rho,\mu_1,\ldots,\mu_{k+l}} | \pi(p) \rangle = i f \pi p_{\rho} p_{\mu_1} \ldots p_{\mu_{k+l}} \langle x^l (1-x)^k \rangle
\]
Lattice operators for the 2\textsuperscript{nd} moment

Two local operators are relevant

\[ O_{\rho \mu \nu}^{-}(x) = \bar{d}(x) \left[ \not{D}_{(\mu} \not{D}_{\nu} - 2 \not{D}_{(\mu} \not{D}_{\nu} + \not{D}_{(\mu} \not{D}_{\nu} \right] \gamma_{\rho} \gamma_{5} u(x) \]

and

\[ O_{\rho \mu \nu}^{+}(x) = \bar{d}(x) \left[ \not{D}_{(\mu} \not{D}_{\nu} + 2 \not{D}_{(\mu} \not{D}_{\nu} + \not{D}_{(\mu} \not{D}_{\nu} \right] \gamma_{\rho} \gamma_{5} u(x) \]

Their bare matrix elements between vacuum and a pion state are proportional to

\[ \langle 0 | O_{\rho \mu \nu}^{-}(x) | \pi \rangle \sim \langle [x - (1 - x)]^2 \rangle = \langle \xi^2 \rangle \]

\[ \langle 0 | O_{\rho \mu \nu}^{+}(x) | \pi \rangle \sim \langle [x + (1 - x)]^2 \rangle = \langle 1^1 \rangle \]
Lattice operators for the 2\textsuperscript{nd} moment

We estimate the following correlation functions

\[ C_\rho(t, p) = a^3 \sum_x e^{-i px} \langle O_\rho(x, t) J_{\gamma_5}(0) \rangle \]

\[ C^\pm_{\rho \mu \nu}(t, p) = a^3 \sum_x e^{-i px} \langle O^\pm_{\rho \mu \nu}(x, t) J_{\gamma_5}(0) \rangle \]

and construct ratios

\[ R^\pm_{\rho \mu \nu, \sigma}(t, p) = \frac{C^\pm_{\rho \mu \nu}(t, p)}{C_\sigma(t, p)} \sim \rho_\mu \rho_\nu R^\pm \]

which exhibit plateaux and which we fit to extract the value \( R^\pm \).
Lattice operators for the $2^{\text{nd}}$ moment

Finally,

$$\langle \xi^2 \rangle_{\text{MS}} = \zeta_{11} R^- + \zeta_{12} R^+, $$

$$a_{2\text{MS}} = \frac{7}{12} \left[ 5\zeta_{11} R^- + (5\zeta_{12} - \zeta_{22}) R^+ \right]$$

where $\zeta_{ij}$ are renormalization constants estimated non-perturbatively in the RI’/SMOM scheme and matched to the MSbar scheme at NLO.
Pion distribution amplitude on the lattice

New momentum combination

\[ \Theta(Y_{\mu}D_{\nu}D_{\rho}) \]

\[ t \]

\[ 2.5 \quad 5.0 \quad 7.5 \quad 10.0 \quad 12.5 \quad 15.0 \quad 17.5 \quad 20.0 \]

\[ 0.7 \quad 0.8 \quad 0.9 \quad 1.0 \quad 1.1 \quad 1.2 \]

(412, 142)
(421, 214)
(124, 241)
New momentum combination

\[ \langle 1^2(t) \rangle \]

\[ \pi_{4ij}, \pi_{123} \]
\[ K_{4ij}, K_{123} \]
Two trajectories lead to the physical point, a third trajectory has $m_I = m_5$. 

Credit: W. Söldner, Univ. Regensburg
Coordinated Lattice Simulations collaboration

Credit: J. Simeth, Univ. Regensburg
Combined fit

We perform a combined fit to all data points (all lattice spacings and all pion/kaon masses along the three trajectories) with continuum ChPT formula (no chiral logs) supplemented with cutoff effects parametrization

\[
\langle \xi^2 \rangle_\pi = (1 + c_0 a + c_1 a\overline{M}^2 + c_2^\pi a\delta M^2) \langle \xi^2 \rangle_0 + \overline{A}\overline{M}^2 - 2\delta A\delta M^2,
\]

\[
\langle \xi^2 \rangle_K = (1 + c_0 a + c_1 a\overline{M}^2 + c_2^K a\delta M^2) \langle \xi^2 \rangle_0 + \overline{A}\overline{M}^2 + \delta A\delta M^2,
\]

with \( \overline{A}, \delta A \) being low energy constants and

\[
\overline{M}^2 = \frac{2m_K^2 + m_\pi^2}{3}, \quad \delta M^2 = m_K^2 - m_\pi^2.
\]

\( \Rightarrow \) 7 fit parameters
Extrapolation of $\langle 1^2 \rangle$: check

Figure: Old vs. new momentum combination
\( \langle \xi^2 \rangle \)
2nd moment of the pion distribution amplitude

\[ \langle \xi^2 \rangle_{\text{ms}} = \text{phys.} \quad \beta = 3.4 \ a \approx 0.0854 \ \text{fm} \]

\[ m_s = \text{phys.} \quad \beta = 3.46 \ a \approx 0.076 \ \text{fm} \]

\[ m_s = \text{phys.} \quad \beta = 3.7 \ a \approx 0.05 \ \text{fm} \]

\[ m_s = \text{phys.} \quad \beta = 3.85 \ a \approx 0.039 \ \text{fm} \]
2nd moment of the pion distribution amplitude

\[ \langle \xi^2 \rangle \]

Symmetric, \( \beta = 3.4 \ a \approx 0.0854 \ \text{fm} \)

\[ \langle \xi^2 \rangle \]

Symmetric, \( \beta = 3.46 \ a \approx 0.076 \ \text{fm} \)

\[ \langle \xi^2 \rangle \]

Symmetric, \( \beta = 3.55 \ a \approx 0.0644 \ \text{fm} \)

\[ \langle \xi^2 \rangle \]

Symmetric, \( \beta = 3.7 \ a \approx 0.05 \ \text{fm} \)

\[ \langle \xi^2 \rangle \]

Symmetric, \( \beta = 3.85 \ a \approx 0.039 \ \text{fm} \)
Continuum extrapolation: preliminary results, systematics under investigation

In the continuum limit $a \to 0$ we obtain:

$\langle \xi^2 \rangle_\pi = 0.2289(68)$
$\langle \xi^2 \rangle_K = 0.2204(42)$
$\alpha_2^\pi = 0.0847(198)$
$\alpha_2^K = 0.0599(128)$
Conclusions

**Pion distribution amplitude**

Our preliminary results (statistical uncertainties only)

\[ \langle \xi^2 \rangle_\pi = 0.2289(68) \text{ corresponding to } a_2^\pi = 0.085(20) \]

\[ \langle \xi^2 \rangle_K = 0.2204(42) \text{ corresponding to } a_2^K = 0.060(13) \]

Compare with the previous value at finite lattice spacing and for \( N_f = 2 \):

\[ \langle \xi^2 \rangle_\pi = 0.236 \pm 0.008, \text{ (Braun et al. '15).} \]

**Full x dependence of the pion DA**

In a separate project we are currently estimating non-perturbatively the position space DA (that carries information on the full x dependence)

⇒ Philipp Wein talk: Wednesday, 14:40, lecture hall 106

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