HIGHER MOMENTS OF PARTON DISTRIBUTION FUNCTIONS FROM LATTICE QCD

ZOHREH DAVOUDI
UNIVERSITY OF MARYLAND, RIKEN FELLOW

ZD and Savage, PRD 86, 054505 (2012).
ZD, Will Detmold, Mike Endres, Andrew Pochinsky and Phiala Shanahan, work in progress.
CONSTRUCTING PDFS FROM MOMENTS:

\[
\langle x^n \rangle_{q, \mu^2} = \int dx x^n q(x; \mu^2)
\]

\[
\langle p, s \mid \mathcal{O}_{\mu_1 \mu_2 \ldots \mu_n} \mid p, s \rangle |_{\mu^2} = 2 \langle x^n \rangle_{q, \mu^2} p\{\mu_1 \mu_2 \ldots \mu_n\}
\]

LQCD IS IDEAL FOR EVALUATING SUCH MES.

PHENOMENOLOGICALLY 6-8 MOMENTS APPEAR TO BE SUFFICIENT.

HOWEVER, ONLY UP TO THE FIRST THREE MOMENTS HAVE BEEN ACCESSIBLE WITH LQCD DUE TO A POWER-DIVERGENCE MIXING WITH LOWER DIMENSIONAL OPERATORS.
LESSON FROM MODERN LQCD SPECTROSCOPY STUDIES:

![Graphs and diagrams]

**SMEARED OPERATORS FROM A CONTINUUM OPERATOR WITH A GIVEN J**

\[ O_{\Lambda, \lambda}^{[J]} \equiv \sum_M S_{\Lambda, \lambda}^{J,M} O_{J,M} \]

\[ S_{\Lambda, \lambda}^{J,M} = \langle \Lambda, \lambda | J, M \rangle \]

\[ O_{J,M} = (\Gamma \times D_{n_D})^{J,M} \]
VII Am but in short it is usually necessary for us to use the value of a situation where accurate description of... In particular see figure w in Refo [v] where the generally improves as one increases... will follow the "reconstruction" scheme outlined therein... have... from fits to the principal correlators and the... equation t to "reconstruct" the correlator matrix. This... The reconstruction... until at some point... There are two reasons why this technique is not curr... identified... The... (a_1 \times D_{j=1}^1)^J=0}$ $. (a_1 \times D_{j=2}^3)^J=3$ 

RELATED IDEAS:


IN THAT SPIRIT, WE CONSIDERED A SIMPLE OPERATOR:

\[
\hat{\theta}_{L,M}(x; a, N) = \frac{3}{4\pi N^3} \sum_{n} |n| \leq N \phi(x) \phi(x + na) Y_{L,M}(\hat{n})
\]

\[\hat{n} \quad \hat{n} + na \quad \hat{n} + na + a \]

**ZD and Savage, PRD 86, 054505 (2012).**
IN THAT SPIRIT, WE CONSIDERED A SIMPLE OPERATOR:

\[
\hat{\theta}_{L,M} (x; a, N) = \frac{3}{4\pi N^3} \sum_{n}^{\mid n \mid \leq N} \phi (x) \phi (x + na) Y_{L,M} (\hat{n})
\]

NO!
IN THAT SPIRIT, WE CONSIDERED A SIMPLE OPERATOR:

\[ \hat{\theta}_{L,M}(x; a, N) = \frac{3}{4\pi N^3} \sum_{n}^{\vert n \vert \leq N} \phi(x) \phi(x + na) Y_{L,M}(\hat{n}) \]
CORRECT PROCEDURE:
KEEP THE PHYSICAL SIZE OF THE OPERATOR FIXED, THEN TAKE THE CONTINUUM LIMIT:

$Na = \frac{1}{\Lambda}$

$\left(2N\right)\left(\frac{a}{2}\right) = \frac{1}{\Lambda}$

$\left(4N\right)\left(\frac{a}{4}\right) = \frac{1}{\Lambda}$

ZD and Savage, PRD 86, 054505 (2012).
AN EXAMPLE:

\[
\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{n=0}^{N} \phi(\mathbf{x}) \phi(\mathbf{x} + na) Y_{L,M}(\hat{n})
\]

\[
\hat{\theta}_{3,0}(\mathbf{x}; a, N) = \frac{C^{(1)}_{30;10}(N)}{\Lambda} \mathcal{O}^{(1)}_z(\mathbf{x}; a) + \frac{C^{(3)}_{30;10}(N)}{\Lambda^3} \mathcal{O}^{(3)}_z(\mathbf{x}; a) + \frac{C^{(5)}_{30;10}(N)}{\Lambda^5} \mathcal{O}^{(5)}_z(\mathbf{x}; a) +
\]

\[
\frac{C^{(5;RV)}_{30;10}(N)}{\Lambda^5} \mathcal{O}^{(5;RV)}_z(\mathbf{x}; a) + \frac{C^{(3)}_{30;30}(N)}{\Lambda^3} \mathcal{O}^{(3)}_{zzz}(\mathbf{x}; a) + \frac{C^{(5)}_{30;30}(N)}{\Lambda^5} \mathcal{O}^{(5)}_{zzz}(\mathbf{x}; a) +
\]

\[
\frac{C^{(5)}_{30;50}(N)}{\Lambda^5} \mathcal{O}^{(5)}_{zzzzz}(\mathbf{x}; a) + \mathcal{O}\left(\frac{\nabla^7 \mathcal{O}_z}{\Lambda^7}\right)
\]

HOW DO THE COEFFICIENTS SCALE WITH \( N \) (\( a \))? BETTER HAVE:

\[
C^{(d)}_{30;L'0}(N) \text{ IS FINITE FOR } L' = 3
\]

\[
C^{(d)}_{30;L'0}(N) \rightarrow 0 \text{ FOR } L' \neq 3
\]

\[
C^{(d;RV)}_{30;L'0}(N) \rightarrow 0 \text{ AS } N \rightarrow \infty.
\]
THE COEFFICIENT OF DESIRED OPERATOR:

The numerical values of the coefficients in eq. (4), at the classical level, as a function of the maximum shell included in the sum in eq. (2) are shown in fig. 2 and fig. 3. From these plots it is clear that while the coefficients \( C^{(3)}_{30;30} \) and \( C^{(3)}_{30;30} \) reach a finite value for large \( N \), the coefficients of lower and higher angular momentum operators, as well as the

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ZD and Savage, PRD 86, 054505 (2012).
THE COEFFICIENT OF LOWER-DIMENSIONAL OPERATOR:

![Graph showing the coefficient of lower-dimensional operator as a function of N, with lines for $C_{30;10}$ and $C_{30;10}$ Continuum Value.]
THE COEFFICIENT OF HIGHER-DIMENSIONAL OPERATOR:

ZD and Savage, PRD 86, 054505 (2012).
THE COEFFICIENT OF LORENTZ-BREAKING OPERATOR:

ZD and Savage, PRD 86, 054505 (2012).
RECALLING THE EXPANSION OF OUR CHOSEN OPERATOR…

\[
\hat{\theta}_{3,0}(x; a, N) = \frac{C^{(1)}_{30;10}(N)}{\Lambda} \mathcal{O}^{(1)}_{x}(x; a) + \frac{C^{(3)}_{30;10}(N)}{\Lambda^3} \mathcal{O}^{(3)}_{x}(x; a) + \frac{C^{(5)}_{30;10}(N)}{\Lambda^5} \mathcal{O}^{(5)}_{x}(x; a) +
\]

\[
\frac{C^{(5;RV)}_{30;10}(N)}{\Lambda^5} \mathcal{O}^{(5;RV)}_{x}(x; a) + \frac{C^{(3)}_{30;30}(N)}{\Lambda^3} \mathcal{O}^{(3)}_{zz}(x; a) + \frac{C^{(5)}_{30;30}(N)}{\Lambda^5} \mathcal{O}^{(5)}_{zz}(x; a) +
\]

\[
\frac{C^{(5)}_{30;50}(N)}{\Lambda^5} \mathcal{O}^{(5)}_{zzzz}(x; a) + \mathcal{O}\left(\frac{\nabla^7}{\Lambda^7}\right)
\]

ZD and Savage, PRD 86, 054505 (2012).
\[ \Lambda^3 \hat{\theta}_{3,0} (x; a, N) = \alpha_1 \frac{\Lambda^2}{N^2} \mathcal{O}^{(1)}_z (x) + \alpha_2 \frac{1}{N^2} \mathcal{O}^{(3)}_z (x) + \alpha_3 \frac{1}{\Lambda^2 N^2} \mathcal{O}^{(5)}_z (x) + \]
\[ \alpha_4 \frac{1}{\Lambda^2 N^2} \mathcal{O}^{(5; RV)}_z (x) + \alpha_5 \mathcal{O}^{(3)}_{zzz} (x) + \alpha_6 \frac{1}{\Lambda^2} \mathcal{O}^{(5)}_{zzz} (x) + \]
\[ \alpha_7 \frac{1}{\Lambda^2 N^2} \mathcal{O}^{(5)}_{zzzzz} (x) + \mathcal{O} \left( \frac{\nabla^7_z}{\Lambda^4} \right) \]

POWER DIVERGENCE OF THE NAIVE OPERATOR EVIDENT:
\[ \alpha_1 \frac{1}{a^2} \mathcal{O}^{(1)}_z + \alpha_2 \mathcal{O}^{(3)}_z + \alpha_3 a^2 \mathcal{O}^{(5)}_z + \alpha_4 a^2 \mathcal{O}^{(5; RV)}_z + \]
\[ \alpha_5 \mathcal{O}^{(3)}_{zzz} + \alpha_6 a^2 \mathcal{O}^{(5)}_{zzz} + \alpha_7 a^2 \mathcal{O}^{(5)}_{zzzzz} + \mathcal{O} \left( a^4 \nabla^7_z \right) \]

\[ N = 1 \]

ZD and Savage, PRD 86, 054505 (2012).
SEE THE PAPER FOR CAREFUL TREATMENT OF THESE FEATURES IN LATTICE PERTURBATION THEORY. THE CONCLUSION IS THAT:

SCALING OF ROTATIONAL-IN Variant CONTRIBUTIONS AT 1-LOOP LPT WITH WILSON FERMIONS:
\[ \sim \frac{\alpha_s}{N} \]

SCALING OF NON-ROTATIONAL-IN Variant CONTRIBUTIONS AT 1-LOOP LPT WITH WILSON FERMIONS:
\[ \sim \alpha_s a^2 \Lambda_g^2 \sim \frac{\alpha_s}{N_g^2} \]

DOES THIS WORK NON-PERTURBATIVELY?

ZD and Savage, PRD 86, 054505 (2012).
EVEN A SMALL SHELL LARGELY ELIMINATES THE CONTAMINATION:

\[ N = \frac{1}{\Lambda a} \]

A TREE LEVEL EXPECTATION

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EVEN A SMALL SHELL LARGELY ELIMINATES THE CONTAMINATION:

![Graph showing the expectation of a tree level expectation.](image)

\[ N = \frac{1}{\Lambda a} \]

**ZD and Savage, PRD 86, 054505 (2012).**

IN PRACTICE, HOW LARGE CAN THE OPERATOR BE?

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\mu$</th>
<th>$N^2$</th>
</tr>
</thead>
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<tr>
<td>0.08 fm</td>
<td>$\sim 2$ GeV</td>
<td>1</td>
</tr>
<tr>
<td>0.06 fm</td>
<td>$\sim 2$ GeV</td>
<td>2</td>
</tr>
<tr>
<td>0.05 fm</td>
<td>$\sim 2$ GeV</td>
<td>4</td>
</tr>
<tr>
<td>0.04 fm</td>
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<td>$\sim 2$ GeV</td>
<td>6</td>
</tr>
<tr>
<td>0.03 fm</td>
<td>$\sim 5$ GeV</td>
<td>2</td>
</tr>
<tr>
<td>0.03 fm</td>
<td>$\sim 2$ GeV</td>
<td>11</td>
</tr>
</tbody>
</table>


**Generating Gauge Configurations with a Multi-Scale Algorithm Method**
The calculation of the PDF moments requires evaluating the following correlation function

\[ \hat{\theta}_{n,l,m}(x_\mu; \mu) = \frac{2}{\pi^2 N^4} \sum_{n_\mu} |n_\mu| \leq N \bar{\psi}(x_\mu) U_{x_\mu, x_\mu + n_\mu a} \psi(x_\mu + n_\mu a) \mathcal{V}_{n,l,m}(\hat{n}_\mu) \]

It is clear that evaluating this correlation function requires only a forward quark propagator in the lattice calculation to a pion mass of 600 MeV.

As this is an exploratory calculation aimed at testing operator construction, we will perform the calculation of the twist-2 matrix elements in the pion rather than the more pheno-

Ensembles E8, E4 and E2 will be independent, with E4 generated from a separate stream of (quenched) gauge-field configurations whose topology is well sampled. With six

FIG. 3: A schematic representation of the 3-point correlation function of the pion with an insertion of the operator in Eq. (3). Curved lines represent the quark and antiquark fields, and operator

\[ C_{3p}(t_f, t) = \sum_{x_f,x} \sum_{n.n_t} |n_\mu| \leq N \langle 0 | \chi_\pi(x, t) \hat{\theta}_{n,l,m}(x, t; \mu) \chi_\pi^\dagger(0, 0) |0 \rangle \]
$C_{3pt}(t_f; t)/C_{2pt}(t)$ AS A FUNCTION OF $t$ AT A FIXED $t_f$

$N^2 = 1$

$N^2 = 5$

$N^2 = 10$

$N^2 = 15$

$N^2 = 20$

$N^2 = 25$

ZD, Will Detmold, Mike Endres, Andrew Pochinsky and Phiala Shanahan, work in progress.
THE MATRIX ELEMENT OF A HIGH “ANGULAR MOMENTUM” QUARK BILINEAR OPERATOR IN PION AT REST AS A FUNCTION OF THE OPERATOR SIZE:

\[ \langle \pi | \hat{\theta}_{3,3,0} | \pi \rangle \]

\[ N_f = 0, \quad m_\pi \approx 900 [\text{MeV}] \]

\[ \Lambda^{-1} = Na [\text{fm}] \]

\[ 24^3 \times 48, \ a \approx 0.08 [\text{fm}] \]
\[ 32^3 \times 64, \ a \approx 0.06 [\text{fm}] \]
\[ 48^3 \times 96, \ a \approx 0.04 [\text{fm}] \]
\[ 64^3 \times 128, \ a \approx 0.03 [\text{fm}] \]

ZD, Will Detmold, Mike Endres, Andrew Pochinsky and Phiala Shanahan, work in progress.
CONTINUUM LIMIT OF THE NAIVE OPERATOR

\[ \langle \pi | \hat{\theta}_{3,3,0} | \pi \rangle \]

EXTENDED OPERATOR WITH A FIXED SIZE

\[ N_f = 0, \ m_\pi \approx 900 \ [\text{MeV}] \]
IN SUMMARY

- THE PROPOSED OPERATOR ON THE LATTICE APPROACHES THE CONTINUUM OPERATOR IN A SMOOTH WAY WITH CORRECTIONS THAT SCALE AT MOST BY $a^2$. TADPOLE IMPROVEMENT AND GAUGE-FIELD SMEARING ARE ESSENTIAL FOR RECOVERING ROTATIONAL INVARIANCE IN LATTICE GAUGE THEORIES.

- NO POWER DIVERGENCE! THE SPECTRUM OF EXCITED STATES AND HIGHER MOMENTS OF HADRON DISTRIBUTION FUNCTIONS ARE CALCULABLE FROM LATTICE QCD.
OUTLOOK

- CAN THE OPERATOR BE FURTHER IMPROVED TOWARDS THE CONTINUUM LIMIT?
- RENORMALIZATION OF THE OPERATOR AND MATCHING.
- ARE OTHER SMEARING PROFILES POTENTIALLY MORE USEFUL?
- COMPARISON WITH OTHER METHODS AND PROPOSALS, e.g., DETMOLD AND LIN, JI, MONAHPAN AND ORGINOS.
SUPPLEMENTARY SLIDES
AN EXAMPLE OF OPERATOR BASIS: $L=1$, $m=0$

\[
\mathcal{O}_z^{(1)}(x) = \phi(x) \nabla_z \phi(x)
\]

\[
\mathcal{O}_z^{(3)}(x) = \phi(x) \nabla^2 \nabla_z \phi(x)
\]

\[
\mathcal{O}_z^{(5)}(x) = \phi(x) (\nabla^2)^2 \nabla_z \phi(x)
\]

\[
\mathcal{O}_z^{(5,RV)}(x) = \phi(x) \sum_j \nabla_{\nabla_z}^4 \nabla_z \phi(x)
\]

LORENTZ-VIOLATING OPERATOR
HOW ABOUT QCD AND BEYOND CLASSICAL EFFECTS?

\[ \hat{\theta}_{L,M} (x; a, N) = \frac{3}{4\pi N^3} \sum_{n \leq N} \bar{\psi}(x) U(x, x + na) \psi(x + na) \ Y_{L,M}(\hat{n}) \]

FEATURE 1: SOME EXTENDED LINKS MAXIMALLY BREAK ROTATIONAL SYMMETRY

FEATURE 2: NONVANISHING TADPOLES WITH LATTICE REGULARIZATION

ZD and Savage, PRD 86, 054505 (2012).