HISQ 2+1+1 light quark hadronic vacuum polarization at the physical point

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Lattice 2018, Michigan State University

July 27, 2018
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Using lattice QCD and continuum, \( \infty \)-volume pQED

\[
a_\mu(\text{HVP}) = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dq^2 f(q^2) \hat{\Pi}(q^2)
\]

\( f(q^2) \) is known, \( \hat{\Pi}(q^2) \) is subtracted HVP, \( \hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) \), computed directly on Euclidean space-time lattice

\[
\Pi^{\mu\nu}(q) = \int d^4x e^{iqx} \langle j^\mu(x)j^\nu(0) \rangle \quad j^\mu(x) = \sum_i Q_i \bar{\psi}(x)\gamma^\mu \psi(x)
\]

\[
= \Pi(q^2)(q^\mu q^\nu - q^2 \delta^{\mu\nu})
\]
Interchange order of FT and momentum integrals

\[ \Pi(q^2) - \Pi(0) = \sum_t \left( \frac{\cos qt - 1}{q^2} + \frac{1}{2} t^2 \right) C(t) \]

\[ C(t) = \frac{1}{3} \sum_{x,i} \langle j_i(x) j_i(0) \rangle \]

\[ w(t) = 2\alpha^2 \int_0^\infty d\omega \frac{f(\omega^2)}{\omega} \left[ \frac{\cos \omega t - 1}{(2 \sin \omega/2)^2} + \frac{t^2}{2} \right] \]

\[ a_{\mu}^{\text{HVP}} = \sum_t w(t) C(t) \]

(note double subtraction)
Staggered Dirac operator $M$

Sum of hermitian and anti-hermitian parts, so it satisfies (even-odd ordering)

$$M \begin{pmatrix} n_o \\ n_e \end{pmatrix} = \begin{pmatrix} m & M_{oe} \\ M_{eo} & m \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} = (m + i\lambda_n) \begin{pmatrix} n_o \\ n_e \end{pmatrix}$$  \hspace{1cm} (1)

and

$$\begin{pmatrix} m & -M_{oe} \\ -M_{eo} & m \end{pmatrix} \begin{pmatrix} m & M_{oe} \\ M_{eo} & m \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} =$$

$$\begin{pmatrix} m^2 - M_{oe}M_{eo} & 0 \\ 0 & m^2 - M_{eo}M_{oe} \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} = (m^2 + \lambda_n^2) \begin{pmatrix} n_o \\ n_e \end{pmatrix}$$  \hspace{1cm} (2)

$$\begin{pmatrix} m^2 - M_{oe}M_{eo} & 0 \\ 0 & m^2 - M_{eo}M_{oe} \end{pmatrix} \begin{pmatrix} n_o \\ n_e \end{pmatrix} = (m^2 + \lambda_n^2) \begin{pmatrix} n_o \\ n_e \end{pmatrix}$$  \hspace{1cm} (3)

Compute eigenvectors $n_o(e)$, $m^2 + \lambda^2$ of preconditioned Dirac operator
Eigenvectors of preconditioned operator are eigenvectors of $M$ with squared magnitude eigenvalues, construct the even part from odd,

$$n_e = \frac{-i}{\lambda_n} M_{eo} n_o.$$  

Eigenvalues come in $\pm$ pairs: If $(n_o, n_e)$ is an eigenvector with eigenvalue $\lambda$, then 

$$(-1)^x \psi(x) = (-n_o, n_e)$$

is also an eigenvector with eigenvalue $-\lambda$.

$$\begin{pmatrix} m & M_{oe} \\ M_{eo} & m \end{pmatrix} \begin{pmatrix} -n_o \\ n_e \end{pmatrix} = (m - i\lambda_n) \begin{pmatrix} -n_o \\ n_e \end{pmatrix}, \quad (4)$$

Thus we can construct pairs of eigenvectors with $\pm i\lambda$ for each $\lambda^2, n_o$!
HVP using spectral decomposition of $M^{-1}$

Use conserved current

$$J^\mu(x) = -\frac{1}{2} \eta_\mu(x) \left( \bar{\chi}(x + \hat{\mu}) U^\dagger_\mu(x) \chi(x) + \bar{\chi}(x) U_\mu(x) \chi(x + \hat{\mu}) \right)$$

and spectral decomposition of propagator

$$M_{x,y}^{-1} = \sum_n^{N_{(\text{low})}} \frac{\langle x|n\rangle \langle n|y \rangle}{m + i\lambda_n} + \frac{\langle x|n_-\rangle \langle n_-|y \rangle}{m - i\lambda_n}$$

(5)
HVP using spectral decomposition of $M^{-1}$

\[
4J_{\mu}(t_x)J_{\nu}(t_y) = \sum_{m,n} \sum_{\tilde{x}} \frac{\langle m|x+\mu\rangle U_{\mu}(x)\langle x|n\rangle}{\lambda_m} \sum_{\tilde{y}} \frac{\langle n|y\rangle U_{\nu}(y)\langle y+\nu|m\rangle}{\lambda_n}
\]
\[
+ \sum_{\tilde{x}} \frac{\langle m|x+\mu\rangle U_{\mu}(x)\langle x+\mu|n\rangle}{\lambda_m} \sum_{\tilde{y}} \frac{\langle n|y+\nu\rangle U_{\nu}(y)\langle y+\nu|m\rangle}{\lambda_n}
\]
\[
+ \sum_{\tilde{x}} \frac{\langle m|x+\mu\rangle U_{\mu}^\dagger(x)\langle x|n\rangle}{\lambda_m} \sum_{\tilde{y}} \frac{\langle n|y+\nu\rangle U_{\nu}(y)\langle y|m\rangle}{\lambda_n}
\]
\[
+ \sum_{\tilde{x}} \frac{\langle m|x\rangle U_{\mu}(x)\langle x+\mu|n\rangle}{\lambda_m} \sum_{\tilde{y}} \frac{\langle n|y+\nu\rangle U_{\nu}^\dagger(y)\langle y|m\rangle}{\lambda_n}
\]

$\lambda_n$ shorthand for $m \pm i\lambda_n$, need to construct the matrices (meson fields)

\[
(\Lambda_{\mu}(t))_{n,m} = \sum_{\tilde{x}} \langle n|x\rangle U_{\mu}(x)\langle x+\mu|m\rangle(-1)^{(m+n)x+m}
\]

(order eigenvectors $\lambda_0, -\lambda_0, \lambda_1, -\lambda_1, \ldots, -\lambda_{2N_{low}}$)
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### HISQ 2+1+1 physical point ensembles

MILC [Bazavov et al., 2017]

<table>
<thead>
<tr>
<th>$m_\pi$ (MeV)</th>
<th>$a$ (fm)</th>
<th>size</th>
<th>$L$ (fm)</th>
<th>$m_\pi L$</th>
<th>meas (approx-corr-lma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>133</td>
<td>0.12224(31)</td>
<td>$48^3 \times 64$</td>
<td>5.87</td>
<td>3.9</td>
<td>26-26-26</td>
</tr>
<tr>
<td>130</td>
<td>0.08786(26)</td>
<td>$64^3 \times 128$</td>
<td>5.62</td>
<td>3.7</td>
<td>18-18-40</td>
</tr>
<tr>
<td>135</td>
<td>0.05662(18)</td>
<td>$96^3 \times 192$</td>
<td>5.44</td>
<td>3.7</td>
<td>14-22-18</td>
</tr>
</tbody>
</table>
All mode averaging (AMA) combined with full volume low mode averaging (LMA) can be very effective in reducing statistical errors for HVP (C. Lehner)

\[ \text{AMA} \quad \langle O \rangle = \langle O \rangle_{\text{exact}} - \langle O \rangle_{\text{approx}} + \frac{1}{N} \sum_i \langle O_i \rangle_{\text{approx}} \]

\langle O_i \rangle_{\text{approx}} \text{ constructed from props with } N_{low} \text{ exact low modes, sloppy CG}
All mode averaging (AMA) combined with full volume low mode averaging (LMA) can be very effective in reducing statistical errors for HVP (C. Lehner RBC/UKQCD [Blum et al., 2018])

\[
\text{AMA} \quad \langle O \rangle = \langle O \rangle_{\text{exact}} - \langle O \rangle_{\text{approx}} + \frac{1}{N} \sum_i \langle O_i \rangle_{\text{approx}} \\
\text{+LMA} \quad - \frac{1}{N} \sum_i \langle O_i \rangle_{\text{LM}} + \frac{1}{V} \sum_i \langle O_i \rangle_{\text{LM}}
\]

\(\langle O_i \rangle_{\text{approx}}\) constructed from props with \(N_{\text{low}}\) exact low modes, sloppy CG
Noise reduction: AMA+LMA

RBC/UKQCD [Blum et al., 2013, Giusti et al., 2004, DeGrand and Schaefer, 2005]

Huge reduction in statistical error at long distance from full volume low mode average

c.f. [Blum et al., 2018]
Bounding method RBC/UKQCD [Blum et al., 2018], BMW [Borsanyi et al., 2018]

Lower: $C(t) = 0, \ t > T$ (BMW choice)
Upper: $C(t) = C(T)e^{-E_0(t-T)}, \ E_0 = 2\sqrt{m^2 + (2\pi/L)^2}$

(total $a_\mu$ for choice of $T$ is plotted)
Continuum limit at finite volume ($L \approx 5.5 \text{ fm}$)

- $a_{\mu} + b a^2$
- possible $a^4$ error
- poor statistics on $96^3$ point
Window method \textit{RBC/UKQCD [Blum et al., 2018]} comparison with DWF and R-ratio

\[ a_\mu^W = \sum C(t)w(t)(\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)), \quad \Theta(t, t', \Delta) = 0.5(1 + \text{tanh}((t - t')/\Delta)) \]

- allows precise comparison of continuum limit
- \( t_0 = 0.4, t_1 = 1.0, \Delta = 0.15 \text{ fm} \)
- all points physical
- all \( L \approx 5.5 \text{ fm} \)
- difference is 2-3 \( \sigma \): lattice spacing, statistics may be responsible
Window method RBC/UKQCD [Blum et al., 2018] comparison with DWF and R-Ratio

\[ a_{W}^{\mu} = \sum C(t)w(t)(\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)), \quad \Theta(t, t', \Delta) = 0.5(1 + \tanh((t - t')/\Delta)) \]

- allows precise comparison of continuum limit
- \( t_0 = 0.4, t_1 = 1.0, \Delta = 0.3 \) fm
- all physical points
- all \( L \approx 5.5 \) fm
- difference is \( \sim 2\sigma \)
Window method RBC/UKQCD [Blum et al., 2018] comparison with DWF and R-Ratio

\[ a_{\mu}^W = \sum C(t)w(t)(\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)), \quad \Theta(t, t', \Delta) = 0.5(1 + \tanh((t - t')/\Delta)) \]

- allows precise comparison of continuum limit
- \( t_0 = 0.2, \ t_1 = 1.0, \ \Delta = 0.15 \text{ fm} \)
- all physical points
- all \( L \approx 5.5 \text{ fm} \)
- difference is \( \sim 2\sigma \)
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Lattice calculations of HVP important for muon g-2 SM v. Exp comparison

- Physical point 2+1+1 HISQ HVP calculation on large lattices
- AMA+LMA very effective for achieving small statistical error (c.f. RBC/UKQCD)
- u+d quark connected contribution only

Important to compare different lattice calculations

- Window method by RBC/UKQCD allows precise comparisons in continuum limit
- small difference with DWF in continuum limit, R-ratio (∼ 0.7% of total $a_\mu$)
- need to understand differences
- future
  - improve statistics on 0.06 fm, 96³ ensemble
  - compare with other staggered calculations
Acknowledgments

- This research is supported in part by the US DOE.
- Computational resources provided by the USQCD Collaboration
- We thank the MILC Collaboration for the use of their configurations
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B- and D-meson leptonic decay constants from four-flavor lattice QCD.

Lattice calculation of the lowest order hadronic contribution to the muon anomalous magnetic moment.

Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment.

New class of variance-reduction techniques using lattice symmetries.

Hadronic vacuum polarization contribution to the anomalous magnetic moments of leptons from first principles.
Strong-isospin-breaking correction to the muon anomalous magnetic moment from lattice QCD at the physical point.

Improving meson two-point functions by low-mode averaging.

Low-energy couplings of QCD from current correlators near the chiral limit.

On sixth-order radiative corrections to $a(\mu)-a(e)$.
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Light quark mass dependence of $a_{\mu}$

- The smeared correlators have smaller overlap with excited states than the local-local correlator.
- The light-quark mass dependence is statistically well resolved because the three points are strongly correlated, and is smaller after rescaling.
- Doubling the correlator extent to a temporal size by using the infinite-time fit function and periodic boundary conditions, we correct for the finite lattice extent.
- Although the fit is consistent with the PDG average for the Breit-Wigner $\pi^+$-improved mass.
- For both ensembles and all valence-quark masses, we observe a roughly 80% of the value of the fitted correlator above the 40% level.
- Following Ref. Chakraborty et al., 2018

$\mu_{\text{HVP}}$ = 779 ± 5 fm agrees within 1.5σ of the PDG average light-quark mass.

$1+1+1+1$ ensembles, respectively; these are statistically significant.

Obtaining good correlated fits with stable central values and errors using the variance of the eigenmodes associated with the correlated contribution to the correlation matrix. For each of our six fits, the number of states and fit range based on the stability of the ground-state and first-excited-state eigenvalues and, thus, the errors on the fit parameters. We choose the number of states and fit range based on the variance of the eigenvalues associated with the correlated contribution to the correlation matrix.

We employ the singular-value-decomposition (SVD) method of 0.015, which modifies about 40% of the eigenvalues.

FIG. 2. Valence-quark-mass dependence of the light-quark mass.

- strong isospin breaking study
- $m_{\pi} = 135$ MeV
- $a = 0.15$ fm
- change in $a_{\mu}$ for 130 MeV pion is negligible $\sim -2 \times 10^{-10}$