$Z_S/Z_P$ from three-flavour lattice QCD

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Outline

1. Motivation: Why is $Z_S/Z_P$ important in the context of quark mass calculations
2. A new method to determine $Z_S/Z_P$ based on Ward identities in the Schrödinger functional framework
3. Preliminary results and crosschecks
Calculation of quark masses

Why are we interested in heavy quark masses?

- Fundamental parameters of the Standard Model
  - Main source of uncertainty in Higgs partial widths comes from $m_c$, $m_b$ and $\alpha_s$ (e.g. arXiv:1404.0319)
- Matching parameters for Heavy Quark Effective Theory

Challenges

- Large discretization effects due to the high mass

⇒ Systematic uncertainties have to be treated carefully
How can we calculate quark masses from LQCD

We work on $N_f = 2 + 1$ \textit{CLS} based ensembles with Wilson-clover fermions.

1. Tune the hopping parameter $\kappa$ such that a particle containing the desired quark has its physical mass.

2. Use an appropriate renormalization pattern to relate this to the renormalized quark mass.
How is the quark mass renormalized in our setup

The standard $^{\text{Alpha}}$ Collaboration method uses the PCAC mass

$$m_{12} = \frac{\langle [\tilde{\partial}_0 A_{o}^{12}(x_{0}) + ac_{A}\partial_0^*\partial_0 P^{12}(x_{0})] P^{21}(0) \rangle}{2\langle P^{12}(x_{0}) P^{21}(0) \rangle}$$

and it’s renormalization and improvement pattern to calculate the renormalized quark mass

$$m_{12R} = \frac{M}{\bar{m}(L)} \frac{Z_{A}}{Z_{P}(L)} m_{12} \left[ 1 + a(b_{A} - b_{P})m_{q12} + a(\bar{b}_{A} - \bar{b}_{P})\text{tr}(M) \right] + O(a^2)$$
How is the quark mass renormalized in our setup

The standard method uses the PCAC mass

\[
m_{12} = \frac{\left\langle \left[ \tilde{\partial}_0 A_{12}^{12}(x_0) + ac_A \partial_0^* \partial_0 P^{12}(x_0) \right] P^{21}(0) \right\rangle}{2 \left\langle P^{12}(x_0) P^{21}(0) \right\rangle}
\]

and it’s renormalization and improvement pattern to calculate the renormalized quark mass

\[
m_{12R} = \frac{M}{\bar{m}(L)} \frac{Z_A}{Z_P(L)} \frac{m_{12}}{m_{12}} \left[ 1 + a(b_A - b_P) m_{q12} + a(\bar{b}_A - \bar{b}_P) \text{tr}(M) \right] + \mathcal{O}(a^2)
\]

\[
\frac{m_{3R}}{m_{1R}} = 2 \frac{m_{13}}{m_{12}} \left[ 1 + (b_A - b_P) \frac{(am_{q3} - am_{q2})}{2} \right] - 1 + \mathcal{O}(a^2)
\]
How is the quark mass renormalized in our setup

- Method used in $N_f = 2$, arXiv:1205.5380
- Light and strange quark masses for $N_f = 2 + 1$ simulations with Wilson fermions
  
  *Jonna Koponen, tomorrow 2:40 pm*

- We want to use a complementary method as a cross check for the future computation of the charm quark’s mass (CLS based, partly joint with the Regensburg group)
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Rolf, Sint, $N_f = 0$
How is the quark mass renormalized in our setup

1. **Subtractive renormalization**

\[ a m_{q,f} = \frac{1}{2 \kappa_f} - \frac{1}{2 \kappa_{\text{crit}}} \]
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2. **Multiplicative renormalization** (plus \( O(a) \) improvement)

\[
m_{fR} = \frac{1}{Z_S(L)} \left\{ \left[ m_{q,f} + (r_m - 1) \frac{\text{tr}(M)}{N_f} \right] \\
+ a \left[ b_m m_{q,f} + \bar{b}_m m_{q,f} \text{tr}(M) + (r_m d_m - b_m) \frac{\text{tr}(M^2)}{N_f} + (r_m \bar{d}_m - \bar{b}_m) \frac{\text{tr}(M^2)}{N_f} \right] \right\} + O(a^2)
\]
How is the quark mass renormalized in our setup

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   \]
How is the quark mass renormalized in our setup

We have two different methods to arrive at the renormalized quark mass:

\[
\frac{m_{3R}}{m_{1R}} = 2 \frac{m_{13}}{m_{12}} \left[ 1 + \left( b_A - b_P \right) \frac{am_{q3} - am_{q2}}{2} \right] - 1 + \mathcal{O}(a^2)
\]

\[
m_{1R} - m_{2R} = \frac{1}{Z_S(L)} \left[ m_{q1} - m_{q2} \right] \left[ 1 + ab_m(m_{q1} + m_{q2}) + abm_{q1} \text{tr}(M) \right] + \mathcal{O}(a^2)
\]
Progress on non-perturbative determination of renormalization constants and parameters

- Renormalization and improvement factors for the axial current were previously determined in our group
  (There is also an approach to $Z_A$ in the chirally rotated Schrödinger functional)

- Results for $Z_P$ and the RGI factor $\frac{M}{\overline{m}(L)}$ were published recently by our collaboration
  \[\text{LPHA Collaboration, Campos, Fritzsch, Pena, Preti, Ramos, Vladikas; arXiv:1802.05243}\]

- Determination of $b_m$ and $b_A - b_P$ is almost finished

- $Z_S$ is undetermined so far
Our method for the determination of $Z_S/Z_P$

We start with the general **axial Ward identity**:

$$\int d\sigma_{\mu}(x) \left\langle A_{\mu}^a(x) O_{\text{int}}^b(y) O_{\text{ext}}^c(z) \right\rangle - 2m \int d^4x \left\langle P_a(x) O_{\text{int}}^b(y) O_{\text{ext}}^c(z) \right\rangle$$

$$= - \left\langle \left[ \delta^a_A O_{\text{int}}^b(y) \right] O_{\text{ext}}^c(z) \right\rangle$$
Our method for the determination of $Z_S/Z_P$

We start with the general **axial Ward identity**:

\[
\int \frac{\partial}{\partial R} d\sigma_\mu(x) \left\langle A_\mu^a(x) \mathcal{O}_{\text{int}}^b(y) \mathcal{O}_{\text{ext}}^c(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x) \mathcal{O}_{\text{int}}^b(y) \mathcal{O}_{\text{ext}}^c(z) \right\rangle = -\left\langle \left[ \delta_A^a \mathcal{O}_{\text{int}}^b(y) \right] \mathcal{O}_{\text{ext}}^c(z) \right\rangle
\]

And use the transformation property of the **pseudoscalar density** under small chiral rotations:

\[
\delta_A^a P^b(x) = d^{abc} S^c(x) + \frac{\delta^{ab}}{N_f} \bar{\psi}(x) \gamma_5 \psi(x)
\]
Our method for the determination of $Z_S/Z_P$

We start with the general **axial Ward identity**:

$$\int d\sigma_\mu(x) \left\langle A_\mu^a(x) O_{\text{int}}^b(y) O_{\text{ext}}^c(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x) O_{\text{int}}^b(y) O_{\text{ext}}^c(z) \right\rangle = -\left\langle [\delta_A^a O_{\text{int}}^b(y)] O_{\text{ext}}^c(z) \right\rangle$$

And use the transformation property of the **pseudoscalar density** under small chiral rotations:

$$\delta_A^a P^b(x) = d^{abc} S^c(x) + \frac{\delta^{ab}}{N_f} \bar{\psi}(x) \psi(x)$$

$\triangleright$ $d^{abc} \neq 0$, for $SU(N_f)$ with $N_f \geq 3$
Our method for the determination of $Z_S/Z_P$

Inserting a pseudoscalar density into the chiral Ward identity leads to:

$$
\int d^3 y \int d^3 x \left\langle \left[ A^a_0(y_0+t, x) - A^a_0(y_0-t, x) \right] P^b(y_0, y) O_{\text{ext}} \right\rangle
$$

$$
-2m \int d^3 y \int d^3 x \int_{y_0-t}^{y_0+t} dx_0 \left\langle P^a(x_0, x) P^b(y_0, y) O_{\text{ext}} \right\rangle
$$

$$
= -d^{abc} \int d^3 y \left\langle S^c(y) O_{\text{ext}} \right\rangle
$$
Our method for the determination of $Z_S/Z_P$

When the Ward identity is evaluated on a lattice with Schrödinger functional boundary conditions we end up with:

$$Z_A Z_P \left[ 1 + ab_A m_q + a \bar{b}_A \text{tr}(M) \right] \left[ 1 + ab_P m_q + a \bar{b}_P \text{tr}(M) \right] \times$$

$$\left[ f_{AP}^{abcd} (y_0 + t, y_0) - f_{AP}^{abcd} (y_0 - t, y_0) - 2m f_{PP}^{abcd} (y_0 + t, y_0 - t) \right]$$

$$= - Z_S \left[ 1 + ab_S m_q + a \bar{b}_S \text{tr}(M) \right] f_S^{abcd} (y_0) + O(a^2) + O(am)$$
Our method for the determination of $Z_S/Z_P$

When the Ward identity is evaluated on a lattice with Schrödinger functional boundary conditions we end up with:

$$Z_A Z_P \left[ f_{AP}^{l,abcd}(y_0 + t, y_0) - f_{AP}^{l,abcd}(y_0 - t, y_0) - 2m \bar{f}_{PP}^{abcd}(y_0 + t, y_0 - t) \right] = -Z_S f_s^{abcd}(y_0) + O(a^2)$$
Our method for the determination of $Z_S/Z_P$

When the Ward identity is evaluated on a lattice with Schrödinger functional boundary conditions we end up with:

$$Z_A Z_P \times \left[ f^{l,abcd}_{AP}(y_0 + t, y_0) - f^{l,abcd}_{AP}(y_0 - t, y_0) - 2m \tilde{f}^{abcd}_{PP}(y_0 + t, y_0 - t) \right] = -Z_S$$

$$f^{abcd}_S(y_0) + O(a^2)$$
Figure: Graphical representation of the Wick contractions contributing to $f_{\Gamma\bar{\Gamma}}$ (arXiv:hep-lat/9611015).
\[ f^{abcd}_{\Gamma_1 \Gamma_2} (x, y) \propto - \langle \mathcal{O}^{\dagger a} \bar{\psi}(x) \Gamma^{b} \psi(x) \bar{\psi}(y) \Gamma^{c} \psi(y) \mathcal{O}^{d} \rangle 
\]
\[ = - a^{12} \text{Tr} \left( \tau^{a} \tau^{b} \tau^{c} \tau^{d} \right) \sum_{u,v,u',v'} \left\langle \text{tr} \left\{ \left[ \zeta'(v') \bar{\psi}(x) \right] \Gamma \left[ \psi(x) \bar{\psi}(y) \right] \bar{\Gamma} \left[ \psi(y) \bar{\zeta}(u) \right] \gamma_{5} \left[ \zeta(v) \bar{\zeta}'(u') \right] \gamma_{5} \right\} \right\rangle 
\]
\[ - a^{12} \text{Tr} \left( \tau^{a} \tau^{b} \tau^{d} \tau^{c} \right) \sum_{u,v,u',v'} \left\langle \text{tr} \left\{ \left[ \zeta'(v') \bar{\psi}(x) \right] \Gamma \left[ \psi(x) \bar{\zeta}(u) \right] \gamma_{5} \left[ \zeta(v) \bar{\zeta}(y) \right] \bar{\Gamma} \left[ \psi(y) \bar{\zeta}'(u') \right] \gamma_{5} \right\} \right\rangle 
\]
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\]
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\]
\[ - a^{12} \text{Tr} \left( \tau^{a} \tau^{d} \tau^{c} \tau^{b} \right) \sum_{u,v,u',v'} \left\langle \text{tr} \left\{ \left[ \zeta'(v') \bar{\zeta}(u) \right] \gamma_{5} \left[ \zeta(v) \bar{\zeta}(y) \right] \bar{\Gamma} \left[ \psi(y) \bar{\psi}(x) \right] \Gamma \left[ \psi(x) \bar{\zeta}'(u') \right] \gamma_{5} \right\} \right\rangle 
\]
\[ + a^{12} \text{Tr} \left( \tau^{a} \tau^{b} \right) \text{Tr} \left( \tau^{d} \tau^{c} \right) \sum_{u,v,u',v'} \left\langle \text{tr} \left\{ \left[ \zeta'(v') \bar{\psi}(x) \right] \Gamma \left[ \psi(x) \bar{\zeta}'(u) \right] \gamma_{5} \right\} \text{tr} \left\{ \left[ \psi(y) \bar{\zeta}(u) \right] \gamma_{5} \left[ \zeta(v) \bar{\psi}(y) \right] \bar{\Gamma} \right\} \right\rangle 
\]
\[ + a^{12} \text{Tr} \left( \tau^{a} \tau^{c} \right) \text{Tr} \left( \tau^{d} \tau^{b} \right) \sum_{u,v,u',v'} \left\langle \text{tr} \left\{ \left[ \zeta'(v') \bar{\psi}(y) \right] \bar{\Gamma} \left[ \psi(y) \bar{\zeta}'(u) \right] \gamma_{5} \right\} \text{tr} \left\{ \left[ \psi(x) \bar{\zeta}(u) \right] \gamma_{5} \left[ \zeta(v) \bar{\psi}(x) \right] \Gamma \right\} \right\rangle 
\]
Flavour choices

- We are free to choose the flavour indices $a, b, c, d$ to arrive at different Ward identities differing only by ambiguities proportional to the lattice spacing.

- Two specific choices seem to be beneficial from a numerical point of view:
  - $\text{WI}(83)-\text{WI}(41)$: $[a = c = 8, b = d = 3] - [a = c = 4, b = d = 1]$ leads to a Ward identity where disconnected and one kind of connected diagrams contribute.
  - $\text{WI}(2568)$: $a = 2, b = 5, c = 6, d = 8$ leads to a Ward identity where all connected but no disconnected diagrams contribute.
Schrödinger functional boundary conditions

Line of constant physics (system size $L \approx 1.2$ fm)

Use of wavefunctions to maximize the overlap with the ground state

Table: Summary of simulation parameters of the gauge configuration ensembles used in this study, as well as the number of (statistically independent) replica per ensemble ‘ID’ and their total number of molecular dynamics units.
Preliminary chiral extrapolation of $Z_P/Z_S$ derived from WI(2568) without and with mass term for $g_0^2 = 1.7084$. Data points from Ensemble B2k1, which violates the constant physics condition, for comparison.
Figure: Preliminary results for $Z_P/Z_S$ from WI(83)-WI(41) and WI(2568) with interpolating Padé fits. Dashed lines indicate the bare couplings used in CLS simulations.
Thank you for your attention!