# Resonant dynamics in presence of space charge

G. Franchetti, GSI

# Treatment of nonlinear dynamics

Main effects of Nonlinearities

1) Amplitude dependent detuning
2) Resonances (stable, unstable)
3) DA and all stochastic seas

**Treatment**

**Analytic**

1) Perturbative
2) Normal form

R. Hagedorn, Report No. CERN 57–1, 1957.
R. Hagedorn and A. Schoch, Report No. CERN 57–1
A. Schoch, Report No. CERN 57-23, 1958
G. Guignard, Report No. CERN 76–06, 1976.
G. Guignard, Report No. CERN 78–11, 1978.

**Numerical**

1. Symplectic tracking
2. Early indicators (not so easy)

# Popular tools

Resonance driving terms

- Driving terms Measurement
- Resonance compensation
- Prediction of amplitude dependent detuning

Normal Form

Interpolating Hamiltonian,

Note: DA still computed by brute force tracking

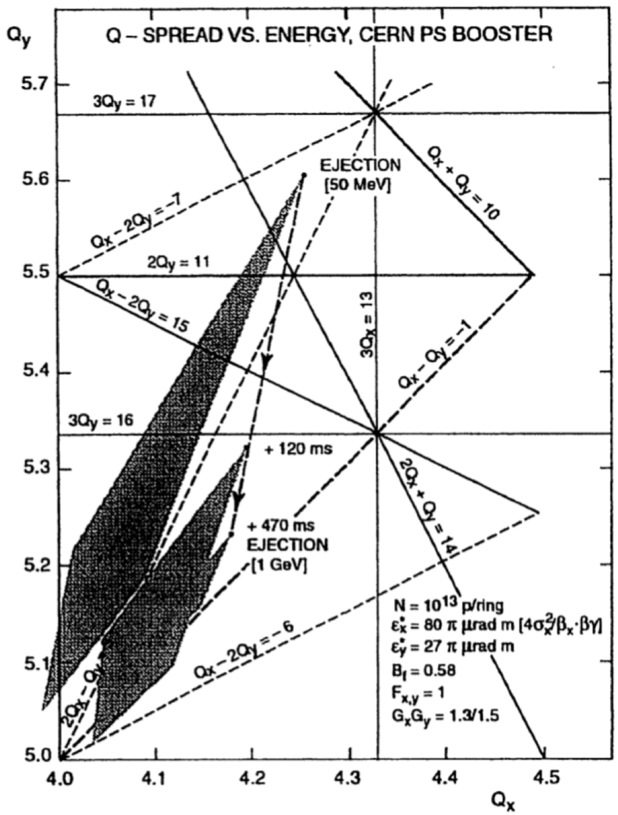
# Space Charge (direct)

| | **Coherent Effects** | **Incoherent Effects** |
|---|---|---|
| Feature | Space charge forces create a collective beam response: | Space charge forces acts only on particles like “external forces” |
| Example | • Envelope oscillations<br>• Envelope instabilities<br>• Coherent tunes<br>• .. | • Amplitude dependent detuning<br>• Tune-spread<br>• Modification of optics<br>• Structure resonances |
| Time scale | Fast | Fast |

Sometimes the distinction is ambiguous!

Interplay: direct vs. indirect, coherent/incoherent subject of investigation

# Space Charge in a circular accelerator




K. Schindl CAS 2003

**Some issues**

1) Space charge + resonances in coasting beams
2) Space charge + resonances in bunched beams
3) Collective beam response to direct space charge forces


# Start of HB series

20th ICFA Advanced Beam Dynamic Workshop on High Intensity and High Brightness Hadron Beams, 8-12 April 2002, Fermilab, Chicago, USA.

# Timeline

**Incoherent**

Incoherent tuneshift
KV
Laslett

**coherent/ core stability**

Emittance preservation
multiparticle simulations
Sacherer's theory

**few % beam loss**

Particle core models
simulations codes

**return of the incoherent**

Machines studies
CERN,GSI, J-PARC, SNS

machine modeling in codes

long term tracking (frozen)

coupling with EC, and other effects

HB starts

~ 1985

~ 1990

~ 2000

Coherent
Vs
Incoherent

Thanks to Shinji

# The framework: perturbative

Ansatz of functional solution: $$x = \sqrt{\beta_x \epsilon_x} \cos(\psi_x(s) + \phi_0)$$

Resonant dynamics: $$x = \sqrt{\beta_x a_x} \cos(\psi_x(s) + \varphi_x)$$

New dynamical variables

Instead of discussing x,x',y,y' we discuss $a_x, \varphi_x, a_y, \varphi_y$

# The resonance → fixed lines

The resonant dynamics is described by the canonical equations of the slow dynamical variables

Distance from the resonance

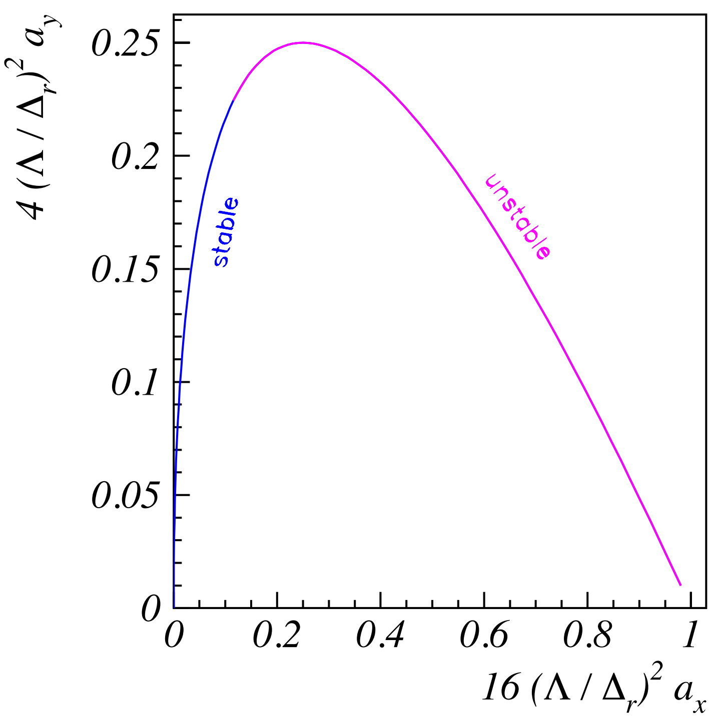
$$\Delta_r = Q_x + 2Q_y - m$$

The fixed lines are the "fixed points" in the "slow" phase space

$$a_x \varphi_x , a_y , \varphi_y$$

There are infinite fixed lines!

Qx + 2Qy=m



# Quantitative prediction

$$0 = \Delta_r \Lambda (-1)^M \left[ \frac{1}{2\sqrt{\tilde{a}_x}} \tilde{a}_y + 2\sqrt{\tilde{a}_x} \right] + \frac{\Delta_r^2}{2}$$

with $\Lambda = \sqrt{\Lambda_c^2 + \Lambda_s^2}$ $\quad \cos\theta = \Lambda_c / \Lambda \quad \sin\theta = \Lambda_s / \Lambda$

$$\Lambda_c = -\sum_j \frac{1}{8L} K_{2j} \sqrt{\beta_{xj}} \beta_{yj} \cos\left[ 2\pi \frac{s_j}{L} N + \mathscr{D}_x(s_j) + 2\mathscr{D}_y(s_j) \right]$$

$$\Lambda_s = -\sum_j \frac{1}{8L} K_{2j} \sqrt{\beta_{xj}} \beta_{yj} \sin\left[ 2\pi \frac{s_j}{L} N + \mathscr{D}_x(s_j) + 2\mathscr{D}_y(s_j) \right]$$

$\Lambda$ Is the amplitude driving term $\qquad$ $\theta$ Is the phase of the driving term

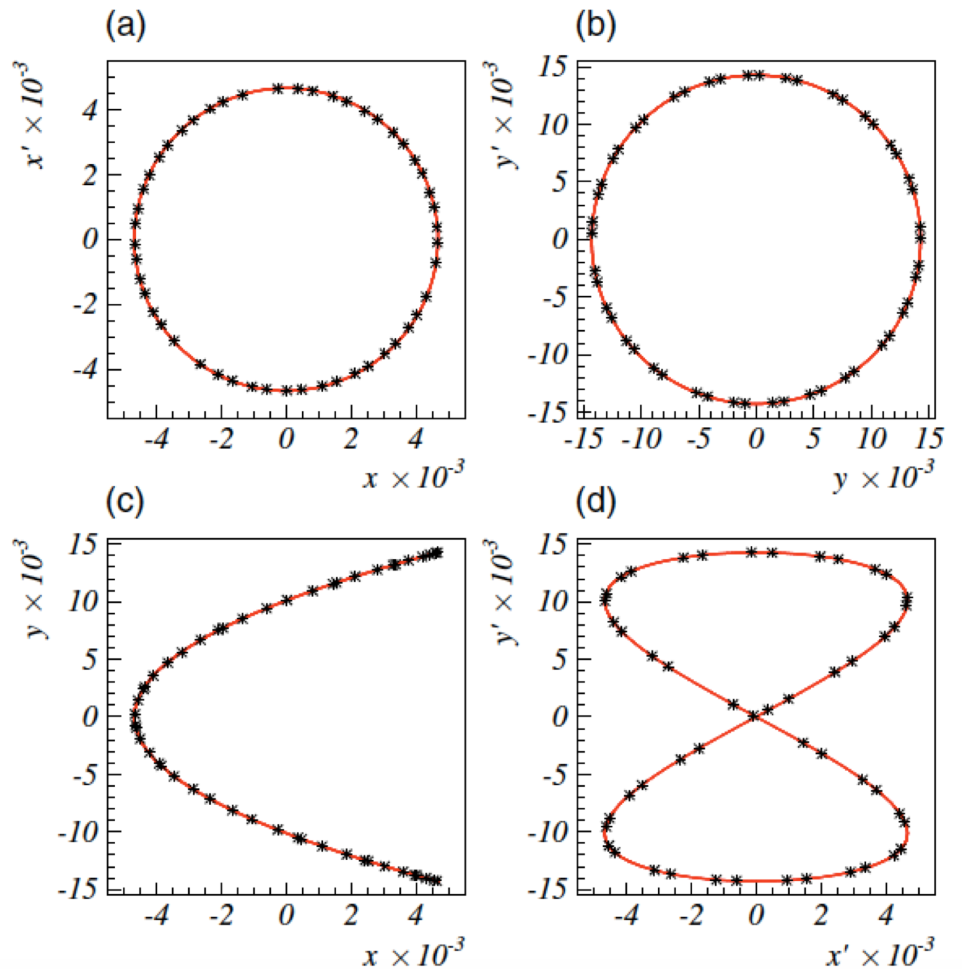
# On the physical space

$$x(t) = \sqrt{\beta_x a_x}\cos(-2t + \pi M),$$
$$y(t) = \sqrt{\beta_y a_y}\cos(t),$$

with

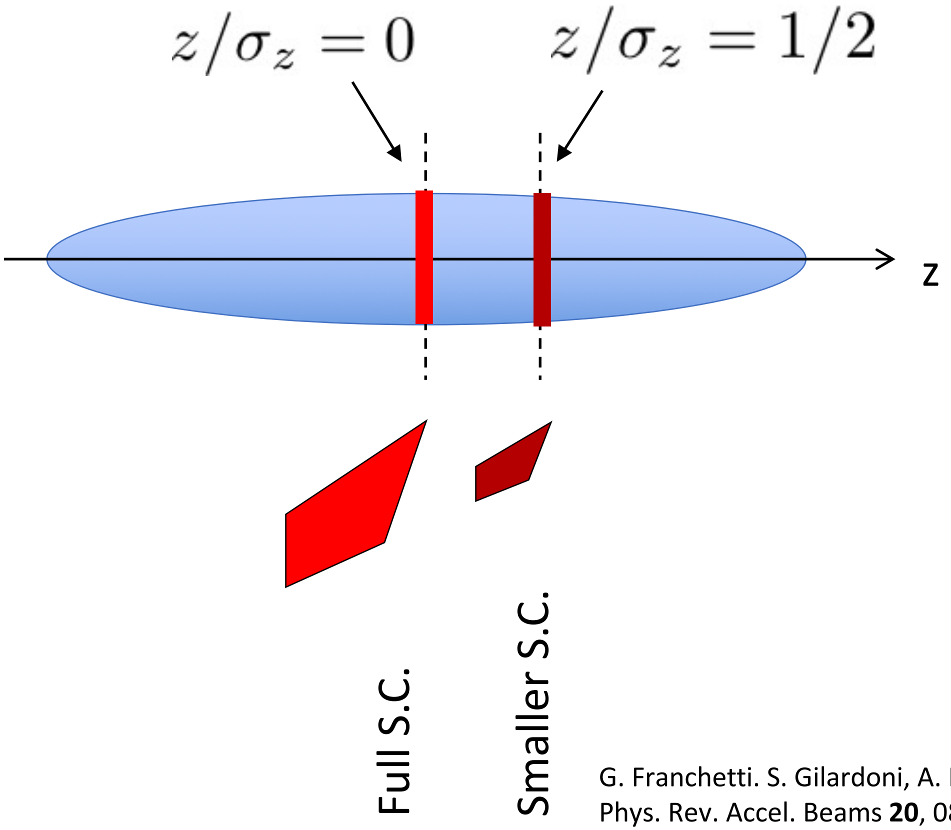
$$\tilde{a}_y = \frac{(2\pi\Delta_r)^2}{4\Lambda^2 L^2}\tau(1-\tau)$$
$$\tilde{a}_x = \frac{(2\pi\Delta_r)^2}{16\Lambda^2 L^2}(1-\tau)^2$$

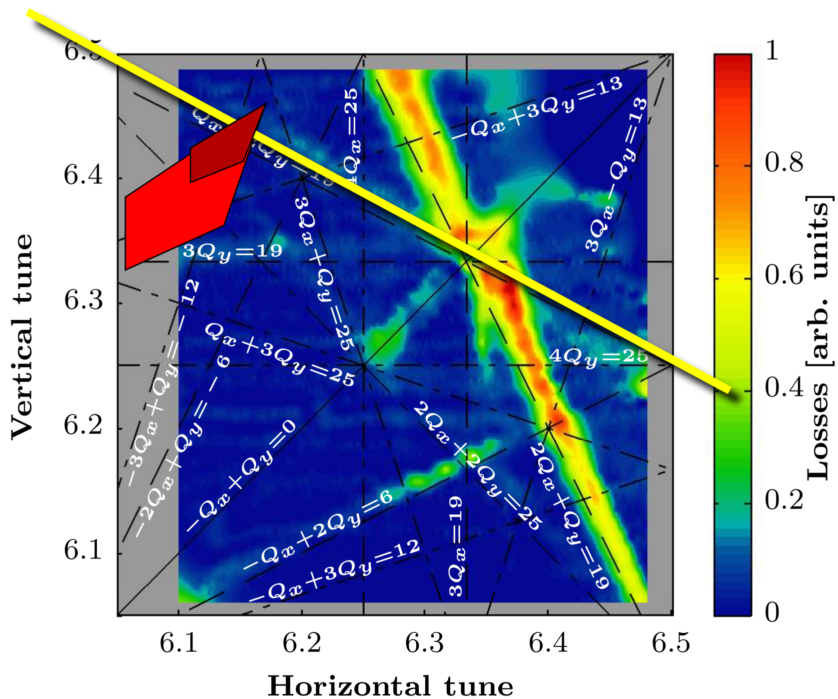
G.Franchetti & F. Schmidt
Phys. Rev. Lett. **114**, 234801 (2015)



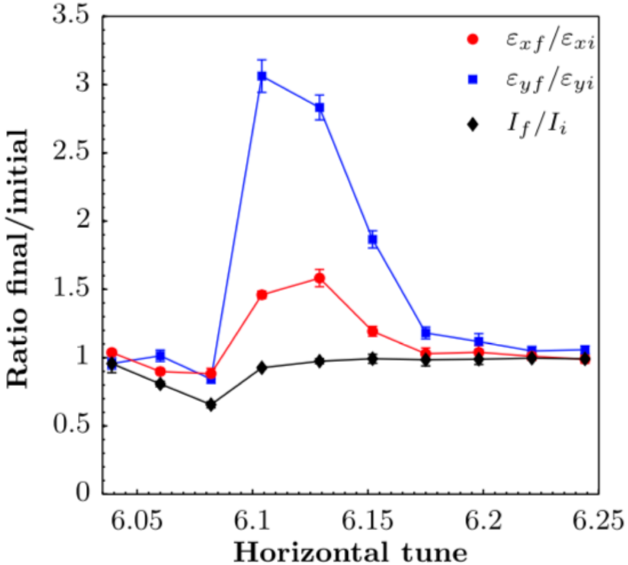
# The effect of space charge

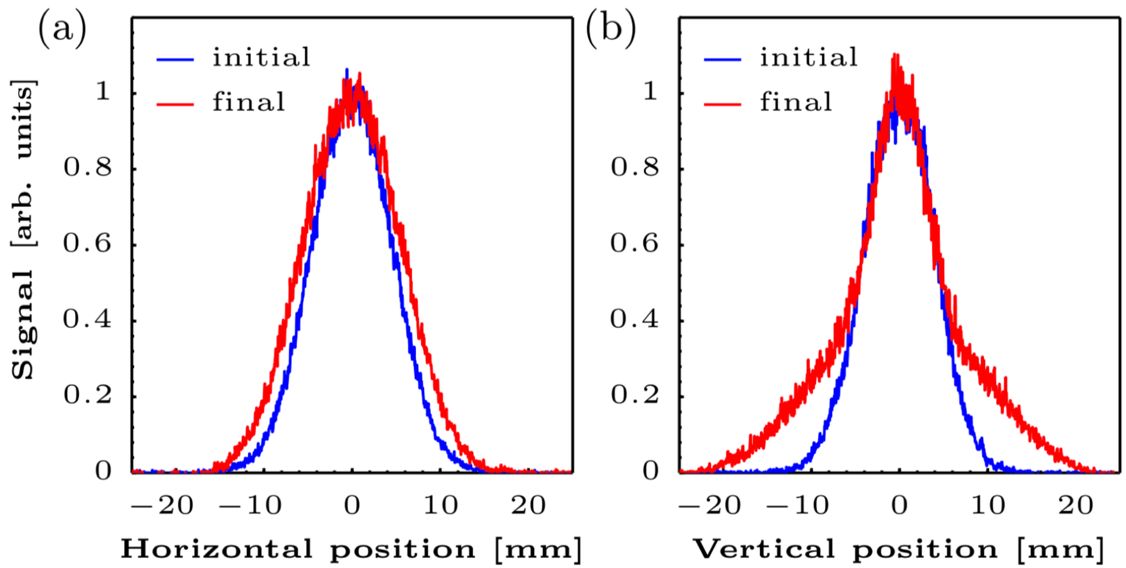
longitudinal motion is kept frozen, so to retrieve Poincare' section orbits




G. Franchetti. S. Gilardoni, A. Huschauer, F. Schmidt, R. Wasef
Phys. Rev. Accel. Beams **20**, 081006 (2017)

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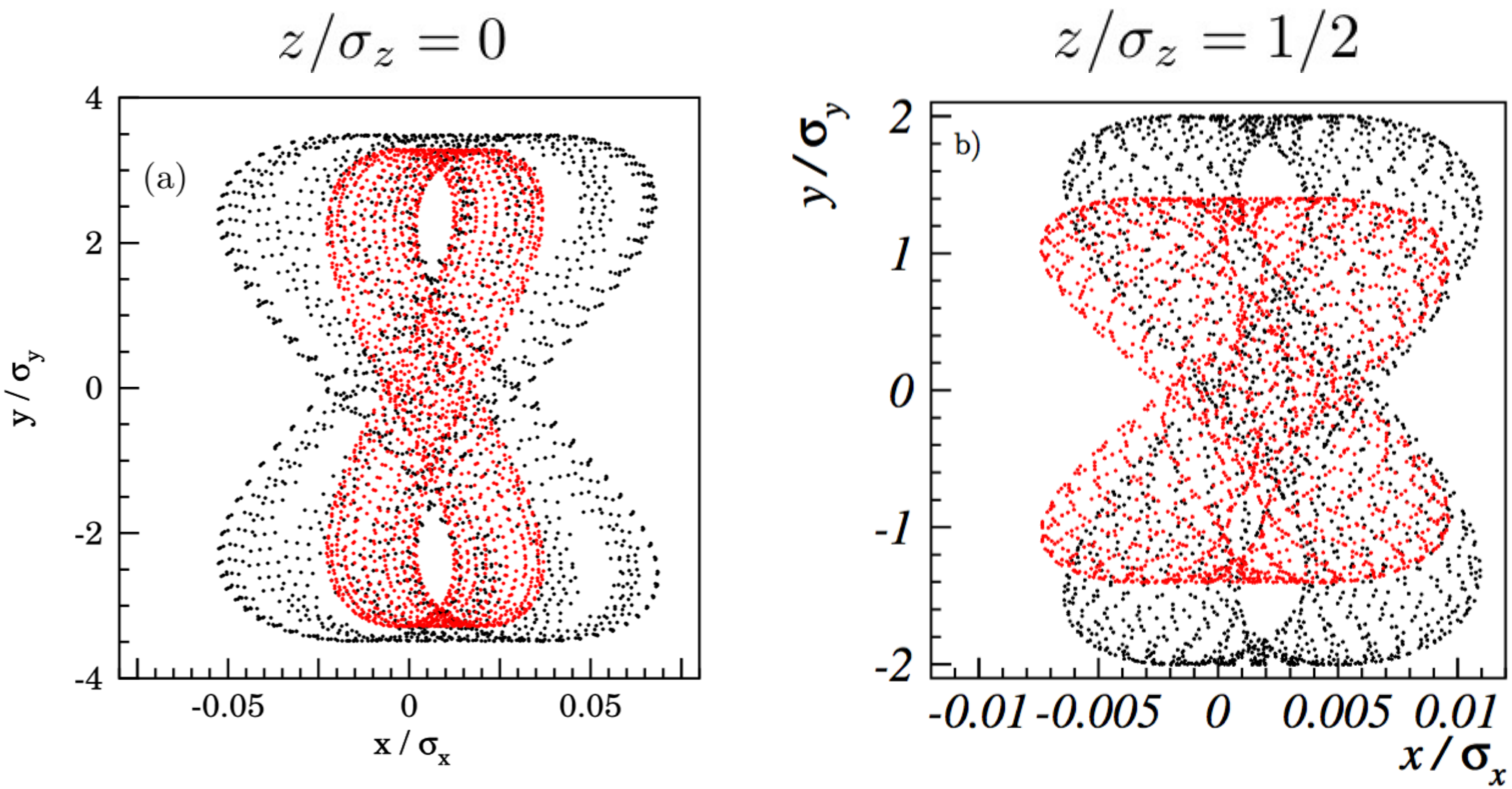


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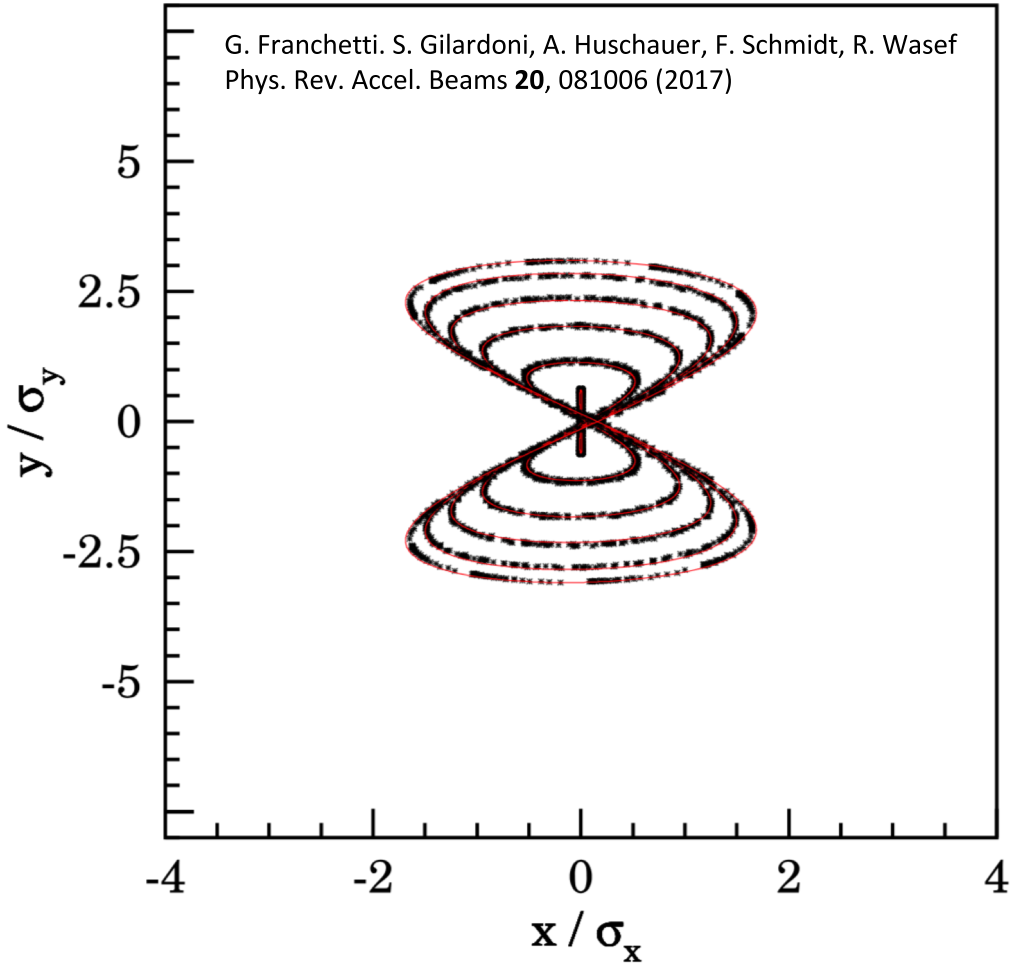
# The adiabatic limit




Any particle at small transverse amplitude is trapped and reached the same ”adiabatic” fixed line

It was concluded that the “adiabatic limit” seems to predict the maximum amplitudes of the diffusing process due to periodic resonance crossing

**Why the adiabatic limit works ?**

# Resonance theory with space charge

To discuss convergence properties one has to consider scaled quantities

$$\rho = \frac{\lambda}{\pi a_0 b_0} F'\left(\frac{x^2}{a_0^2} + \frac{y^2}{b_0^2}\right) \qquad a_0 = \sqrt{\beta_x \epsilon_x}, \quad b_0 = \sqrt{\beta_y \epsilon_y} \qquad \hat{a}_x = a_x / \epsilon_x$$

$$V_{sc} = -\frac{K}{2} \int_0^\infty \frac{F(T(t)) - F(0)}{(a_0^2 + t)^{1/2}(b_0^2 + t)^{1/2}} dt$$

$$C = N_y \hat{a}_x - N_x \hat{a}_y$$

This is an invariant of motion
(in the slow harmonics approximation)

# Dynamics of slow variables

$$\begin{aligned} \mathsf{a}' &= \frac{4\rho_s}{R} N_x \mathsf{a}^{n_x/2} \mathsf{a}_y^{n_y/2} \sin(\Phi) \\ \Phi' &= \frac{2\rho_s}{R} \mathsf{a}^{n_x/2} \mathsf{a}_y^{n_y/2} \left( N_x \frac{n_x}{\mathsf{a}} + N_y \frac{n_y}{\mathsf{a}_y} \right) \cos(\Phi) + \\ &+ \frac{\Delta_r}{R} + \frac{\mathscr{D}_{r,sc}}{R} N_x \frac{d\mathcal{V}_{sc}}{d\mathsf{a}} + \sum_n \frac{\mathscr{D}^n_{r,m}}{R} N_x \frac{d\mathcal{V}^n_m}{d\mathsf{a}} \end{aligned}$$

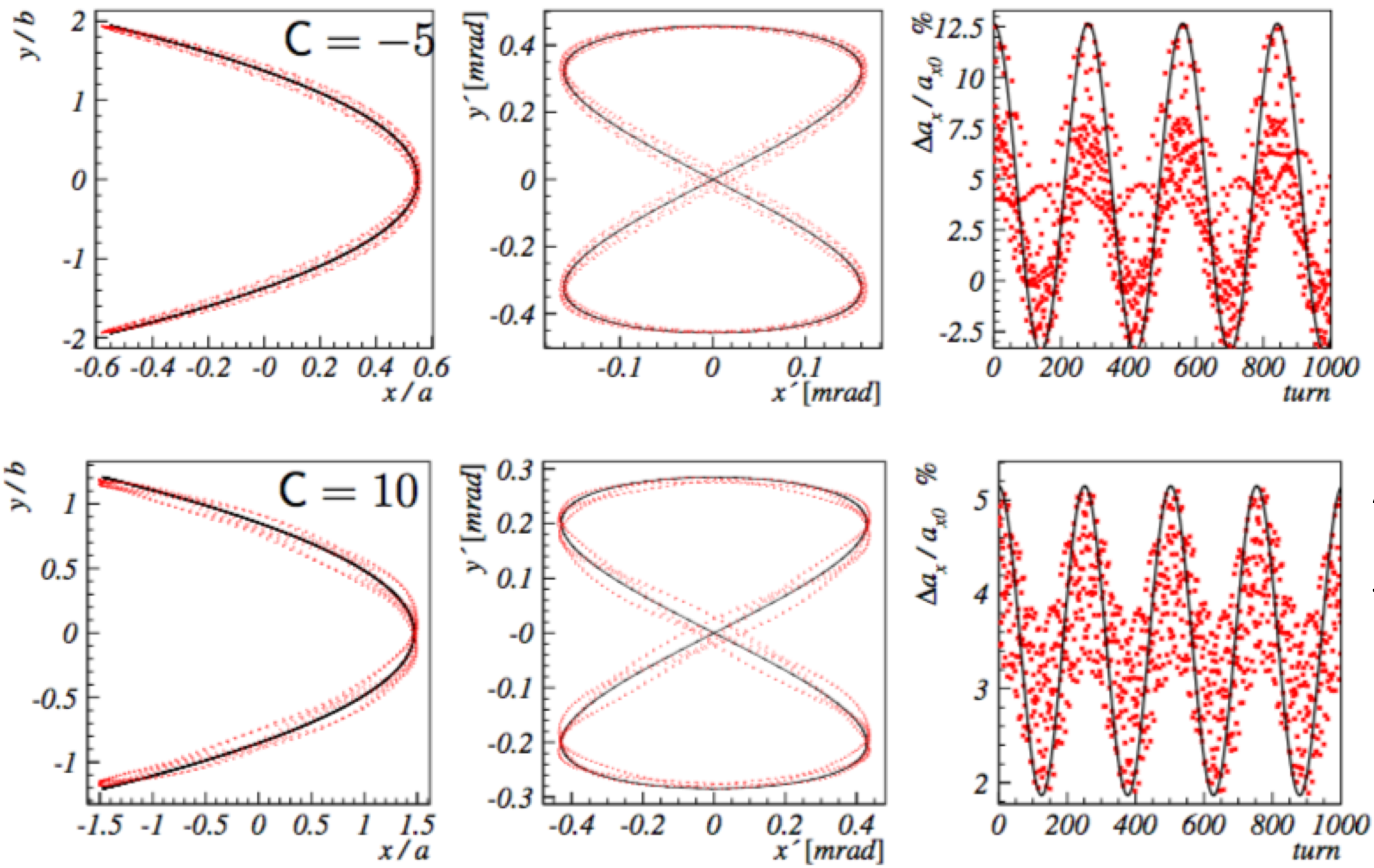
$$\mathscr{D}_{r,sc} = N_x \Delta Q_x + N_y \Delta Q_y$$

"Resonance tune-spread" is naturally obtained from the theory.

$\rho_s$ is a normalized driving term from **lattice nonlinear error** or **space charge** (no dimension)

The infinite, fast, non-resonant harmonics from space charge changes C as $|C - C_0| \propto \Delta Q/Q$

# Fixed lines with space charge




PS-Exp. parameters

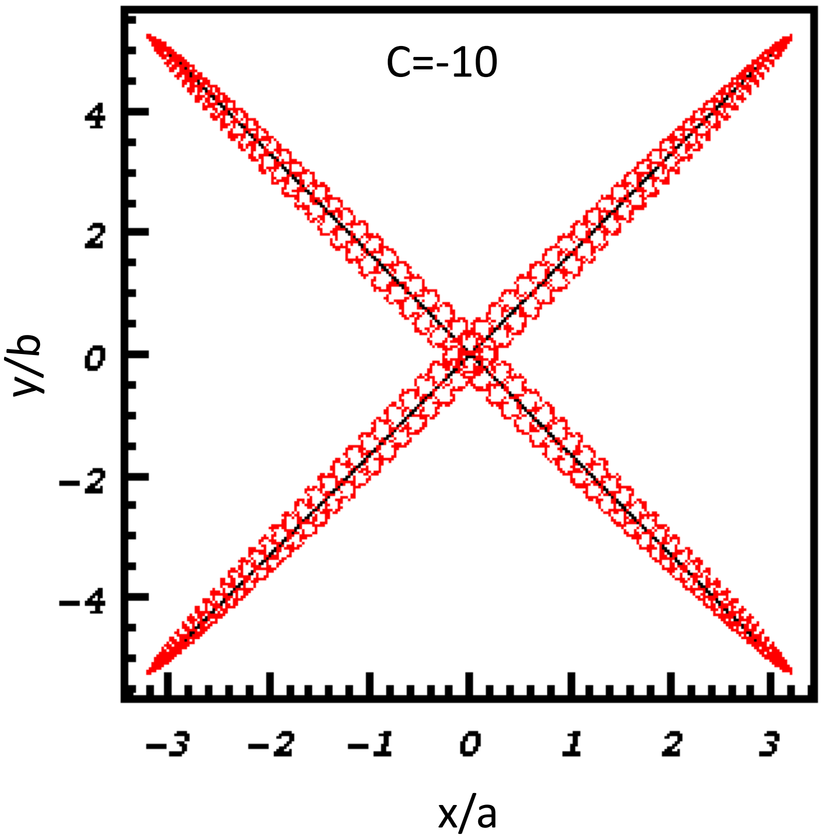
Qx + 2Qy = 19

DQx = -0.05

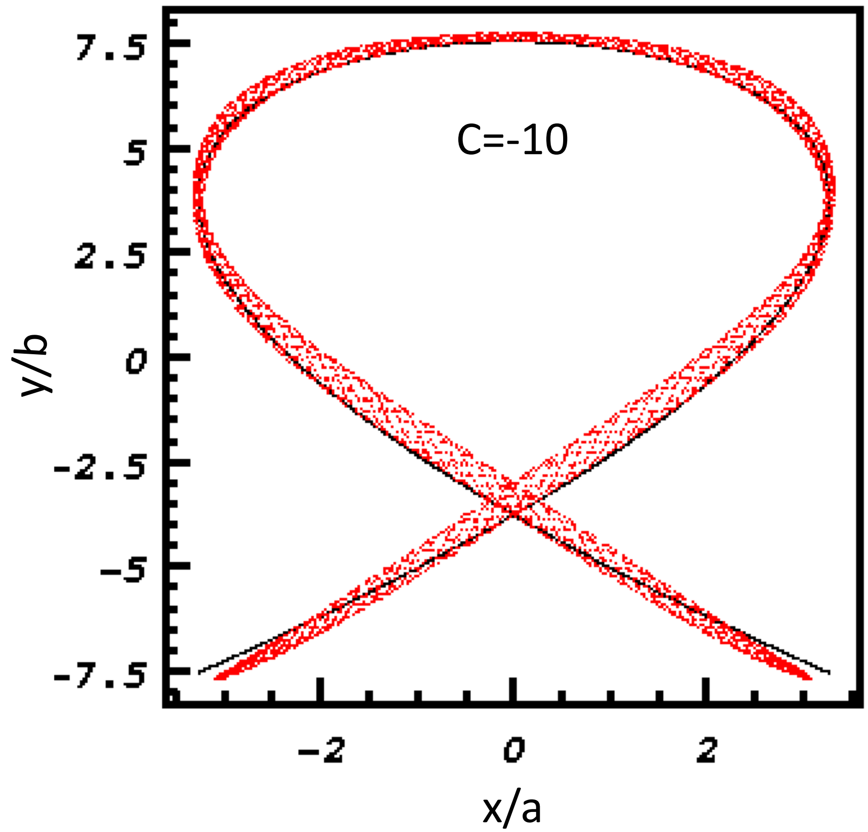
Analytic prediction of the secondary tunes for resonances of any orders … (hence the stability)

# With higher order resonances (fixed lines) and space charge

2 Qx + 2 Qy = 19 (normal)

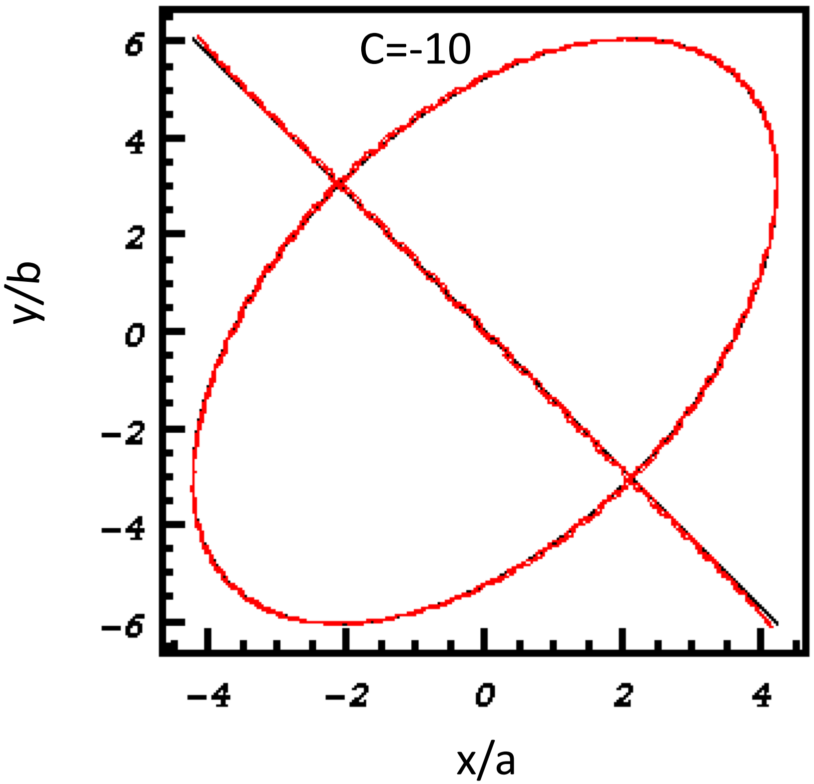



3 Qx + 2 Qy = 19 (skew)

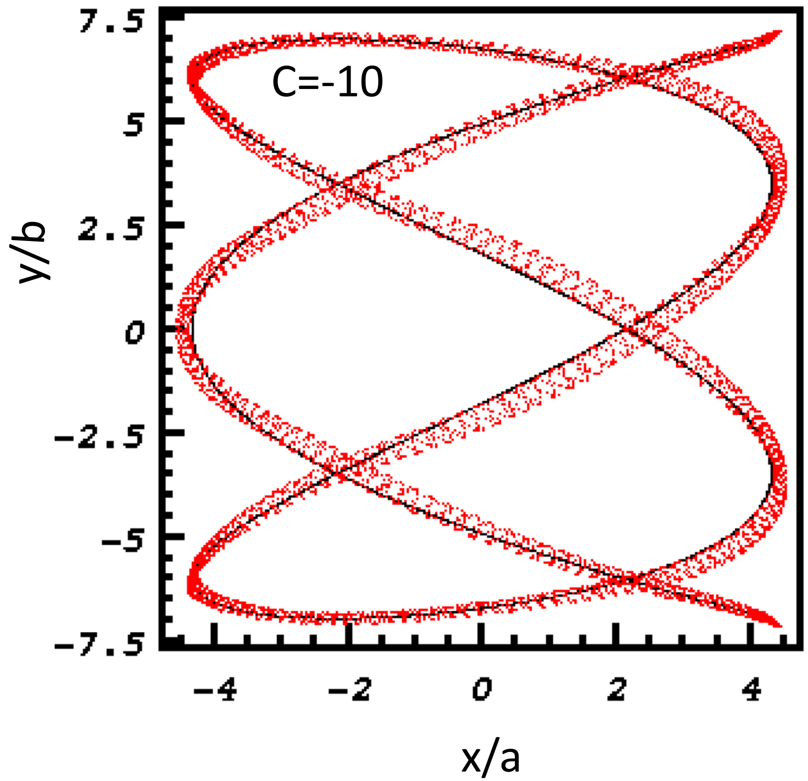


# With higher order resonances (fixed lines) and space charge

3 Qx + 3 Qy = 29 (skew)

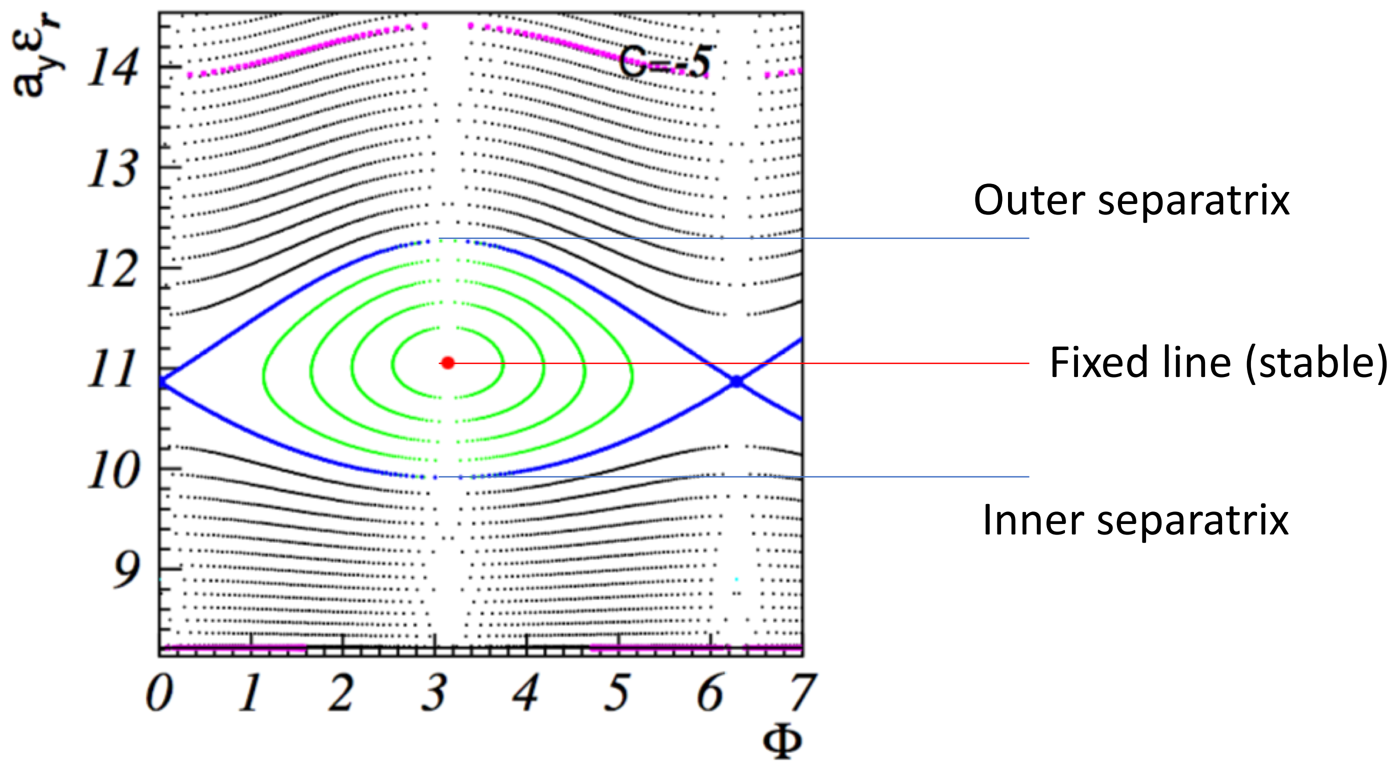



3 Qx + 6 Qy = 49 (normal)



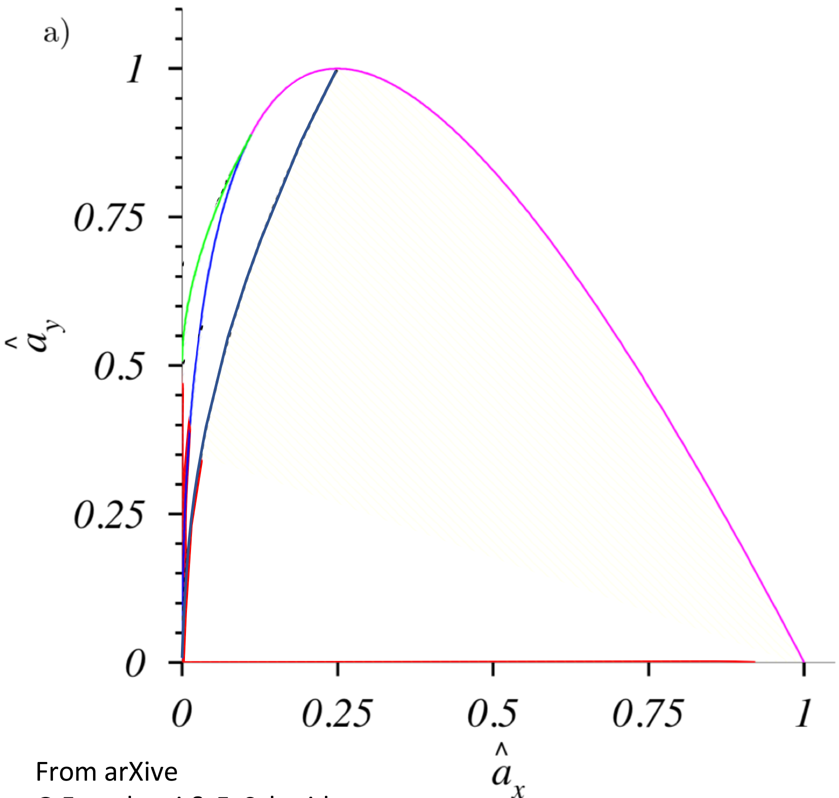
# The shape of resonances

Qx + 2Qy = 19



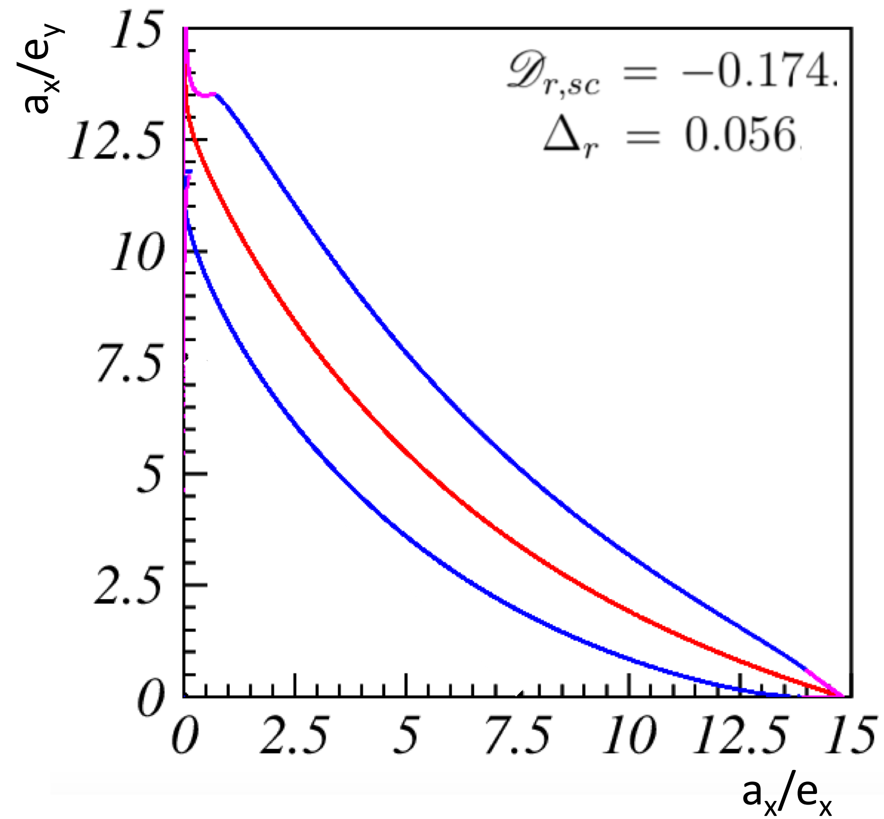
# Effect of space charge on the "infinite" fixed lines

Pure Qx+2Qy=19 No space charge



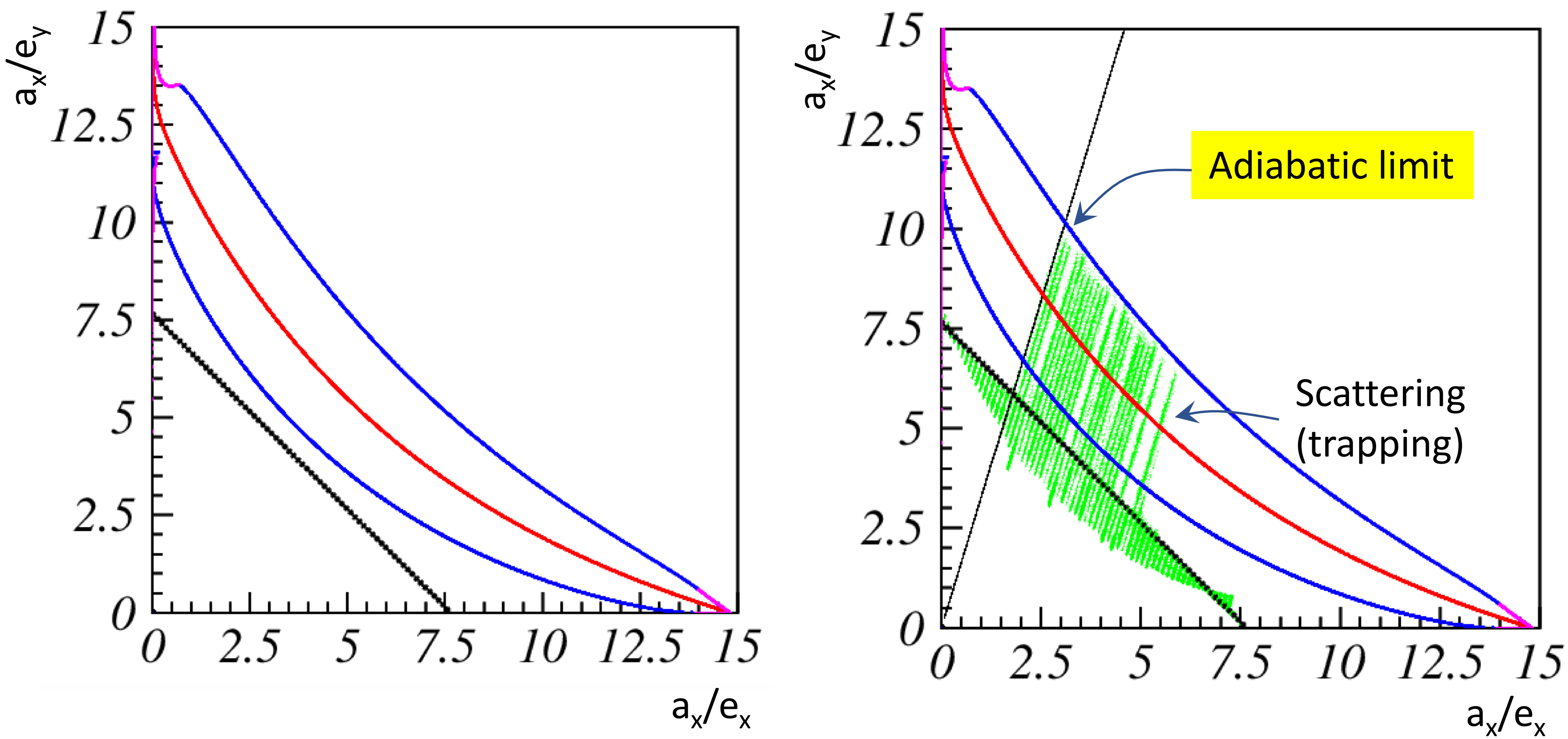

From arXive
G.Franchetti & F. Schmidt

Qx+2Qy=19 with space charge



# Periodic "fixed lines" crossing (periodic resonance crossing)

Qx + 2Qy = 19 $\quad Q_x = 6.104 \quad \Delta_r = 0.056 \quad \Delta Q_x = -0.05 \quad \mathscr{D}_{r,sc} = -0.174.$



# Snags: lack of self-consistency

Comparison with PIC: coasting beam

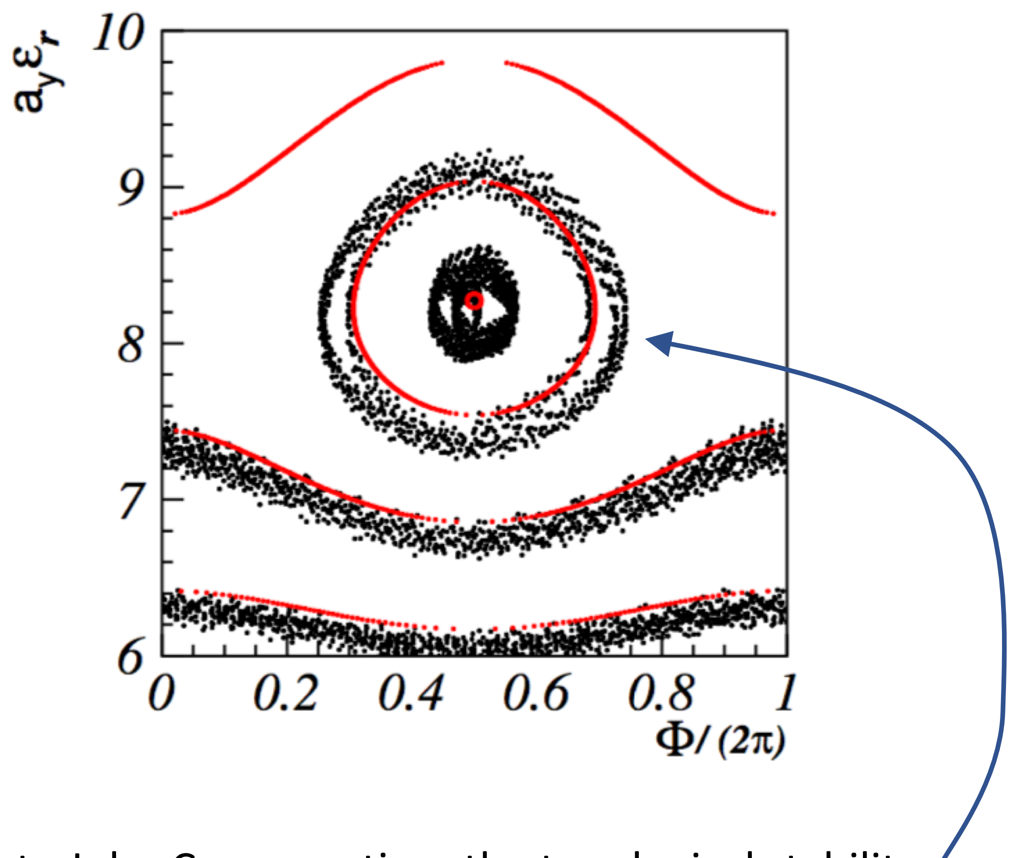
$10^6$ macro-particles

$512 \times 512$ grids

1000 turns

Qx + 2Qy = 19

Constant focusing




Attempt answering to John Cary question: the topological stability

# Summary

Theory of resonances with frozen space charge “almost” complete (4D)

From theory → secondary tunes for all orders

→ SC stabilizes all resonances, which otherwise would be unstable

Prediction of amplitudes of all fixed lines (still under check)

Periodic resonance crossing in a bunch: diffusion bounded by outer separatrix
(further check for large halos)

Some consistency with PIC is verified… open

Further tests for broad range of parameters underway

Interplay coherent & incoherent…. open