Investigating Nonlinear Integrable Optics with Space Charge in IOTA Using IMPACT-Z

Fermilab Workshop on Megawatt Rings & IOTA/ FAST Collaboration Meeting

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Chad Mitchell

Lawrence Berkeley National Laboratory

BERKELEY LAB

U.S. DEPARTMENT OF ENERGY | Office of Science

ACCELERATOR TECHNOLOGY & APPLIED PHYSICS DIVISION ATAP

Outline

- *Introduction and single-particle tracking algorithm*
- *Generation of zero-current matched distribution types*
- *Self-consistent symplectic integration with space charge*
- *Beam stability and invariant preservation (ΔQ=-0.03)*
  - *- approach to Vlasov near-equilibrium (through turn 700)*
  - *- dynamics within a near-equilibrium beam*
  - *- long-term stability behavior (through turn 3K)*
- *Conclusions*

A very brief review of single-particle nonlinear integrable optics in IOTA

- **Dynamics inside the nonlinear magnetic insert:**

$$H_\perp = \frac{1}{2}(P_x^2 + P_y^2) - \frac{\tau c^2}{\beta(s)} U\left(\frac{X}{c\sqrt{\beta(s)}}, \frac{Y}{c\sqrt{\beta(s)}}\right) \rightarrow H_N = \frac{1}{2}(P_{xN}^2 + P_{yN}^2 + X_N^2 + Y_N^2) - \tau U(X_N, Y_N)$$

first invariant

Courant-Snyder transformation, scaling

D&N give in [1] a realizable potential *U* such that H_N admits a second invariant I_N :

$$\{H_N, I_N\} = 0 \,.$$

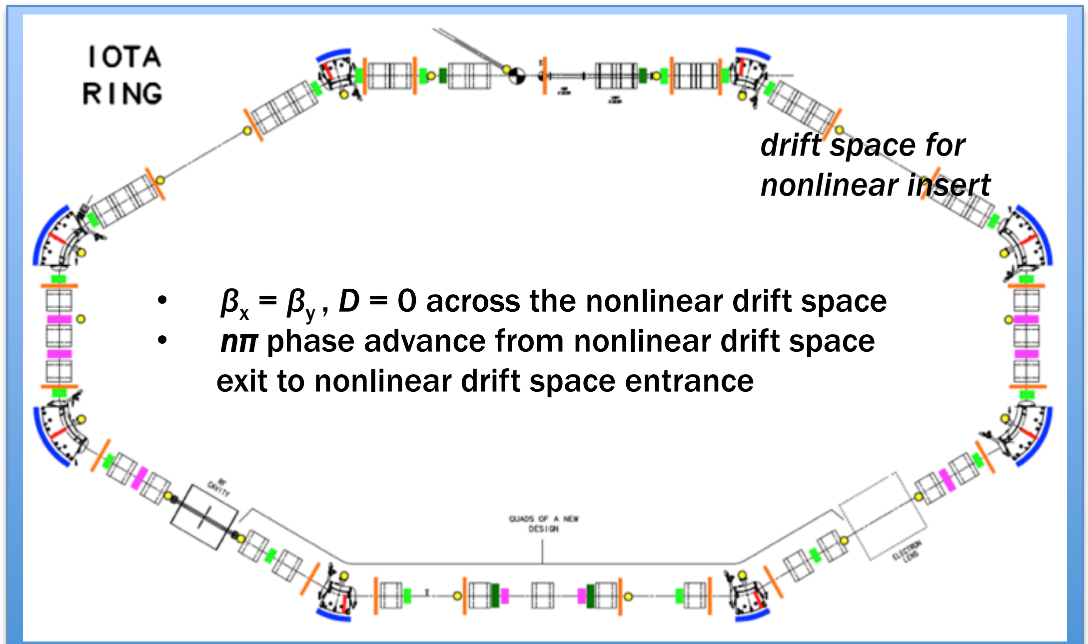
- **Dynamics in the arc external to the nonlinear magnetic insert:**

Assumed *linear* with a map R_N given by:

$$R_N = \pm I$$ (4x4 identity)

Thus, the phase advance must be *nπ*.

H_N, I_N are invariant under the one-turn map.



[1] V. Danilov and S. Nagaitsev, PRAB **13**, 084002 (2010)

Tracking in the nonlinear insert is implemented in IMPACT-Z using a second-order symplectic integrator.

The ideal 2D magnetic field within the nonlinear insert at location s is given by $\vec{B} = \nabla \times \vec{A} = -\nabla\psi$, where the potentials are given in terms of dimensionless quantities:

$$F = \frac{A_s + i\psi}{B\rho}, \quad z = \frac{x+iy}{c\sqrt{\beta(s)}}, \quad \tilde{t} = \frac{\tau c^2}{\beta(s)}$$

using the complex function:

$$F(z) = \left(\frac{\tilde{t}z}{\sqrt{1-z^2}}\right)\arcsin(z) \ .$$

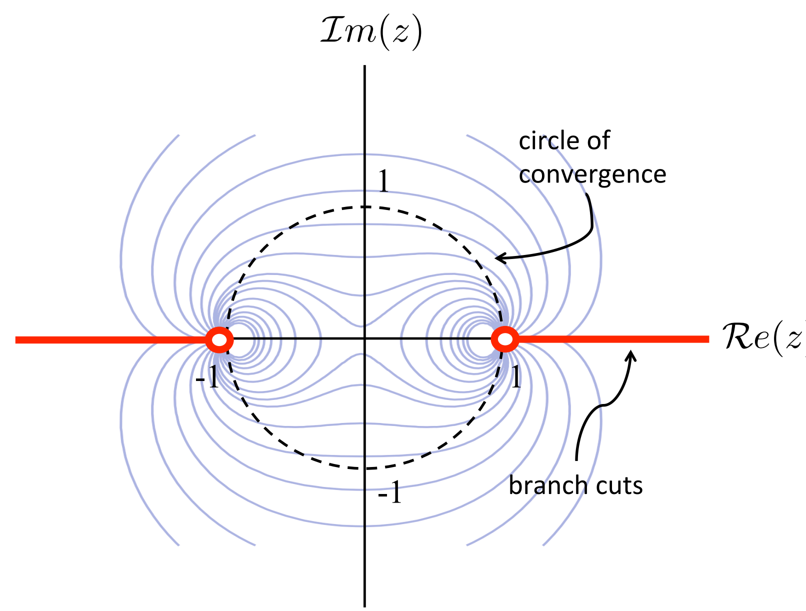
s-Dependent symplectic tracking is performed using:

$$H = H_{drift} + H_{NLL}, \quad H_{NLL} = -A_s/B\rho$$

The map for a single numerical step of size h is:

$$\mathcal{M}(s \to s+h) = \mathcal{M}_{drift}\left(\frac{h}{2}\right)\mathcal{M}_{NLL}\left(h, s+\frac{h}{2}\right)\mathcal{M}_{drift}\left(\frac{h}{2}\right) + O(h^3)$$

Field lines of the nonlinear insert in the transverse plane (blue)



τ – dimensionless insert strength
c – transverse scale parameter [$m^{1/2}$]
$B\rho$ – magnetic rigidity [T-m]
β – betatron amplitude [m]

C. E. Mitchell, THPAK035, IPAC 2018 and LBNL Report LBNL-1007217 (2017)

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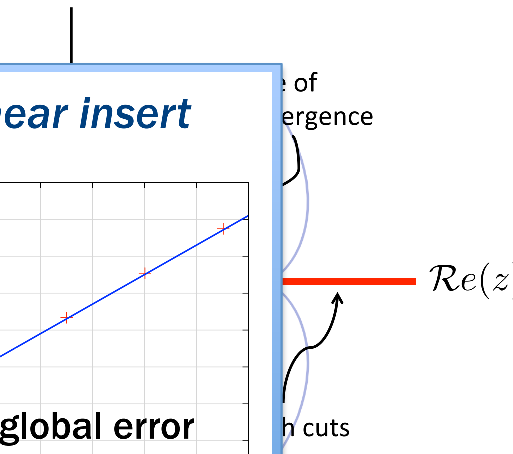


using th

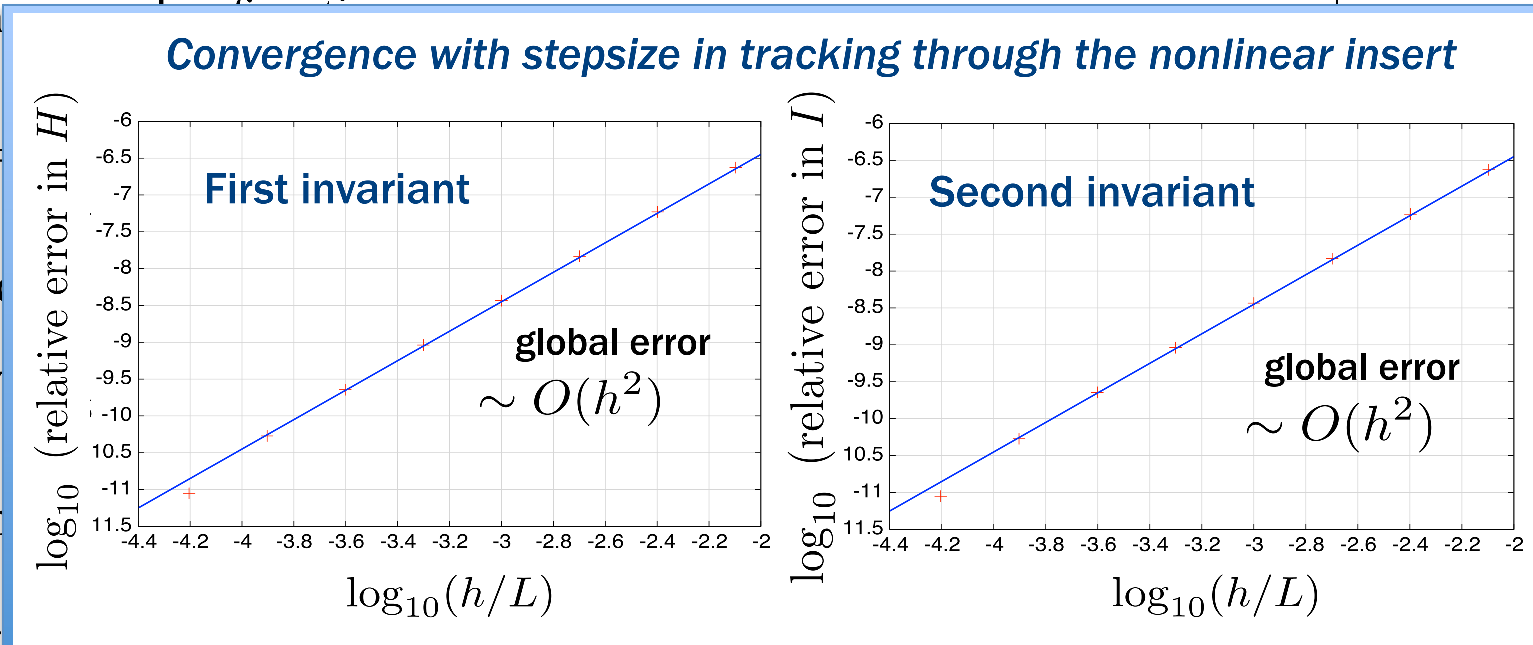
$F(z) =$

s-Depend

$H = H$

The map

$\mathcal{M}(s \rightarrow s$



sert strength
parameter [m$^{1/2}$]
y [T-m]
β – betatron amplitude [m]

- **Generation of zero current matched distribution types**

General procedure for the generation of an initial beam distribution matched to the nonlinear lattice

normalized phase space variables (x_N, p_{xN}, y_N, p_{yN})

"nonlinear KV distribution" [1]

$$f \sim \delta(H - \epsilon_0)$$

"nonlinear waterbag distribution"

$$f \sim \Theta(\epsilon_0 - H)$$

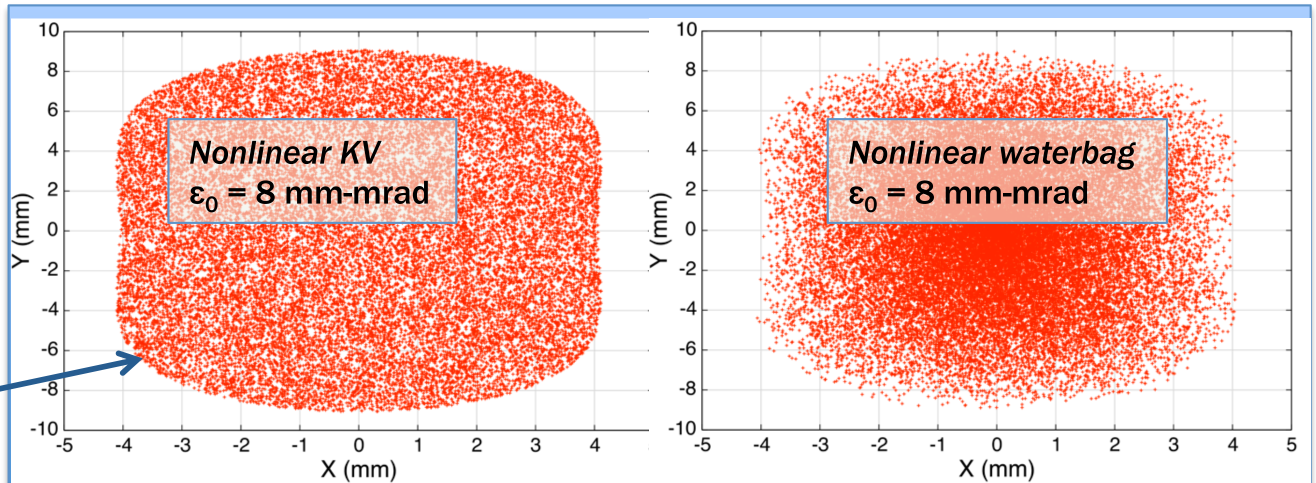
- Hamiltonian is *s*-independent
- distribution function is stationary

$$\begin{pmatrix} x_N \\ p_{xN} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

boundary = equipotential curve of the nonlinear potential

physical phase space variables (x, p_x, y, p_y)

- Hamiltonian is *s*-dependent
- distribution varies periodically in *s*
- parameter ε_0 plays the role of emittance



- Python script developed by the RadiaSoft team is used for matched KV beam generation - populates uniformly a fixed level set of *H*.

[1] S. Webb *et al*, p. 3099, IPAC 2013.

Generation of matched distribution types with general Hamiltonian dependence

Given a probability density f on the phase space M (of dimension $2N$) and a smooth Hamiltonian H, the probability density P_H describing the values of H is given by the co-area formula as:

$$P_H(h) = \int_M f(\zeta)\delta(h - H(\zeta))d\zeta = \int_{H^{-1}(h)} \frac{f(\zeta)}{|\nabla H(\zeta)|} d\sigma^{2N-1}(\zeta)$$

where the surface integral on the right is over the $2N$-1 dimensional level set of H with value h. (This integral is defined and finite for almost all values of h.)

If f is taken to be uniform on the level sets of H, then $f = G \circ H$ for some function G, and so:

$$P_H(h) = G(h)\kappa(h), \qquad \kappa(h) = \int_{H^{-1}(h)} \frac{1}{|\nabla H(\zeta)|} d\sigma^{2N-1}(\zeta)$$

A numerical procedure for generating the beam is to first sample values of H from P_H and then to populate uniformly each level set of H, so the resulting density on the phase space is given by:

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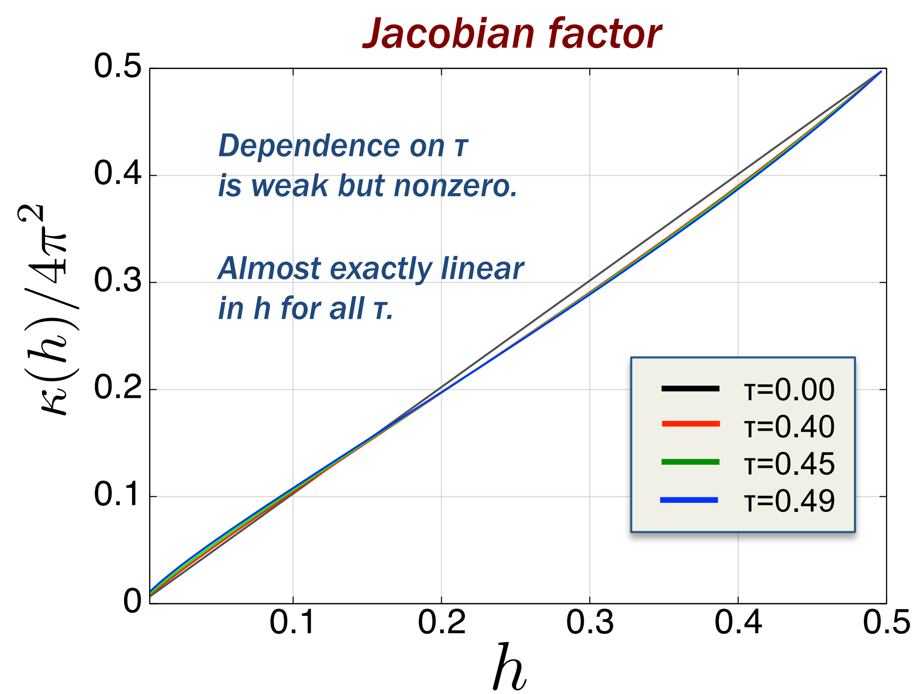
→ This Jacobian factor depends on *h*.

Numerical evaluation of the Jacobian factor for the Danilov & Nagaitsev Hamiltonian

The Jacobian κ is evaluated using a numerical Monte Carlo method, as follows:

- sample 10M points uniformly within the 2D domain A_h for the value h = ½ (dimensionless)
- compute the dimensionless potential value $\Phi(x,y)$ at each point (x,y)
- compute the histogram of Φ values to produce $\kappa_\Phi \propto P_\Phi$, valid for $h \leq 1/2$
- compute the cumulative distribution function of Φ to give κ



Hamiltonian

$$H(x,p_x,y,p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \Phi(x,y)$$

$$A_h = \{(x,y) \in \mathbb{R}^2 : \Phi(x,y) \leq h\}$$

line integral

$$\kappa_\Phi(t) = \int_{\Phi^{-1}(t)} \frac{1}{|\nabla\Phi(x,y)|} d\sigma^1(x,y)$$

Jacobian factor

2D area

$$\kappa(h) = 2\pi m(A_h) = 2\pi \int_0^h \kappa_\Phi(t)dt$$

Definition of a two-parameter family of matched distribution types - smooth (except at the beam edge)

Nonlinear waterbag distribution: $f \propto \Theta(\epsilon_0 - H)$

$$P_H(h) \propto \begin{cases} h, & \text{if} \quad h/\epsilon_0 \leq 1, \\ 0, & \text{else} \end{cases} \qquad \langle H \rangle = \frac{2\epsilon_0}{3}$$

Truncated nonlinear Gaussian (thermal) distribution: $f \propto e^{-H/\epsilon_0}\Theta(\Lambda - H/\epsilon_0)$

$$P_H(h) \propto \begin{cases} he^{-h/\epsilon_0}, & \text{if} \quad h/\epsilon_0 \leq \Lambda \\ 0, & \text{else} \end{cases}$$

ϵ_0 *temperature parameter*

Λ *cutoff parameter*

$$\langle H \rangle = \epsilon_0 \left(2 + \frac{\Lambda^2}{1 - e^{\Lambda} + \Lambda} \right) \longrightarrow 2\epsilon_0 \quad \text{as} \quad \Lambda \to \infty .$$

- Reduce to traditional waterbag and Gaussian distributions, respectively, when insert is off.
- The nonlinear Gaussian reduces to the nonlinear waterbag as Λ goes to zero with the quantity <*H*> held fixed, and provides the ability to control (matched) halo extent.

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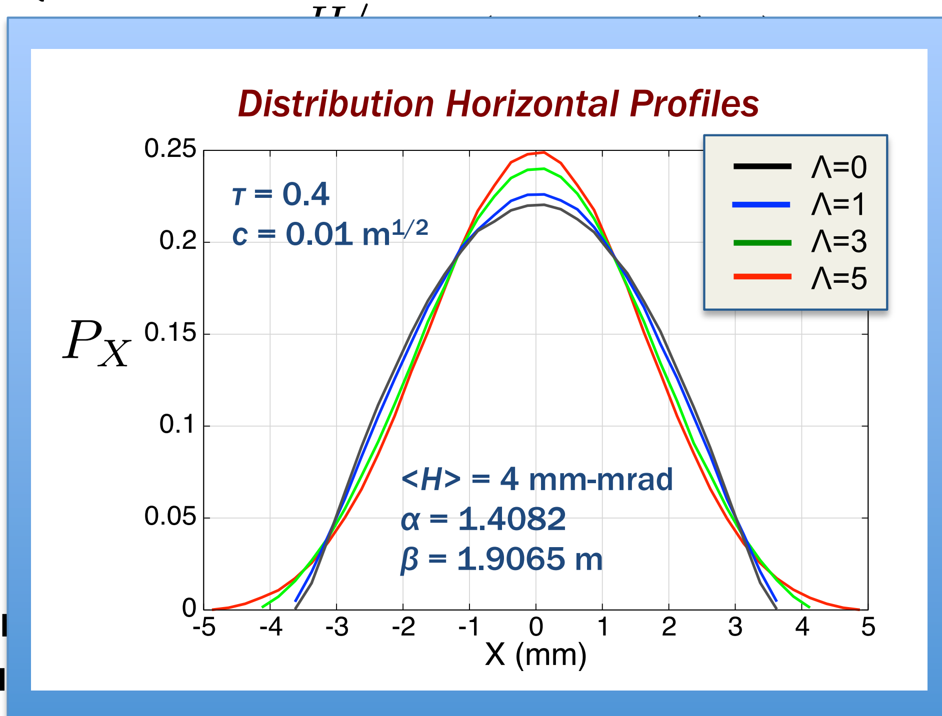
Truncated nonlinear ... $^{H/\epsilon_0}\Theta(\Lambda - H/\epsilon_0)$

$P_H(h) \propto$ ...

... *temperature parameter*

cutoff parameter

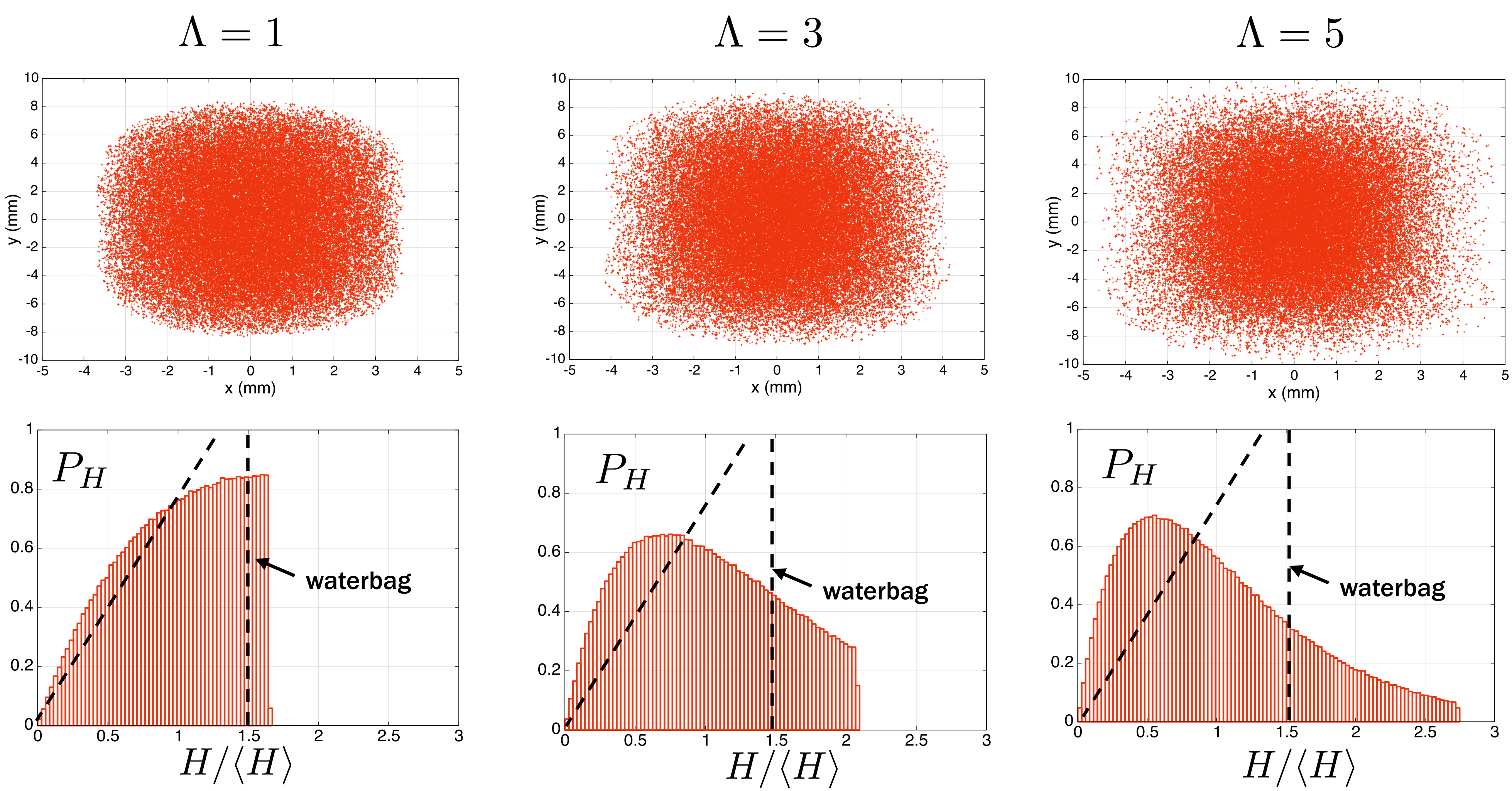
$\langle H \rangle = \epsilon_0 \Big(2$ ... ∞ .



- Reduce to traditio... ...ively, when insert is off.
- The nonlinear Gau... ...es to zero with the quantity <*H*> held fixed, and provides the ability to control (matched) halo extent.

Examples of nonlinear Gaussian (thermal) distributions with cutoffs at Λ = 1, 3, 5 (shown at entrance to NLI)

Distribution parameters are selected to give: $\langle H \rangle = 4\ \text{mm-mrad}$ **with identical Twiss values.**



- **Self-consistent symplectic integration with space charge**

Theory of a symplectic spectral space charge solver for coasting 2D beams (1)

Collective *N*-particle Hamiltonian:

$$H = H_1 + H_2$$

Symplectic map for a single step:

$$\mathcal{M}(\tau) = \mathcal{M}_1(\tau/2)\mathcal{M}_2(\tau)\mathcal{M}_1(\tau/2) + O(\tau^3)$$

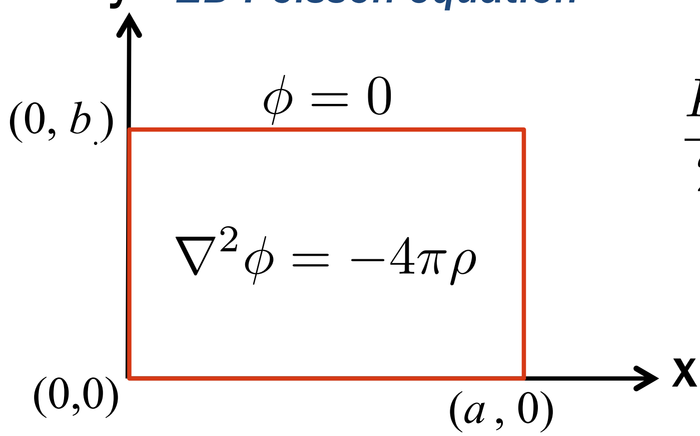
$$H_1(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_p}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{N_p}) = \sum_{j=1}^{N_p} H_{\mathrm{ext}}^{s.p.}(\mathbf{r}_j, \mathbf{p}_j)$$

→ single-particle Hamiltonian w/o space charge ("external")

$$H_2(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_p}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{N_p}) = \frac{K}{2}\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} G(\mathbf{r}_i, \mathbf{r}_j)$$

→ 2D space charge Green function in a rectangular conducting pipe

2D Poisson equation



↓ continuum limit

$$\frac{K}{2}\int d\mathbf{r}\rho(\mathbf{r})\int d\mathbf{r}'\rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}') = \frac{K}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})d\mathbf{r}$$

generalized perveance $K = \dfrac{I}{I_A}\dfrac{2}{\beta^3\gamma^3}$

J. Qiang, Phys. Rev. ST Accel. Beams 20, 014203 (2017).

Theory of a symplectic spectral space charge solver for coasting 2D beams (2)

Spectral approximation of *G* using N_l , N_m Fourier modes in x and y, respectively:

$$G(\mathbf{r}_i,\mathbf{r}_j) = 4\pi\frac{4}{ab}\frac{1}{N_p}\sum_{l=1}^{N_l}\sum_{m=1}^{N_m}\frac{1}{\gamma_{lm}^2}\sin(\alpha_l x_j)\sin(\beta_m y_j)\sin(\alpha_l x_i)\sin(\beta_m y_i) \qquad (\bigstar)$$

for mode (*l*,*m*):

$$\alpha_l = \frac{l\pi}{a}, \quad \beta_m = \frac{m\pi}{b}, \quad \gamma_{lm}^2 = \alpha_l^2 + \beta_m^2$$

The symplectic map $\mathcal{M}_2$ associated with H_2 is given for particle *i* as:

$$p_{xi}(\tau) = p_{xi}(0) - \tau K \sum_{j=1}^{N_p}\frac{\partial G(\mathbf{r}_i,\mathbf{r}_j)}{\partial x_i},$$

$$p_{y_i}(\tau) = p_{yi}(0) - \tau K \sum_{j=1}^{N_p}\frac{\partial G(\mathbf{r}_i,\mathbf{r}_j)}{\partial y_i}$$

Computed directly from particle data in the laboratory frame using (★).

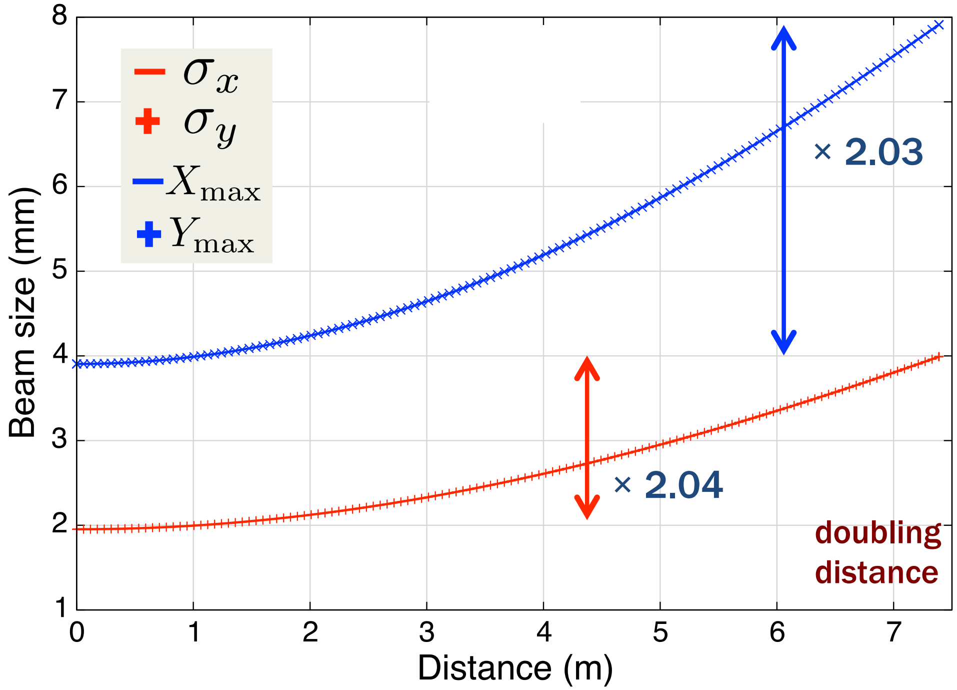
Note: Momenta are normalized by the design momentum p_0.

Computational complexity scales as $O(N_l \times N_m \times N_p)$. ***See also [1] and the talk by N. Cook.***

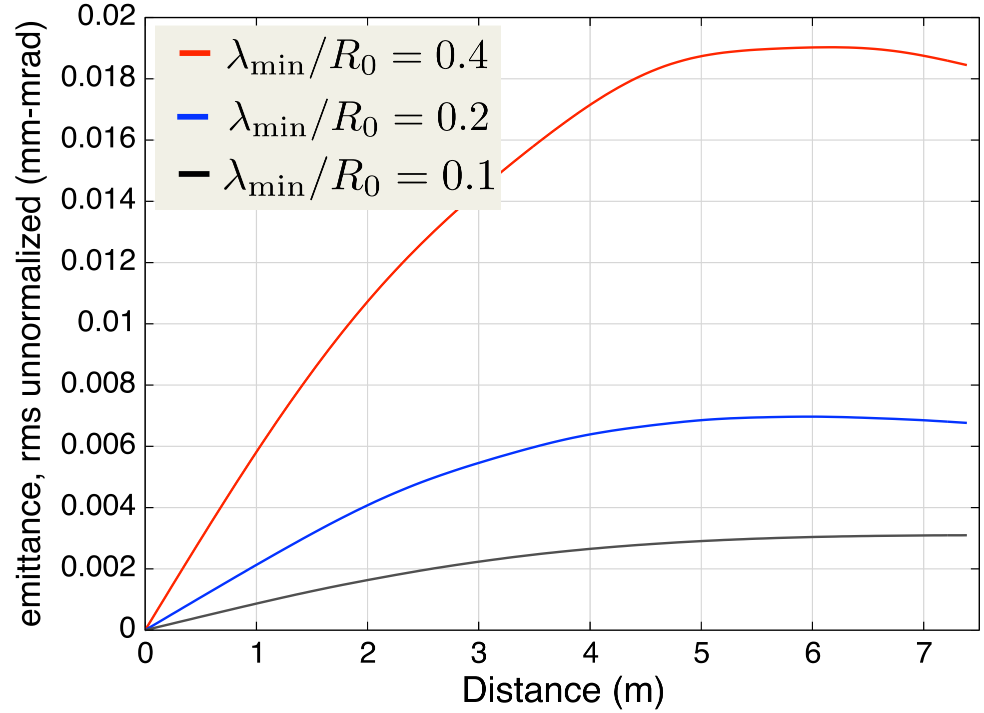
[1] S. Webb, Plasma Phys. Control. Fusion **58**, 034007 (2016).

Benchmark 1: Expansion in free space of a cold uniform cylinder beam (1).

Beam size evolution



Emittance evolution



Linear drift **(kinematic nonlinearities off)**

***KE* = 2.5 MeV p** **(equal to IOTA benchmark value)**

R_0 = 3.905 mm **(equal to IOTA benchmark value)**

***I* = 4.113 mA** **(10 × IOTA benchmark value)**

***a* = *b* = 5 cm** **(chosen to be >> R_0)**

Measures of numerical resolution:

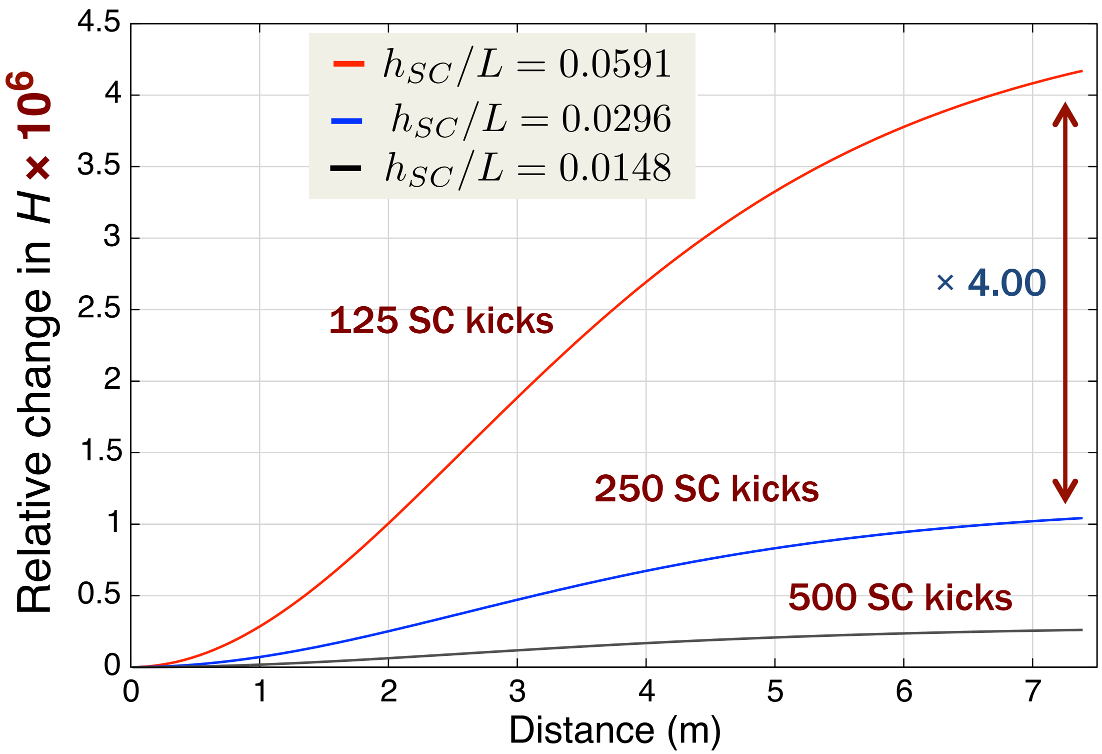
$$\lambda_{\min}/R_0 = 0.1 \qquad h_{SC}/L = 0.0591$$

$$R_0/a = 0.0781$$

Benchmark 1: Expansion in free space of a cold uniform cylinder beam (2): Hamiltonian preservation

$$H = \sum_{i=1}^{N_p} \frac{\mathbf{p}_i^2}{2} + 4\pi \frac{K}{2} \frac{4}{ab} \frac{1}{N_p} \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sin(\alpha_l x_j) \sin(\beta_m y_j) \sin(\alpha_l x_i) \sin(\beta_m y_i)$$

Evolution of the N-particle Hamiltonian



Initial value of *H* :

$$H/N_p = 9.694248 \times 10^{-7}$$

Numerical resolution:

$$\lambda_{\min}/R_0 = 0.1$$

$$R_0/a = 0.0781$$

$$N_p = 1.024 \text{ M}$$

The error in H scales as expected:

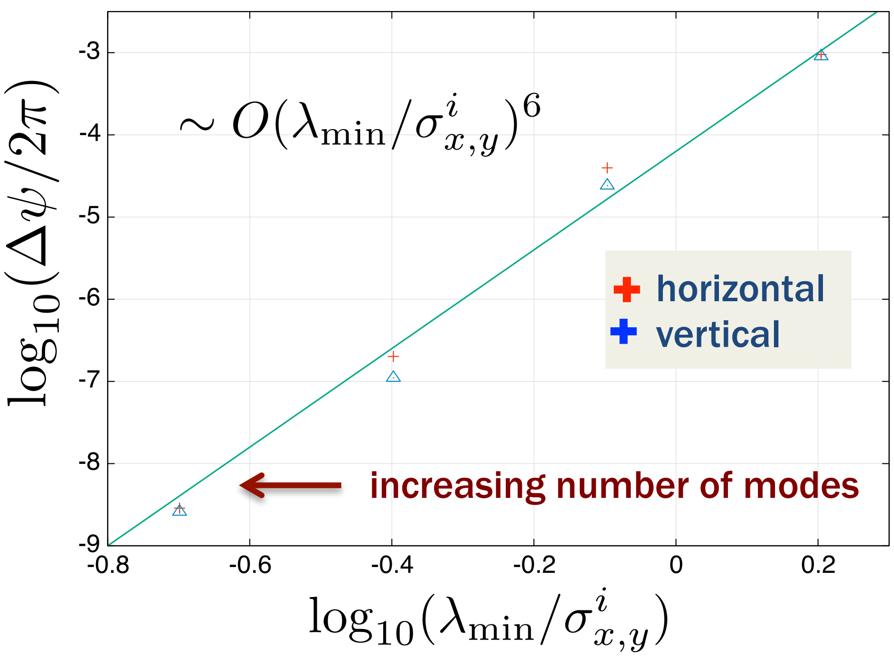
$$\sim O(h_{SC}/L)^2$$

Benchmark 2: A matched waterbag beam in the linear IOTA lattice.

- IOTA lattice retuned for space charge tune depression of ΔQ = -0.03
- Nonlinear insert turned OFF (linear lattice)

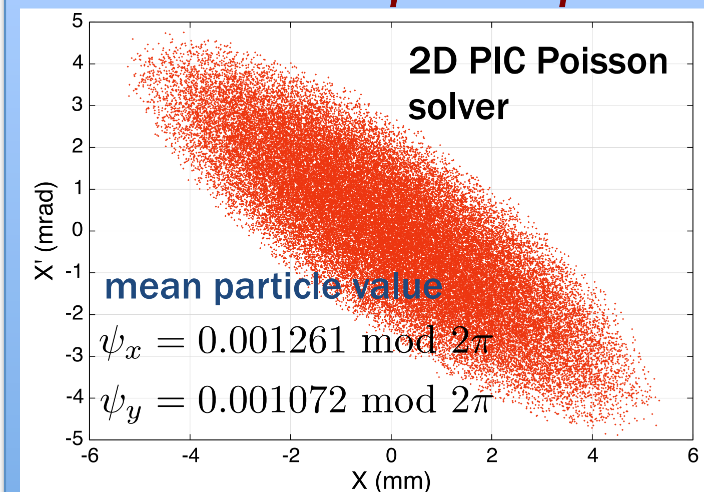
Waterbag beam, 1.024 M particles
I = 0.4113 mA, <*H*> = 4 mm-mrad

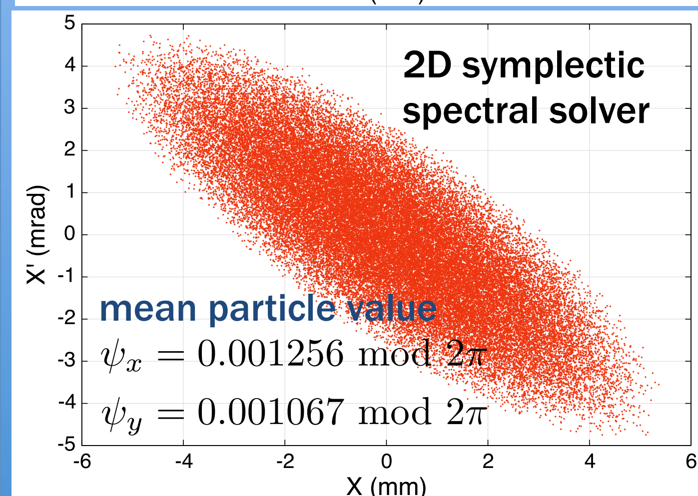
Convergence of phase advance across the arc



Emittance evolution: $\Delta\epsilon_{x,y}/\epsilon_{x,y} \approx 4 \times 10^{-5}$

Final beam phase space





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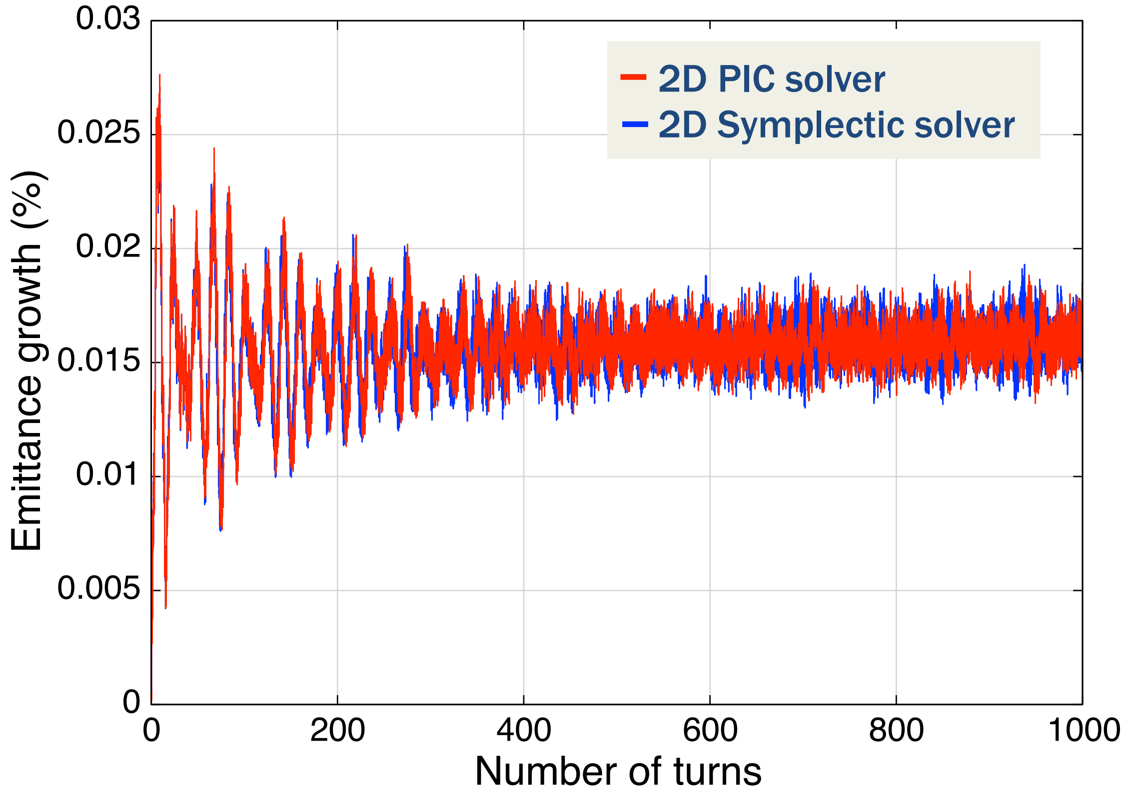
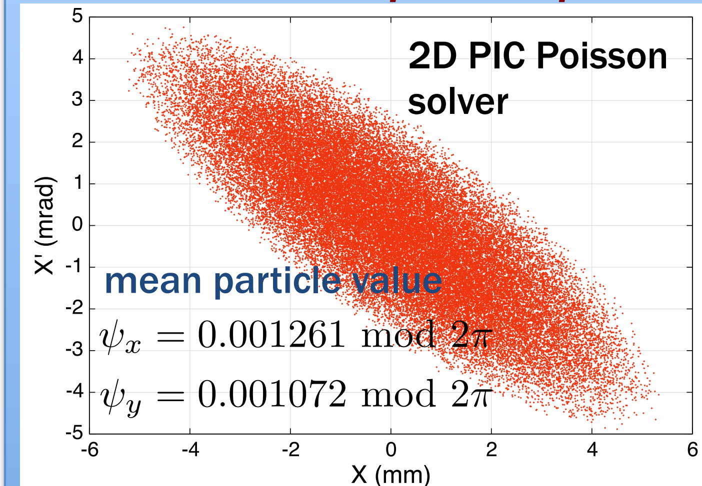
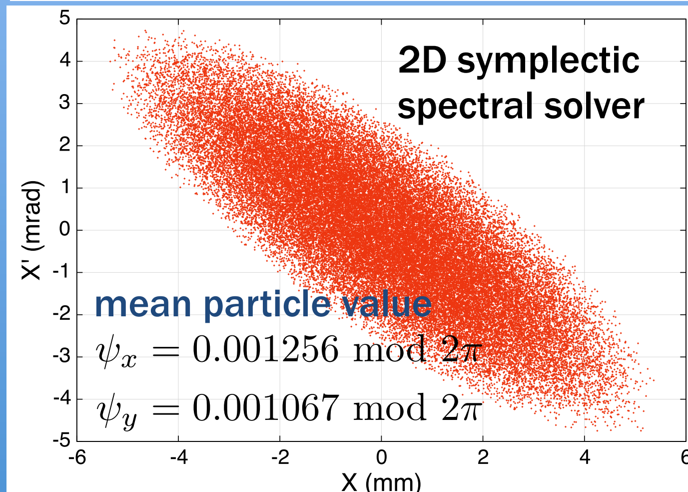
Relative emittance growth over 1K turns



Final beam phase space





- **Beam stability and invariant preservation**

Tracking in the IOTA Lattice with Space Charge - Assumptions and Simulation Parameters

Objective: To isolate and understand the perturbative effects of space charge on the ideal integrable single-particle dynamics at moderate space charge tune depression.

- Elements external to the nonlinear insert are sliced longitudinally and treated as symplectic maps alternating with space charge momentum kicks (split-operator approach): *linear order*.
- Space charge is included self-consistently throughout the lattice using the symplectic spectral solver with a rectangular boundary of large aperture to emulate free-space boundary conditions.
- We consider a long, unbunched beam with zero energy spread to remain near the ideal integrable working point.
- Quadrupole settings are retuned to provide $n\pi$ phase advance across the arc after including the linearized space charge fields at the desired value of beam current (A. Romanov, [1]).
- Twiss functions with linearized space charge included must be appropriately matched to the nonlinear insert. *See also [2].*

Insert parameters: $\tau = 0.4$, $c = 0.01\ \mathrm{m}^{1/2}$, $\mu_0 = 0.30345$, $L = 1.8$ m
Beam parameters: $KE = 2.5$ MeV, $I = 0.4113$ mA, $\langle H \rangle = 4.0$ mm-mrad, $\Delta Q_x = \Delta Q_x = -0.03$

[1] A. Romanov *et al*, THPOA23, NAPAC 2016. [2] C. Hall *et al*, WEA4CO02, NAPAC 2016.

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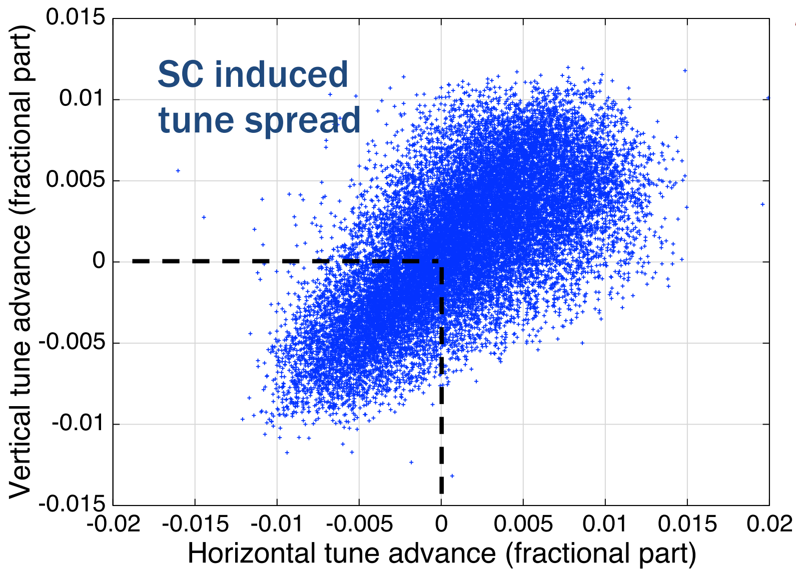
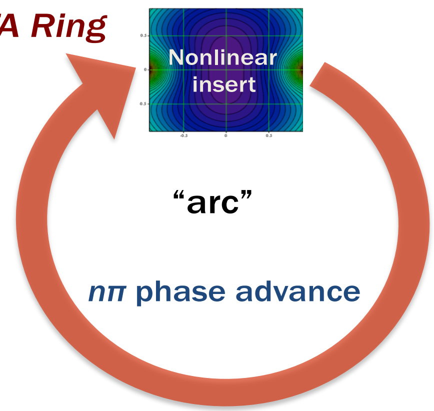
- We con … ideal integra …
- Quadru … including the line … ov, [1]).
- Twiss fu … hed to the nonline …

Insert pa …
Beam pa … 3

Tune advance footprint for 0.4113 mA
(waterbag beam, <H> = 4 mm-mrad)





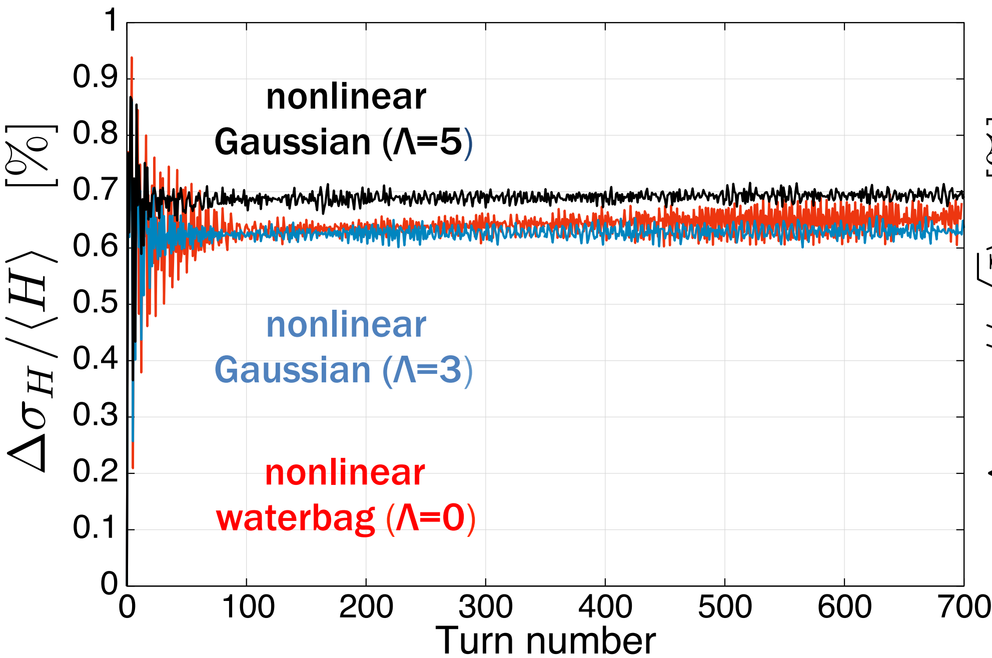
Settings designed for a space charge tune depression of -0.03.

[1] A. Romanov *et al*, THPOA23, NAPAC 2016. [2] C. Hall *et al*, WEA4CO02, NAPAC 2016.

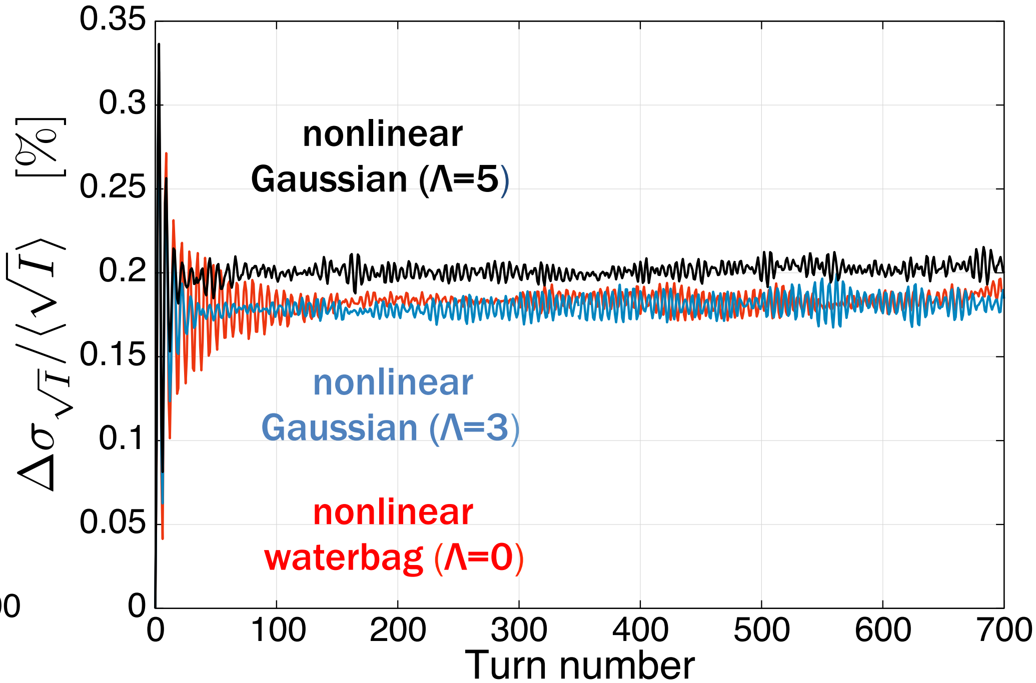
Tracking in the IOTA Lattice with Space Charge – First 700 Turns (1)

Evolution of the standard deviation of the two invariants of single-particle motion for 700 turns.

First invariant



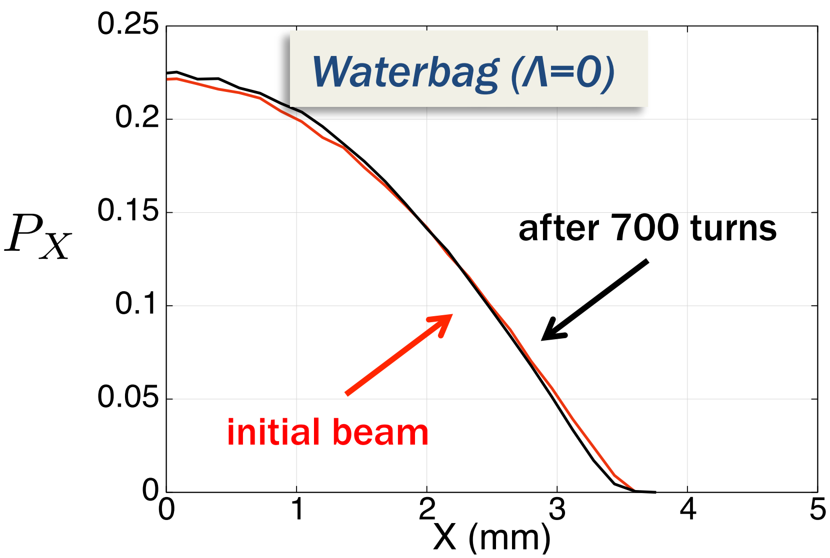
Second invariant

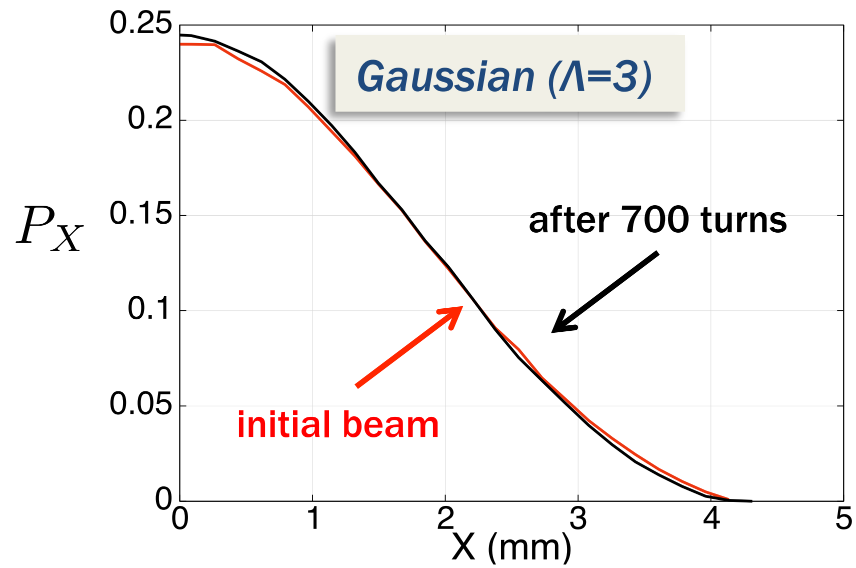


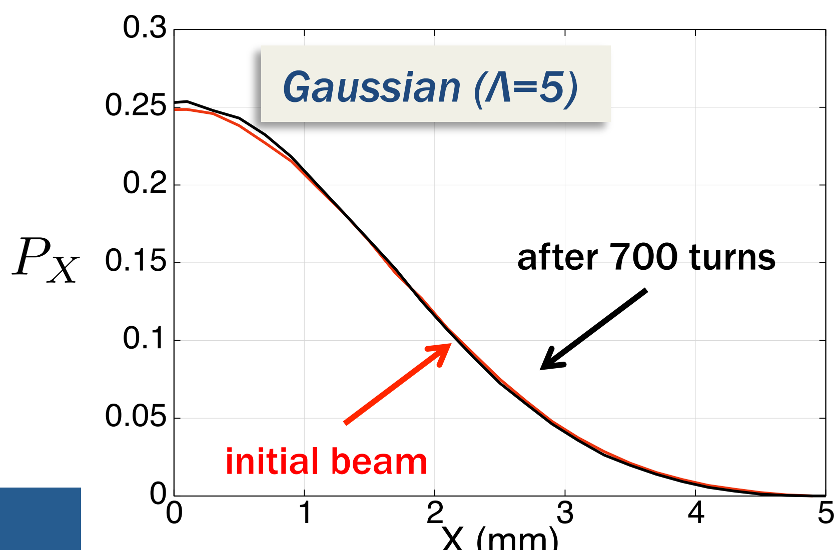
- **In all cases, <*H*> = 4 mm-mrad. Growth during the initial period of nonlinear mixing and phase space filamentation (turns 1-100) appears to depend only weakly on the cutoff parameter Λ.**
- **The time scale for this initial mixing decreases with increasing Λ, as larger-amplitude particles in the tail contribute to stronger nonlinear damping.**

Tracking in the IOTA Lattice with Space Charge – First 700 Turns (2)

Evolution of the horizontal density profile showing stability of low-density tail







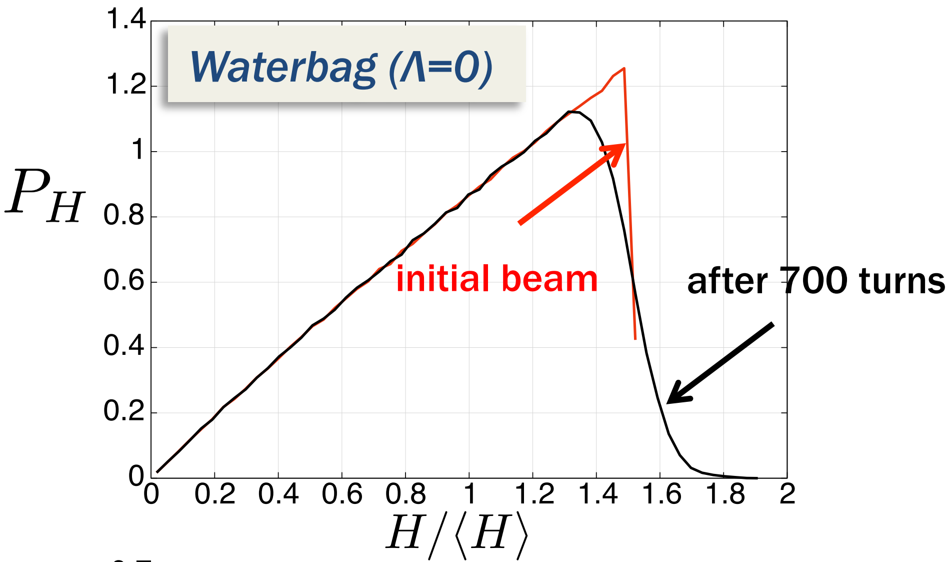
All three spatial profiles appear quite stable.

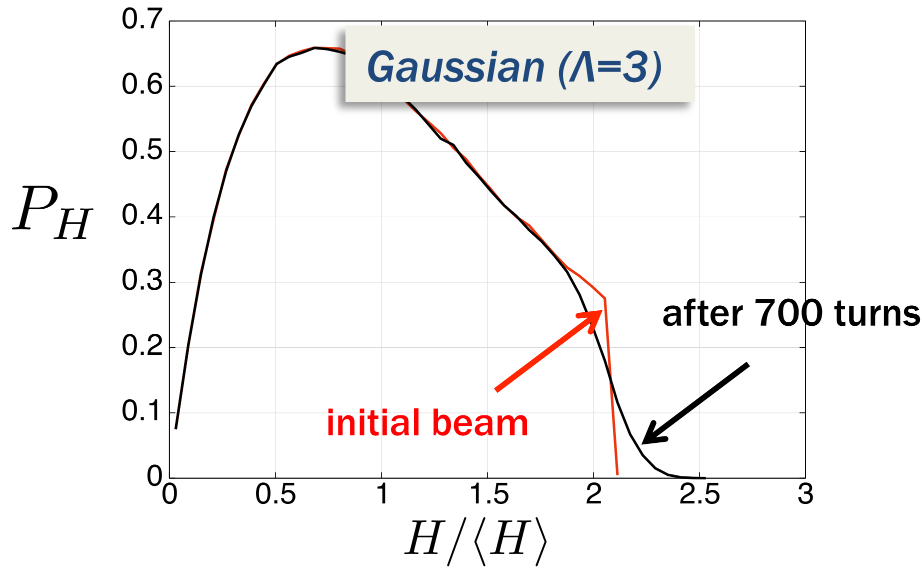
Perhaps a minor change in the curvature of the profile near the outer beam edge in the waterbag case?

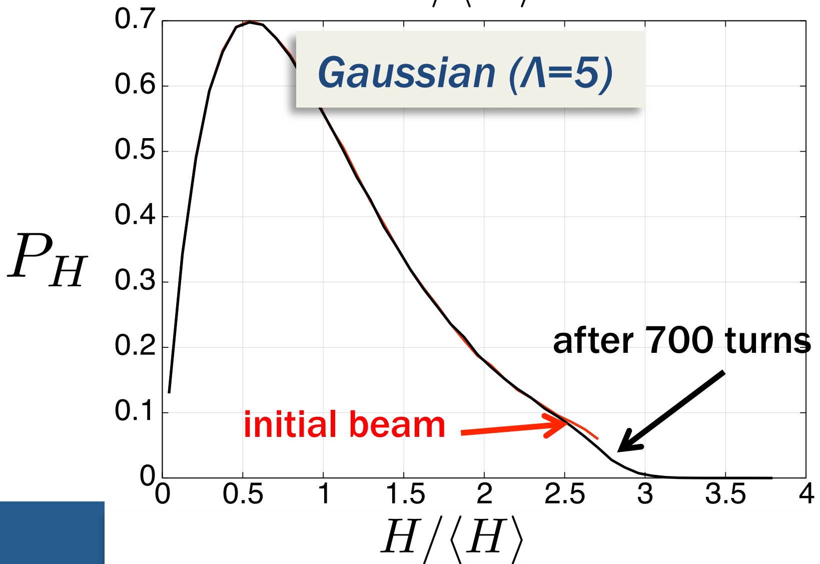
1.024 M particles, 64 x 64 spectral modes
480 steps through the nonlinear insert

Tracking in the IOTA Lattice with Space Charge – First 700 Turns (3)

Evolution of the first invariant (Hamiltonian) profile showing the effects of space charge





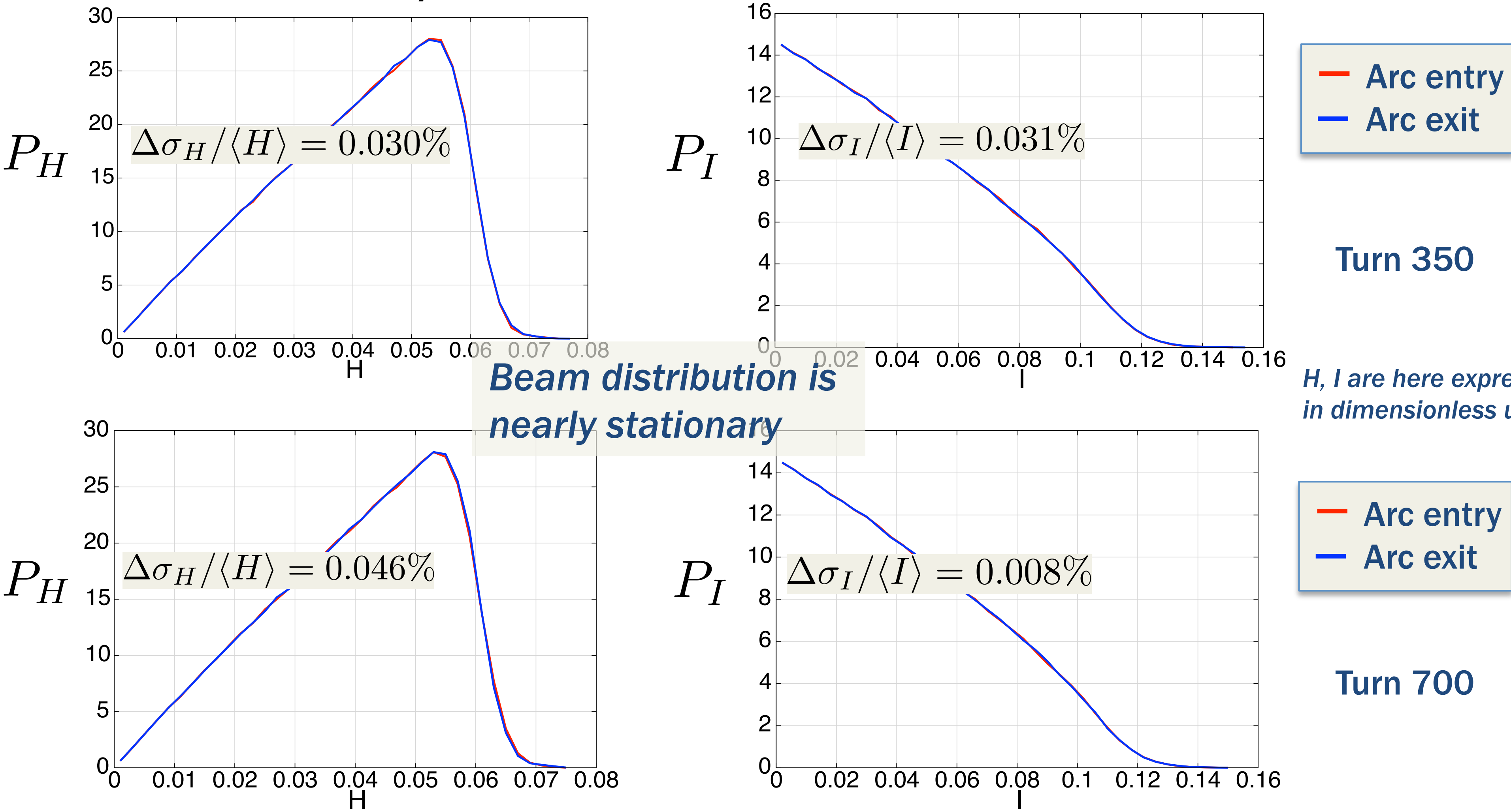


Redistribution of the single-particle invariants occurs primarily near the outer beam edge through profile smoothing near the discontinuity at $H=H_{max}$.

Similar behavior is apparent in the evolution of the profile of the second invariant.

Dynamics within a near-equilibrium nonlinear waterbag beam – Invariant profiles are well-preserved across the arc

First invariant profile ***Second invariant profile***



Beam distribution is nearly stationary

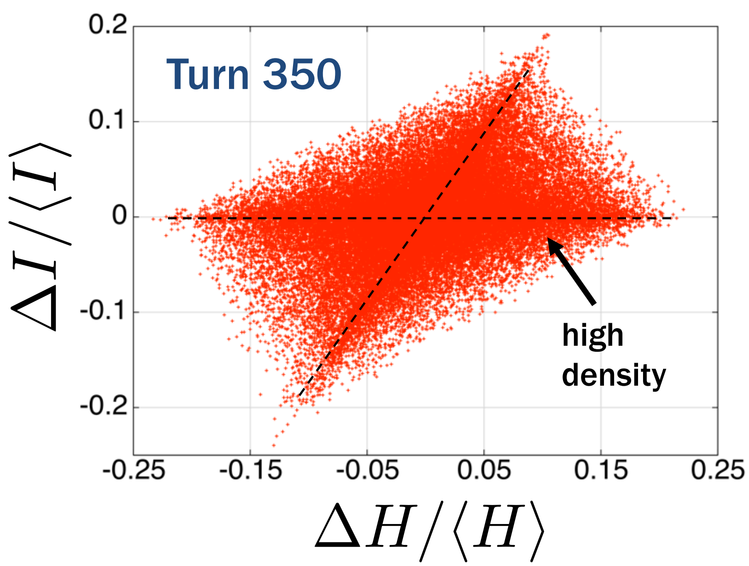
Turn 350

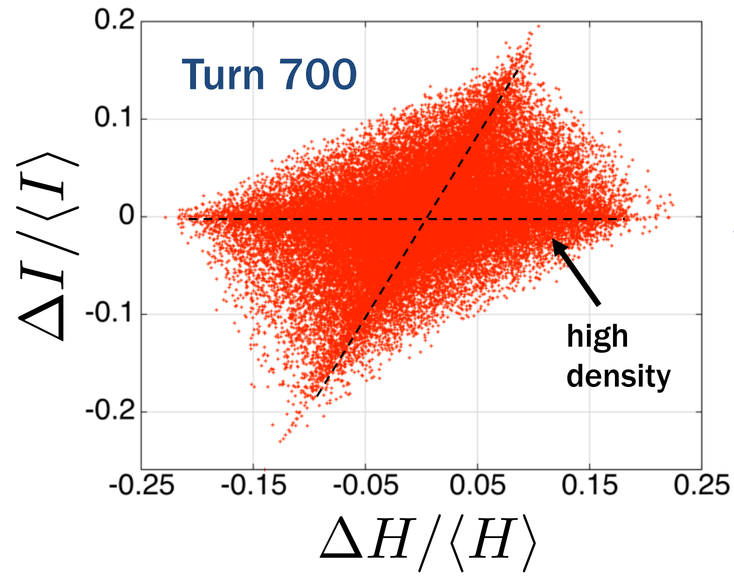
H, I are here expressed in dimensionless units

Turn 700

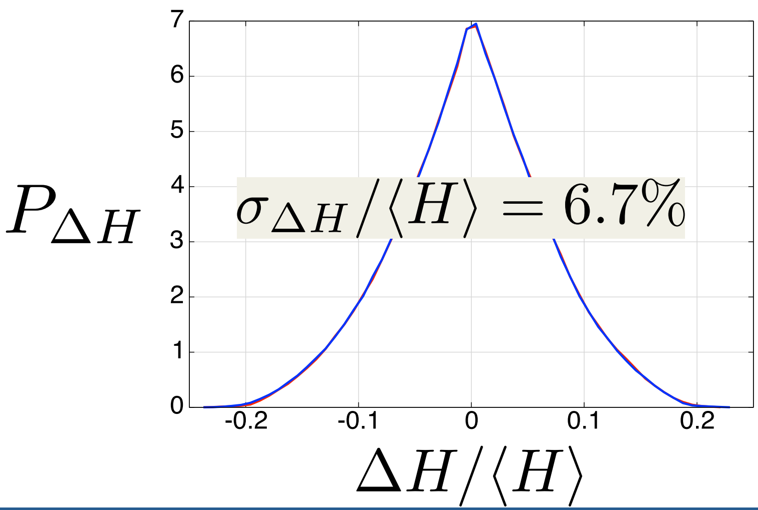
Dynamics within a near-equilibrium nonlinear waterbag beam – Advance of invariant values across the arc

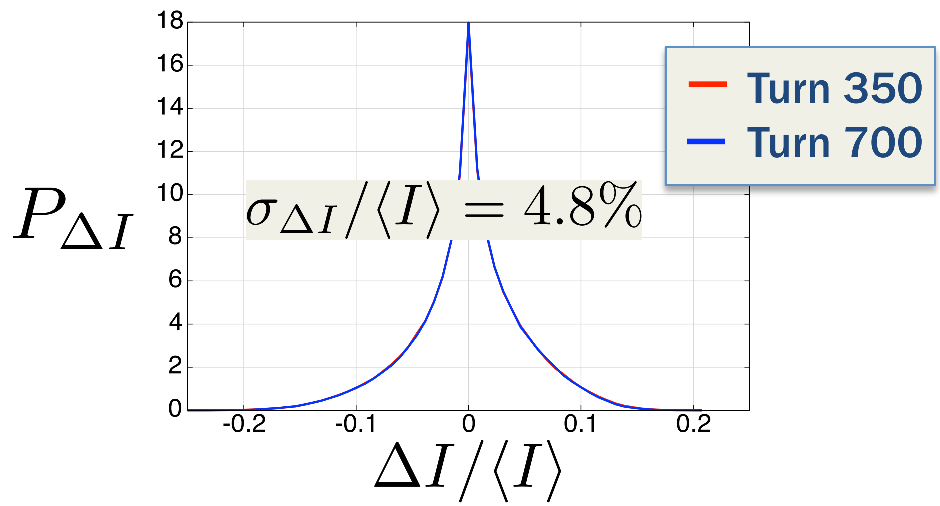
Individual particle changes in H and I are as large as 20%





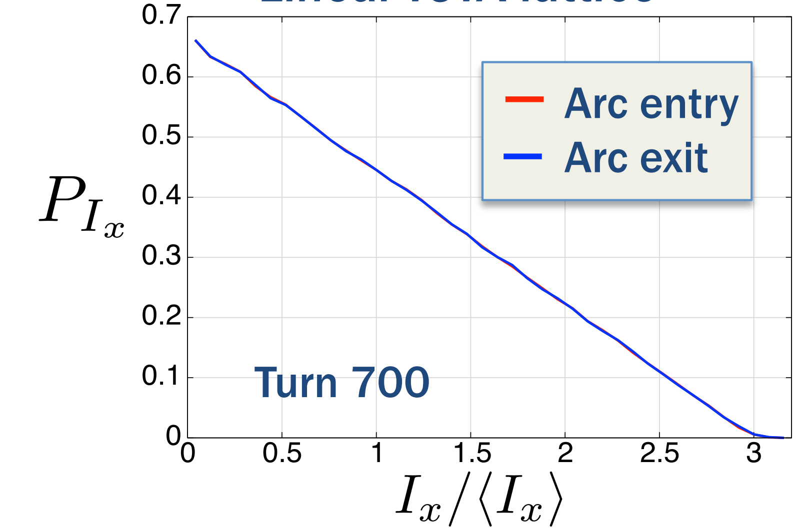
Footprint is stable over 350 turns



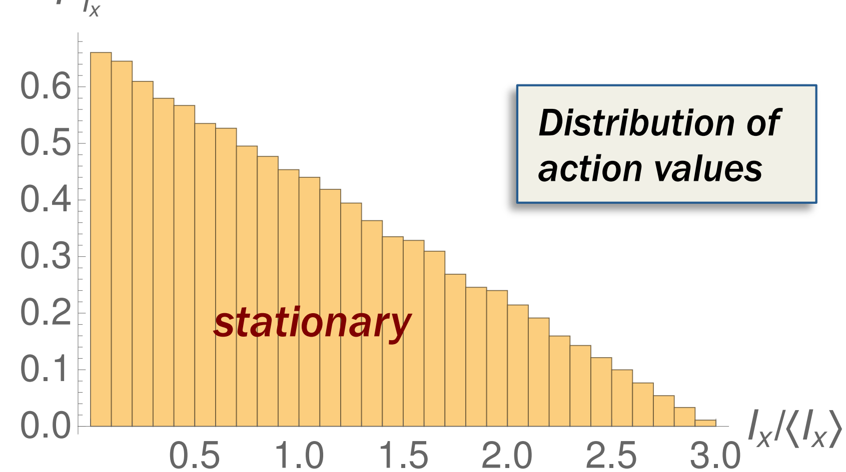


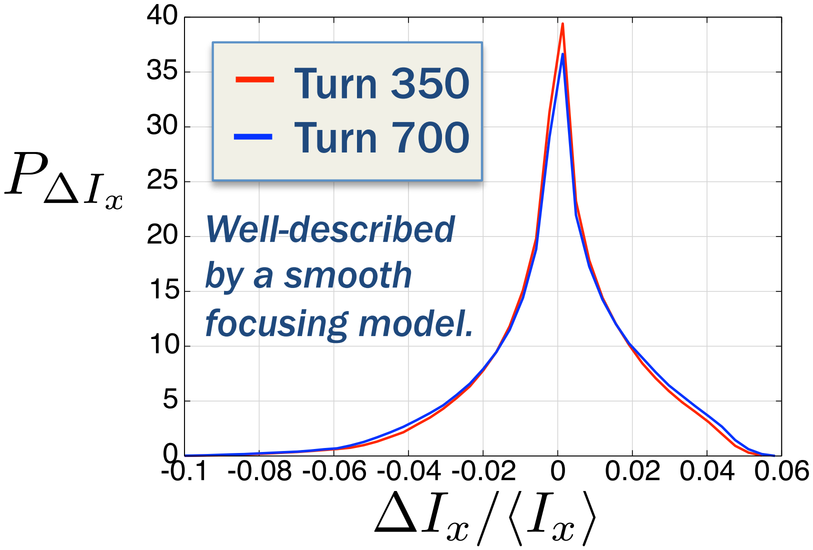
Smooth focusing waterbag model of equilibrium in the linear IOTA lattice (action advance across the arc)

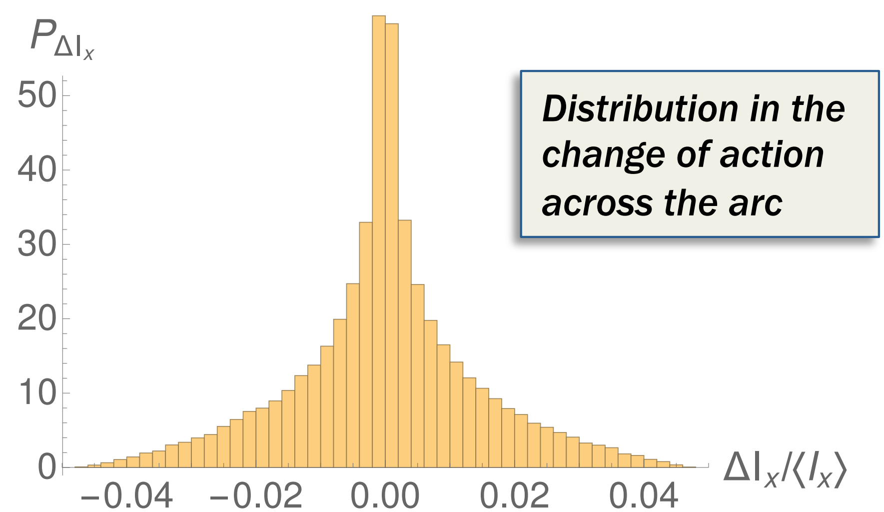
Linear IOTA lattice



Smooth focusing model[1]





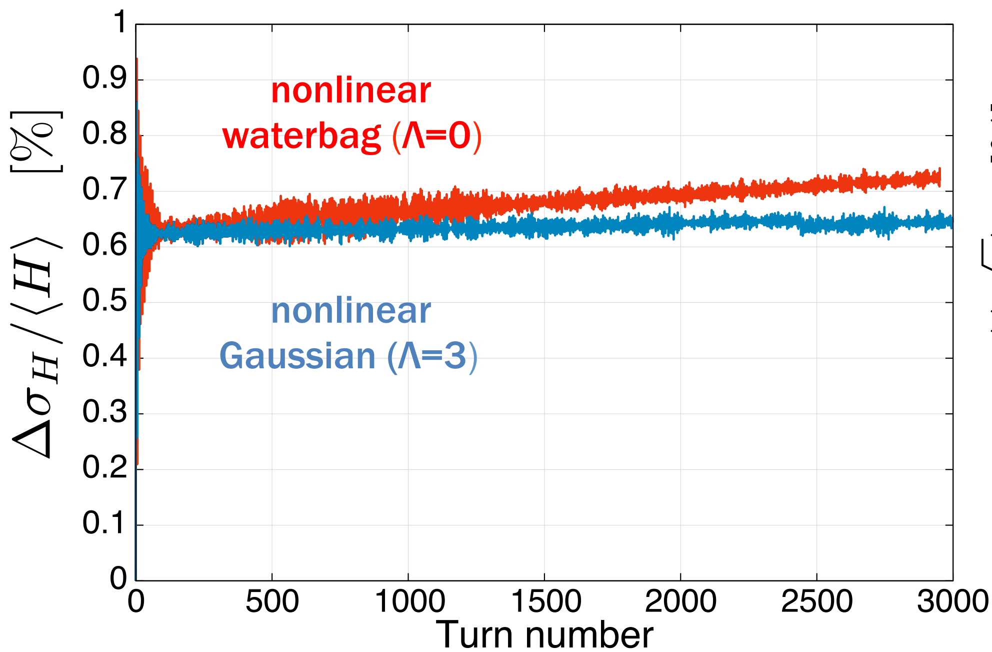


[1]Assuming a stationary waterbag beam with emittance, phase advance, and tune depression used in the IOTA lattice.

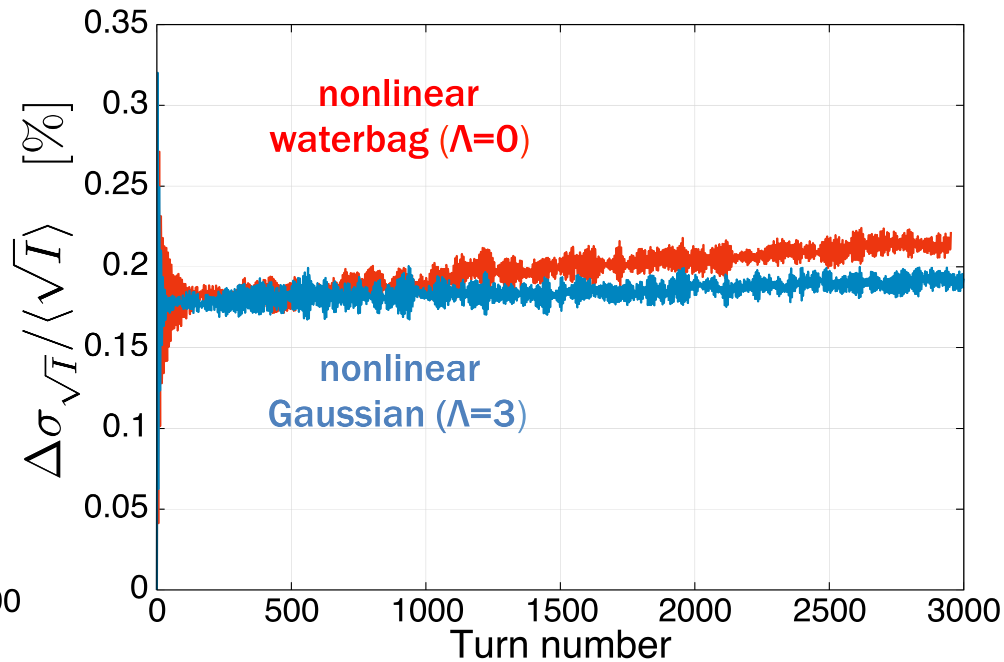
Tracking in the IOTA Lattice with Space Charge – First 3,000 Turns

Long-term tracking over 3,000 turns showing sensitivity of diffusion rates to distribution details.

First invariant



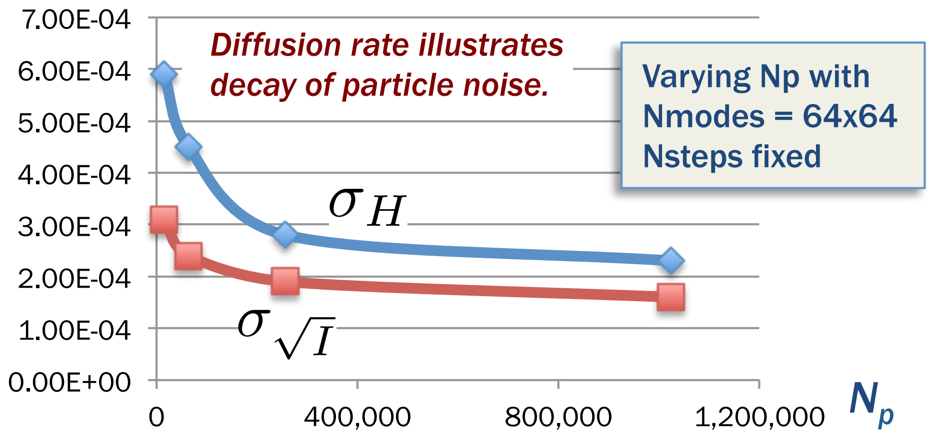
Second invariant



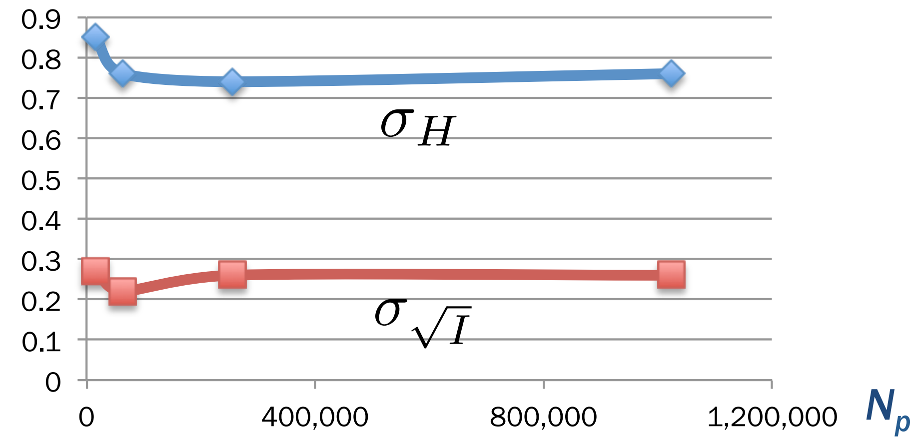
In both cases, <*H*> = 4 mm-mrad. The beams are well-matched in horizontal and vertical rms beam sizes, but the current density differs by >10% at the beam center → differences in tune advance for particles in the core. Greater nonlinearity in the space charge fields for the Gaussian beam does not appear to result in more rapid diffusion in this case.

Convergence of the Standard Deviation of Single-Particle Invariants with Numerical Resolution of Space Charge

Rate of relative growth (%/turn)



Relative growth at turn 300 (%)

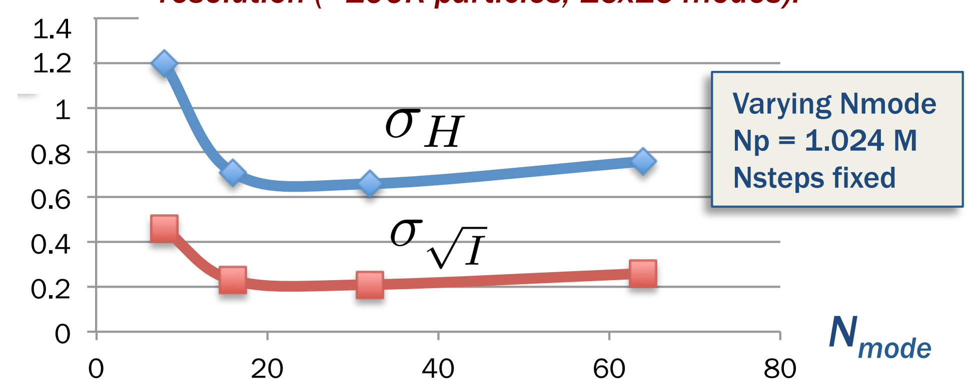


Dynamics of initial mixing and phase space filamentation are well-resolved using ~100K particles and 16x16 spectral modes.

Rates of diffusion (above) are computed using linear interpolation of data taken over 200 turns.

Due to particle noise, ~1M particles are required to begin to approach convergence of the diffusion rates. This could potentially be improved using higher-order macroparticle shapes.

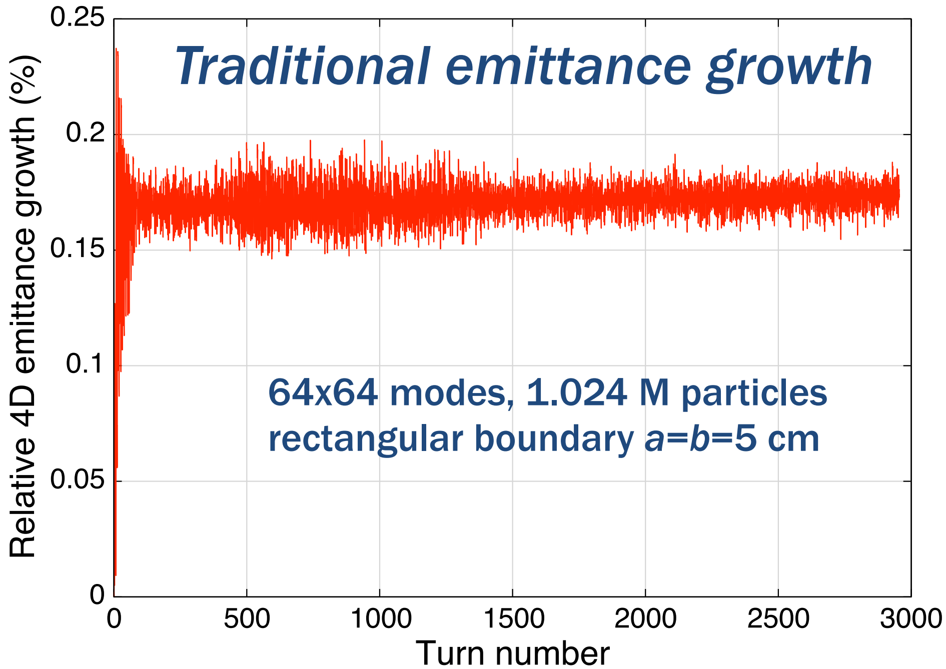
Initial mixing behavior saturates at low resolution (~100K particles, 16x16 modes).

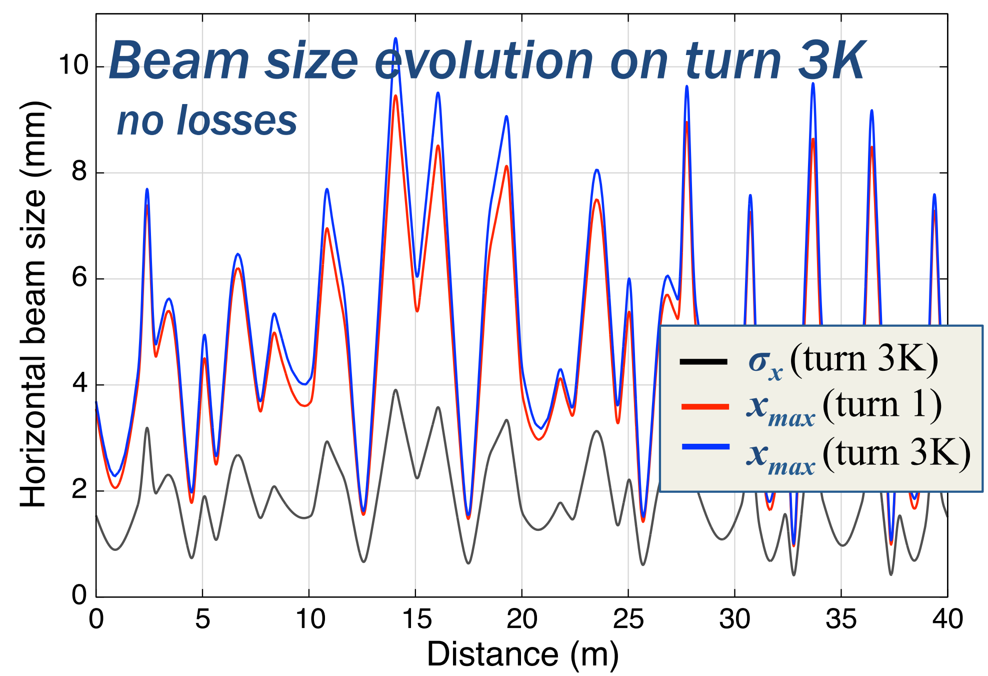


Traditional rms emittance growth and beam size evolution for the nonlinear waterbag beam

- Emittance is well-preserved after initial redistribution due to space charge.
- Growth in the outer beam edge at the 10% level over 3K turns (stable beyond turn 350).

Matched nonlinear waterbag beam
I = 0.4113 mA, <*H*> = 4.0 mm-mrad
τ = 0.4, c = 0.01, μ_0 = 0.30345, L = 1.8 m





X and *Y* are strongly coupled by the nonlinear insert. We show the change in $\sqrt{\epsilon_x \epsilon_y}$.

Change in rms beam size: 3% (0.1 mm)
Change in maximum *x*-deviation: 11% (1.0 mm)

Conclusions

- A family of initial beam distribution types controlled by two parameters <H> (generalized emittance) and Λ (cutoff) allows us to investigate sensitivity to distribution details while remaining matched to the (ideal) nonlinear lattice.

- Tracking is performed using a symplectic integrator within the nonlinear magnetic insert, coupled with a symplectic spectral solver for self-consistent space charge tracking to avoid non-Hamiltonian sources of numerical noise. Particle noise has a significant impact on the observed stochatic diffusion. Here 1.024M particles are used.

- We focus on isolating the perturbative effects of space charge for moderate tune depression (ΔQ=-0.03).

- Initial nonlinear mixing leads to a near-matched equilibrium by turn 350. The largest visible effect is smoothing of the hard outer beam edge. The distribution of invariants is well-preserved both 1) across the nonlinear insert and 2) across the arc. However, the single-particle invariants along each orbit fluctuate significantly (~20%). Nevertheless, the invariants provide a sensitive measure of beam quality.

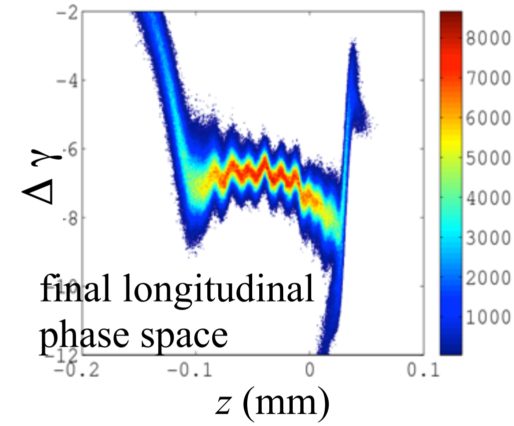
- There remains evidence of slow stochastic diffusion, while both rms beam sizes remain well-controlled. An 11% increase in maximum horizontal particle amplitude is visible, and a larger number of spectral modes may be needed to verify that the space charge fields at the beam edge are well-resolved.

- The dynamics appear robust in the presence of nonlinear space charge fields caused by the presence of a matched, low-density tail, which also results in more rapid nonlinear damping.

- **Backup material**

Overview of advanced computing/modeling using IMPACT-Z

- **The IMPACT-Z code & physics model**
  - *s-based symplectic particle tracking using maps*
  - *Poisson solvers for 6 distinct boundary conditions*
  - *standard beamline elements, RF and RW wakefields*
  - *field, misalignment, and rotation errors*
  - *multi-turn tracking with simulation restart*
  - *efficient parallelization, access to NERSC*



- **The IMPACT code suite is used by > 40 institutes worldwide**
  - *successfully applied to both electron & proton machines:*
    - *CERN PS2 ring*
    - *LCLS-II linac*
  - *unprecedented resolution: ~2B macroparticles*

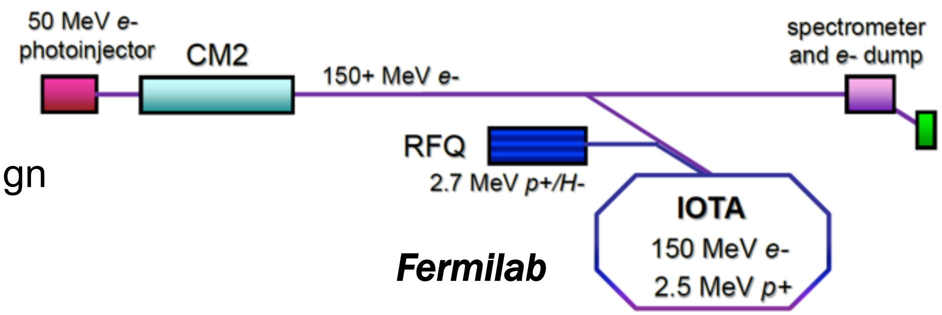
Collaboration with teams at RadiaSoft and FNAL, who are modeling IOTA using SYNERGIA.

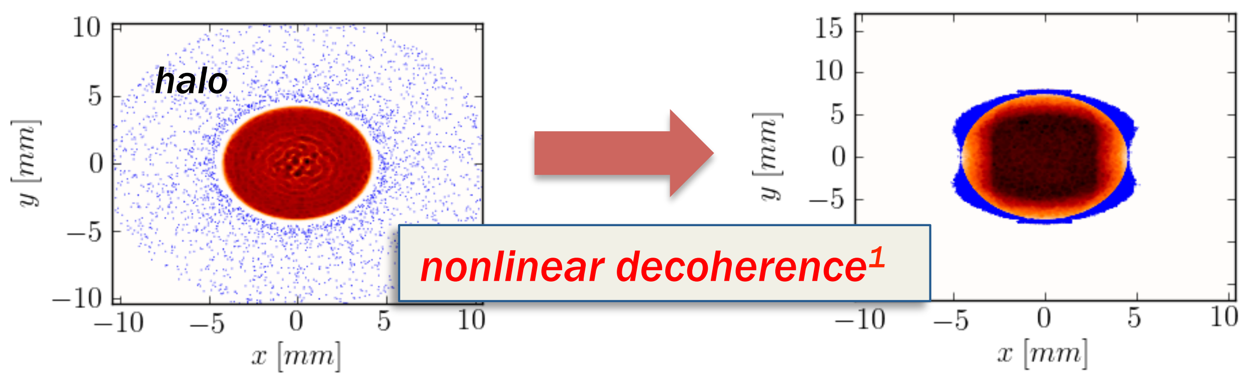
The IOTA ring : a test bed for strategies designed to mitigate space charge-induced beam halo.

- **Possible strategies: electron lenses/columns, nonlinear integrable lattices**

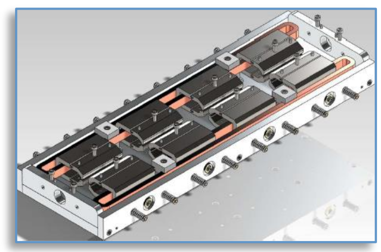
- **Integrable Optics Test Accelerator (IOTA)**

  - Novel accelerator physics: strongly nonlinear design
  - Experimental test bed for SC mitigation schemes
  - Run first with electrons, then low-energy protons





using conventional linear design   *using nonlinear integrable design*



nonlinear magnetic insert

- **Nonlinearity ➡ tune spread "washes out" coherent space charge instabilities**
- **Integrability ➡ ensures orbits are regular and remain bounded (no chaos)**

[1]S. Webb *et al*, p. 2961, IPAC 2012

Summary of spatial projections for revised matched distributions in the D&N potential

Nonlinear KV $G(h) \propto \delta(h-\epsilon_0)$

$$P_{XY}(x,y) \propto \begin{cases} 1, & \Phi(x,y)/\epsilon_0 \leq 1, \\ 0, & \text{else} \end{cases}$$

Nonlinear Waterbag $G(h) \propto \Theta(\epsilon_0 - h)$

$$P_{XY}(x,y) \propto \begin{cases} 1-\Phi(x,y)/\epsilon_0, & \Phi(x,y)/\epsilon_0 \leq 1, \\ 0, & \text{else} \end{cases}$$

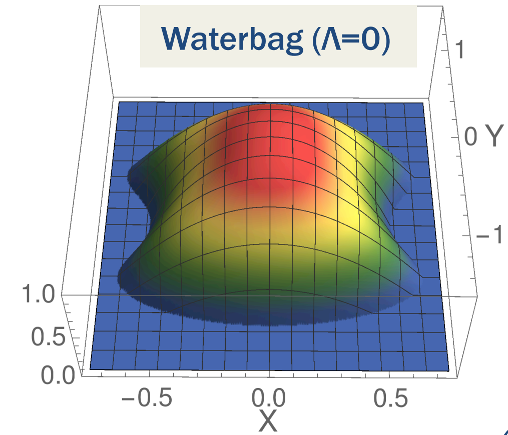
Nonlinear Gaussian $G(h) \propto e^{-h/\epsilon_0}\Theta(\Lambda - h/\epsilon_0)$

$$P_{XY}(x,y) \propto \begin{cases} e^{-\Phi(x,y)/\epsilon_0} - e^{-\Lambda}, & \Phi(x,y)/\epsilon_0 \leq \Lambda, \\ 0, & \text{else} \end{cases}$$

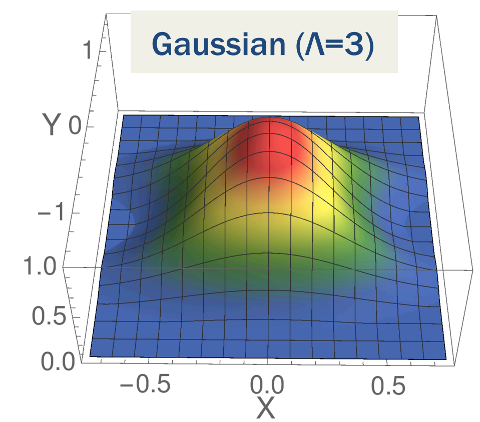
Potential (dimensionless form)

$$\Phi(x,y) = \frac{1}{2}(x^2+y^2) + \tau \mathcal{R}e\left[F(x+iy)\right]$$

$$F(z) = \left(\frac{z}{\sqrt{1-z^2}}\right)\arcsin(z)$$

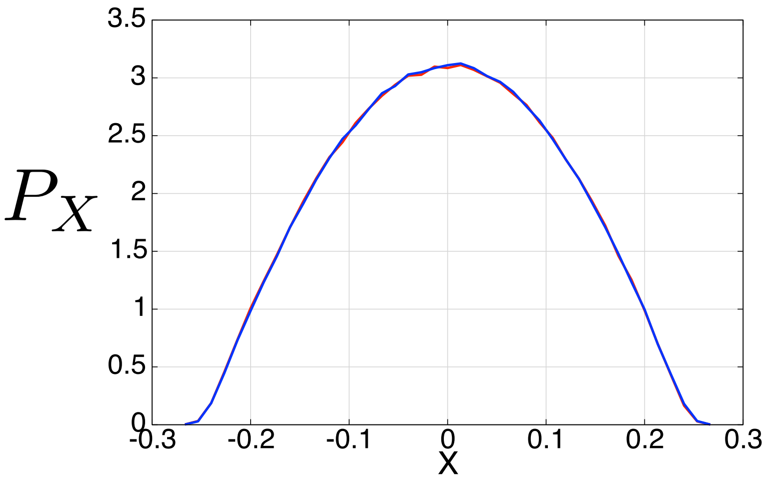


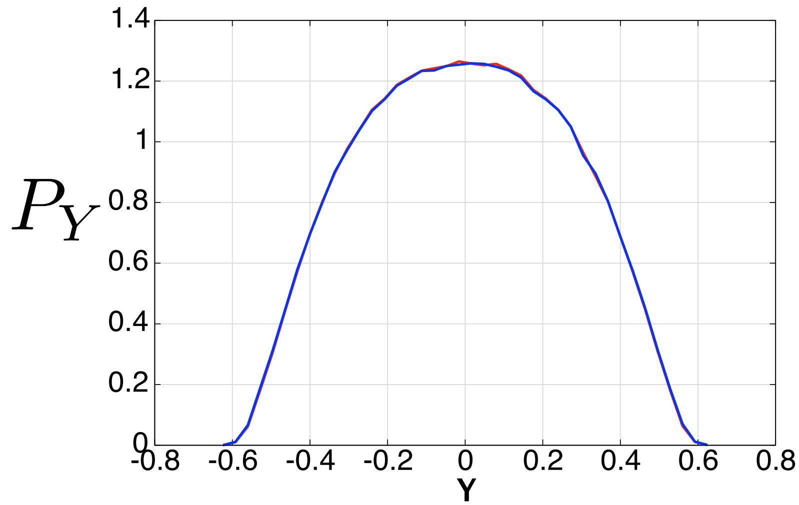
Shown for $\tau = 0.4$, $\langle H \rangle = 0.2$ (dimensionless)



Nonlinear Waterbag Beam in the IOTA Lattice – Spatial profiles are well-preserved across the arc

Horizontal profile | ***Vertical profile***

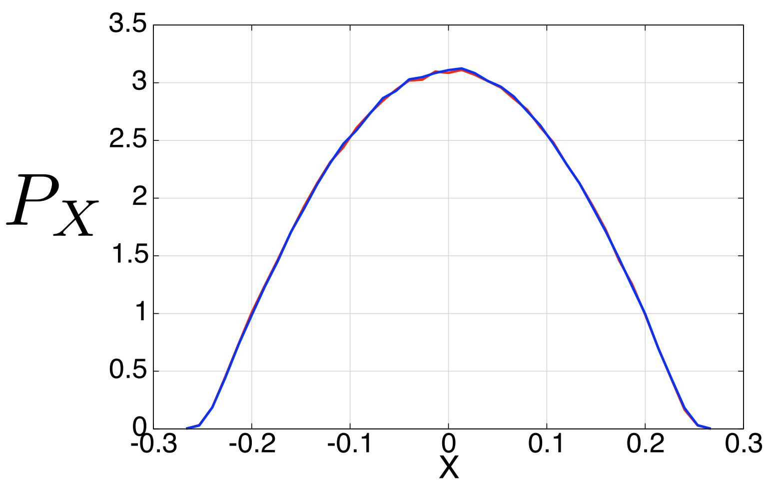


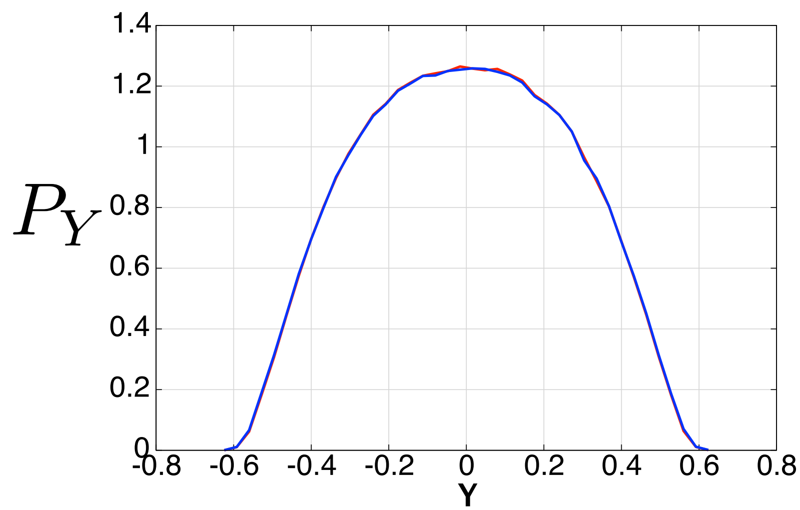


Arc entry
Arc exit

Turn 350

X, Y are dimensionless; singularities located at (X,Y)=(±1,0)





Arc entry
Arc exit

Turn 700

Smooth focusing waterbag model of equilibrium in the linear IOTA lattice

Hamiltonian:

$$H(x,p_x,y,p_y) = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}k_0^2(x^2+y^2) + \frac{q\phi_s(x,y)}{\beta^2\gamma^3 mc^2}$$

← self-consistent space charge potential

$$p_x = \gamma m\dot{x}/p^0, \quad p_y = \gamma m\dot{y}/p^0$$

Due to symmetry under rotation, *H* is *integrable* with a second invariant given by $J_z = xp_y - yp_x$.

Distribution function:

$$f = G \circ H$$

$$G(h) \propto \Theta(H_{max} - h)$$

Spatial density:

$$P_{XY}(x,y) \propto \begin{cases} 1 - W(|\vec{r}|)/H_{max}, & W(|\vec{r}|)/H_{max} \leq 1, \\ 0, & \text{else} \end{cases}$$

Self-consistent potential:

$$W(r) = \frac{1}{2}k_0^2 r^2 + \frac{q\phi_s(r)}{\beta^2\gamma^3 mc^2} = \frac{1}{2}k_0^2 a^2\left[1 - \frac{4}{k_1^2 a^2}\left(1 - \frac{I_0(k_1 r)}{I_0(k_1 a)}\right)\right]$$

Generalized space charge perveance: $K_{perv} = k_0^2 a^2 \frac{I_2(k_1 a)}{I_0(k_1 a)}$

Smooth focusing waterbag model of equilibrium in the linear IOTA lattice

Hamiltonian expressed using zero-current action-angle variables:

$$x_N + ip_{xN} = \sqrt{2I_x}e^{-i\phi_x}, \quad x_N = x\sqrt{k_0}, \quad p_{xN} = p_x/\sqrt{k_0}$$

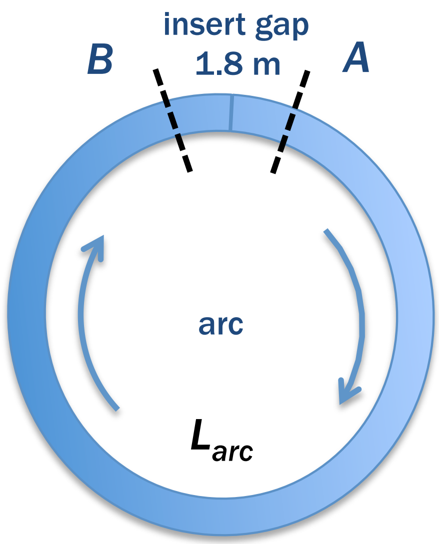
normalized coordinates

$$H(\phi_x, I_x, \phi_y, I_y) = k_0(I_x + I_y) + V(R(\vec{\phi}, \vec{I}))$$

$$V(r) = W(r) - \frac{1}{2}k_0^2 r^2 \qquad R(\vec{\phi}, \vec{I}) = \left[\frac{2I_x}{k_0}\cos^2\phi_x + \frac{2I_y}{k_0}\cos^2\phi_y\right]^{1/2}$$

Input parameters:

1) rms emittance: ε_x = 2 mm-mrad
2) undepressed tune advance *A* to *B*: Q_x = 5.03
3) space charge tune depression: ΔQ_x = -0.03
4) length of arc *A* to *B*: L_{arc} = 38.1682 m



Matched to an rms equivalent KV beam.[1]

Examine the evolution of the zero-current invariants (actions).

Model parameters:

k_0 = 0.828 m^{-1}, k_1 = 66.355 m^{-1}
a = 3.817 mm, H_{max} = 4.995 mm-mrad

Checks: σ_x = 3.118 mm, *I* = 4.85 mA

[1]S. M. Lund *et al*, Phys. Rev. ST – Accel. Beams 12, 114801 (2009)