Transverse Mode Coupling Instability and Space Charge

Tim Zolkin[1], Alexey Burov[1], Bipul Pandey[2]

[1]Fermi National Accelerator Laboratory
[2]The University of Chicago

February 1, 2018

PART I. Introduction

M. Blaskiewicz AirBag Square well (ABS) model:

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS, VOLUME **1**, 044201 (1998)

Fast head-tail instability with space charge

M. Blaskiewicz

Strong Space Charge (SSC) theory:

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **12**, 044202 (2009)

Head-tail modes for strong space charge

A. Burov

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **12**, 124402 (2009)

Transverse modes of a bunched beam with space charge dominated impedance

V. Balbekov*

1.1 SSC Theory (no-wake)

$$\boxed{\frac{1}{Q_{\mathrm{eff}}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\frac{\mathrm{d}\,Y(\tau)}{\mathrm{d}\tau}\right)+\nu\,Y(\tau)=0,\qquad Y'(\pm\infty)=0,}$$

where $\rho(\tau)$ is the normalized longitudinal line density and $u^2(\tau)$ is the local average of the longitudinal velocity squared

$$\rho(\tau)=\int f(\tau,\mathrm{v})\,\mathrm{dv},\qquad u^2(\tau)=\int f(\tau,\mathrm{v})\,\mathrm{v}^2\,\mathrm{dv}\Big/\rho(\tau).$$

$Q_{\mathrm{eff}}(\tau)=Q_{\mathrm{eff}}(0)\rho(\tau)/\rho(0)$ is the space charge tune shift averaged over the slice at the position τ. For the Gaussian transverse distribution $Q_{\mathrm{eff}}(\tau)=0.52Q_{\max}(\tau)$.

Solutions of this S-L problem constitute an orthogonal basis, which when normalized will be referred to as the *SSC harmonics*

$$\boxed{[\nu_k,Y_k(\tau)]:\quad\int_{\mathrm{SB}}\rho(\tau)\,Y_l(\tau)\,Y_m(\tau)\,\mathrm{d}\tau=\delta_{lm},\quad k=0,1,2,3,\ldots.}$$

1.2 SSC Theory (zero chromaticity, $\zeta = 0$)

Wake function $W(\tau)$ modifies the collective dynamics as follows:

$$\boxed{\frac{1}{Q_{\text{eff}}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\frac{\mathrm{d}\,\mathcal{Y}(\tau)}{\mathrm{d}\tau}\right) + \Delta q\,\mathcal{Y}(\tau) = \kappa\,\widehat{\mathrm{W}}\,\mathcal{Y}(\tau)},$$

with

$$\widehat{\mathrm{W}}\;Y(\tau) = \int_\tau^\infty W(\tau-\sigma)\,\rho(\sigma)\,Y(\sigma)\,\mathrm{d}\sigma, \quad \text{and} \quad \kappa = N_{\text{b}}\,\frac{r_0 R_0}{4\,\pi\,\gamma\,\beta^2\,Q_\beta}.$$

Expansion over the basis of SSC harmonics, $\mathcal{Y}_k(\tau) = \sum_{i=0}^{\infty}\mathbf{C}_i^{(k)}\,Y_i(\tau)$, leads to the eigenvalue problem

$$\boxed{\mathbf{M}\cdot\mathbf{C}^{(k)} = \Delta q_k\mathbf{C}^{(k)}, \qquad \mathbf{M}_{lm} = \nu_l\,\delta_{lm} + \kappa\,\widehat{\mathrm{W}}_{lm}}$$

with matrix elements of the wake operator being

$$\widehat{\mathrm{W}}_{lm} = \int_{-\infty}^{\infty}\int_\tau^\infty W(\tau-\sigma)\,\rho(\tau)\,\rho(\sigma)\,Y_l(\tau)\,Y_m(\sigma)\,\mathrm{d}\,\sigma\,\mathrm{d}\,\tau.$$

1.3 TMCI Theorem: only two types of TMCI are possible

$$\boxed{\frac{1}{Q_{\text{eff}}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\frac{\mathrm{d}\,\mathcal{Y}(\tau)}{\mathrm{d}\tau}\right)+\Delta q\,\mathcal{Y}(\tau)=\kappa\,\widehat{\mathrm{W}}\,\mathcal{Y}(\tau)}$$

Scaling with respect to the SC: if $\{\mathcal{Y},\Delta q\}$ are an eigenfunction and eigenvalue for certain SC tune shift Q_{eff} and wake W, then for α times larger SC, $\alpha\,Q_{\text{eff}}$, and α times smaller wake, W/α, the eigenvalues scale as $\Delta q/\alpha$, while eigenfunctions $\mathcal{Y}$ remain the same.

1. SSC TMCI

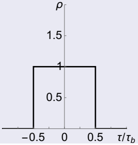
The instability threshold, if there is any, scales inversely proportional to the SC tune shift, $\kappa W_{\rm th}\propto 1/Q_{\text{eff}}(0)$.

2. Vanishing TMCI

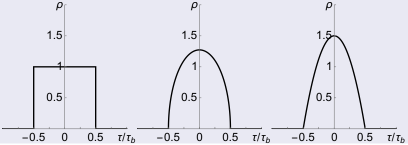
If modes never couple, it means that TMCI can only be outside the SSC applicability area, when some of wake tune shifts are comparable with $Q_{\text{eff}}(0)$. When the SC-asymptotic of the TMCI threshold is considered, if it does exist at all, it asymptotically has to be proportional to the SC tune shift, $\kappa W_{\rm th}\propto Q_{\text{eff}}(0)$.

1.4 SSC models

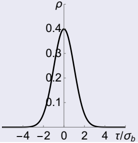
SSCSW: $f_{\mathrm{SW}}(\tau, \mathrm{v}) = \Theta\left[1 - 4\,\tau^2/\tau_\mathrm{b}^2\right] \mathrm{V}(\mathrm{v})/\tau_\mathrm{b}$



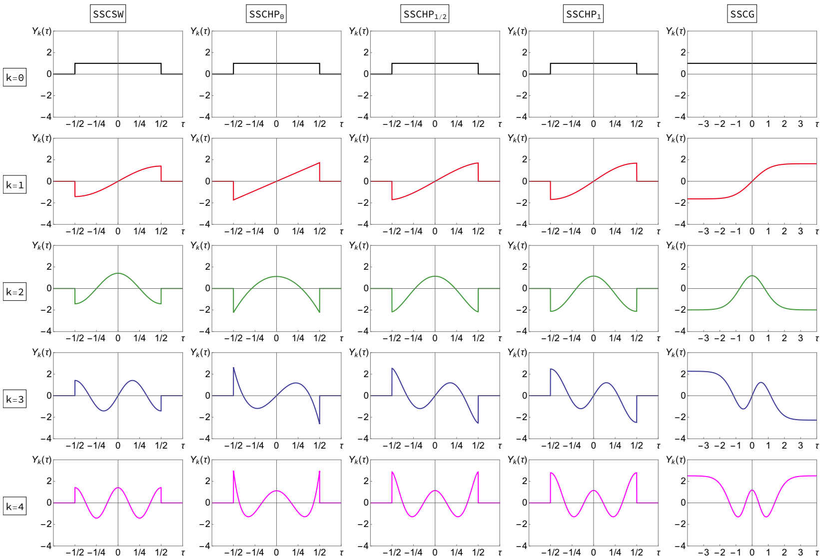
SSCHP: $f_{\mathrm{HP}}^{(n)}(\tau, \mathrm{v}) \propto \left(1 - 4\,\tau^2/\tau_\mathrm{b}^2 - 4\,\mathrm{v}^2/\mathrm{v}_\mathrm{b}^2\right)^{n-1/2}$



SSCG: $f_{\mathrm{G}}(\tau, \mathrm{v}) = \exp\left[-(\tau^2/\sigma_\mathrm{b} - \mathrm{v}^2/\mathrm{v}_\mathrm{b})/2\right]/(2\,\pi\,\sigma_\mathrm{b}\,\mathrm{v}_\mathrm{b})$



1.5 SSC harmonics for SW, $HP_{0,1/2,1}$ and Gaussian bunch



1.6 M. Blaskiewicz AirBag Square well model (ABS)

$$\boxed{f_{\mathrm{ABS}}(\tau, \mathrm{v}) \propto [\delta(\mathrm{v}-\mathrm{v}_0)+\delta(\mathrm{v}+\mathrm{v}_0)]\,\Theta\left[1-4\,\frac{\tau^2}{\tau_{\mathrm{b}}^2}\right]}$$

Equations for the amplitudes along the bunch $x_\pm(\tau)$ are given by

$$\frac{\mathrm{d}\,x_\pm}{\mathrm{d}\tau} = \pm\frac{i}{\mathrm{v}_0}\left[\left(\frac{\Delta Q_{\mathrm{sc}}}{2}+\Delta Q_k\right)x_\pm - \frac{\Delta Q_{\mathrm{sc}}}{2}\,x_\mp - F\right]$$

with the boundary conditions $x_+(\pm\tau_{\mathrm{b}}/2) = x_-(\pm\tau_{\mathrm{b}}/2)$.
The force is defined by a wake function

$$F(\tau) = \kappa\int_\tau^{\tau_{\mathrm{b}}/2} W(\tau-\sigma)\,\overline{x}(\sigma)\,\mathrm{d}\sigma, \qquad \overline{x} = (x_+ + x_-)/2,$$

and satisfies an integro-differential equation:

$$\frac{\mathrm{d}\,F(\tau)}{\mathrm{d}\tau} = -\kappa\,W(0)\,\overline{x}(\tau) + \kappa\int_\tau^{\tau_{\mathrm{b}}/2}\frac{\partial}{\partial\tau}W(\tau-\sigma)\overline{x}(\sigma)\,\mathrm{d}\sigma.$$

1.7 ABS model: no-wake solution

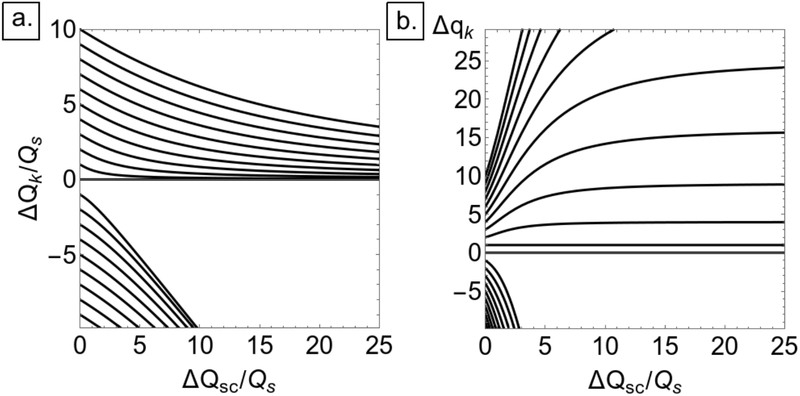


Figure: Eigenvalues (a.) and normalized eigenvalues (b.) as functions of SC parameter for no-wake case, $k = -9, \ldots, 10$. (ABS).

$$\frac{\Delta Q_k}{Q_s} = -\frac{\Delta Q_{sc}}{2\,Q_s} \pm \sqrt{\left(\frac{\Delta Q_{sc}}{2\,Q_s}\right)^2 + k^2}, \qquad \Delta q_k = \frac{\Delta Q_k}{\Delta Q_1}.$$

PART II. Negative wakes

2.1 Spectra for const-wake (ABS)

In order to describe the instability, we introduce a dimensionless *wake parameter* χ and its normalized value χ^*:

$$\chi = \frac{\kappa\, W_0}{Q_s} \qquad \text{and} \qquad \chi^* = \frac{\kappa^*\, W_0}{Q_s},$$

where W_0 is the wake amplitude and $\kappa^* = \dfrac{\kappa}{-\frac{\Delta Q_{\rm SC}}{2\,Q_s} + \sqrt{\left(\frac{\Delta Q_{\rm SC}}{2\,Q_s}\right)^2 + 1}}$.

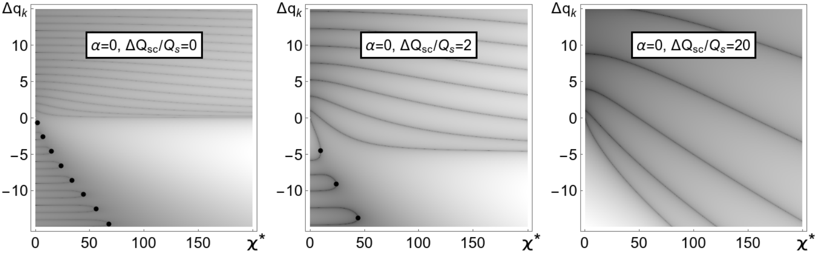


Figure: Spectra for ABS model and const-wake $(\Delta Q_{\rm sc}/Q_s = 0, 2, 20)$.

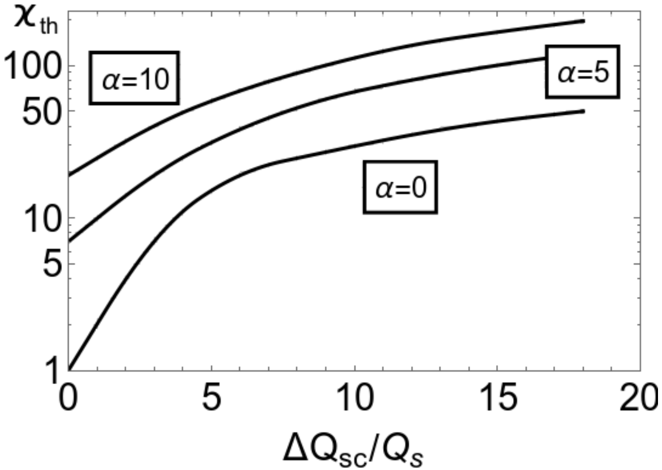


Figure: TMCI threshold as a function of space charge for ABS model with constant and exponential wakes (after M. Blaskiewicz).

2.3 Convergence for SSC models.

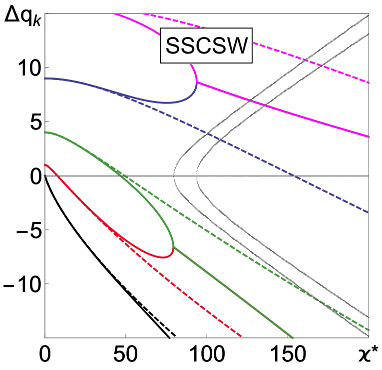


Figure: Real (shown in color) and imaginary (gray colors) parts of the spectra for SSCSW and SSCHP_0 with a constant wake. Solid and dashed lines are obtained using 5 or 50 modes respectively. For sufficiently large number of modes and sufficiently good accuracy of the matrix element computations, TMCI vanishes.

2.4 Convergence for SSC models (continued).

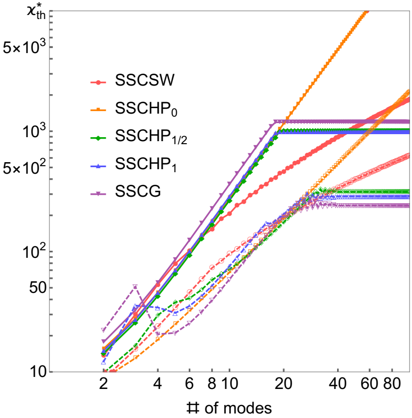


Figure: TMCI threshold as a function of number of modes for constant and resistive wall wakes (solid and dashed lines respectively) at SSC. The unlimited growth with the truncation parameter means that there is no TMCI. The saturation of the threshold for numerical models ($\mathrm{SSCHP}_{1/2,1}$ and SSCG) does not reflect the instability and is due to limited precision of calculated matrix elements, $\widehat{\mathrm{W}}_{lm}$.

2.5 Vanishing TMCI

The vanishing TMCI is confirmed for SSCSW, $\mathrm{SSCHP}_{0,\frac{1}{2},1}$ and SSCG models with

- Delta wake
- Constant wake
- Step wake
- Exponential wake
- Thick resistive wall wake

See also:

V. Balbekov, arXiv preprint arXiv:1608.06243 (2016).
V. Balbekov, arXiv preprint arXiv:1607.06076 (2016).
V. Balbekov, Physical Review Accelerators and Beams 20, 034401 (2017).
V. Balbekov, arXiv preprint arXiv:1706.01982 (2017).
V. Balbekov, arXiv preprint arXiv:1801.04311 (2018).

PART III. Oscillating wakes

3.1 Spectra for cos-wake (ABS & SSCSW)

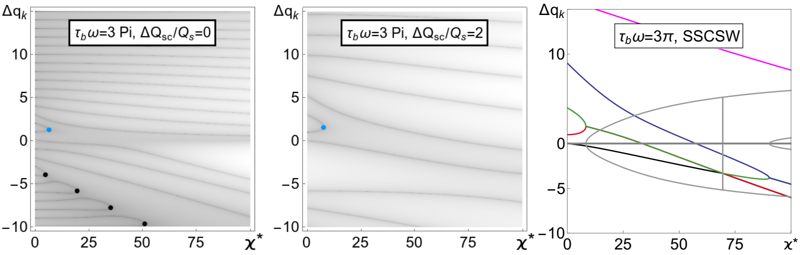


Figure: Spectra for the ABS ($\Delta Q_{\rm sc}/Q_{\rm s} = 0, 2$) and SSCSW models.

SSC makes the beam less stable: the threshold is found to be inversely proportional to the SC parameter when the wake phase advance is large enough, $\omega\,\tau_{\rm b} > \pi$. When the wake oscillations are not pronounced (either the phase advance is insufficient, $\omega\,\tau_{\rm b} < \pi$, or the wake oscillations are overshadowed by their exponential decay) the TMCI vanishes at the SSC limit, as it can be expected.

3.2 TMCI threshold as a function of SC (ABS)

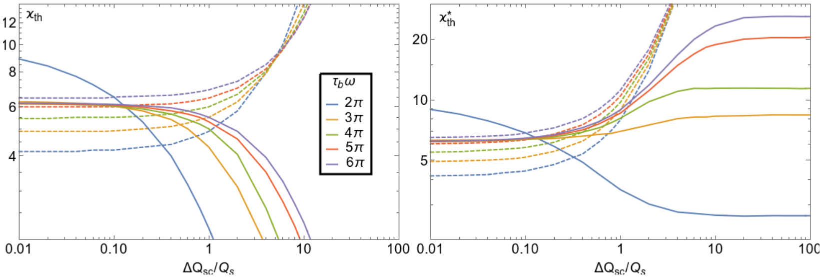


Figure: TMCI threshold as a function of space charge for the ABS model and a cosine wake. The dashed and solid lines correspond to the vanishing and non-vanishing TMCIs. The left figure shows thresholds in regular units. The right figure shows the same in terms of the normalized wake parameter.

3.3 cos-wake and convergence at SSC

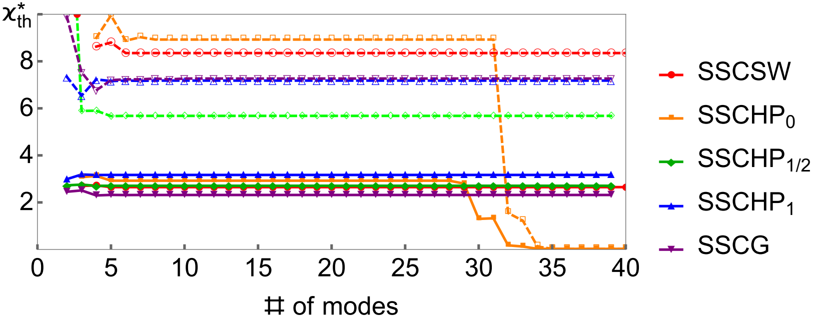


Figure: Instability threshold as a function of the number of modes taken into account at the SSC for the cosine wake. Solid and dashed lines correspond to different values of wake phase advance, $\omega\,\tau_{\text{b}}$: $2\,\pi$ and $3\,\pi$ respectively.

3.4 SSC TMCI threshold as a function of $\omega\,\tau_{\rm b}$

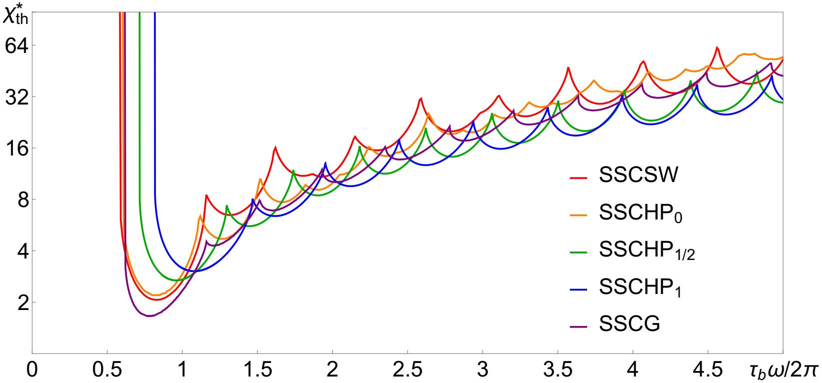


Figure: TMCI threshold for all SSC models with cosine wake as a function of wake phase advance $\omega\,\tau_{\rm b}$. The vertical lines reflect an absolute threshold with respect to $\omega\,\tau_{\rm b}$.

4.1 Spectra for sin-wake (ABS)

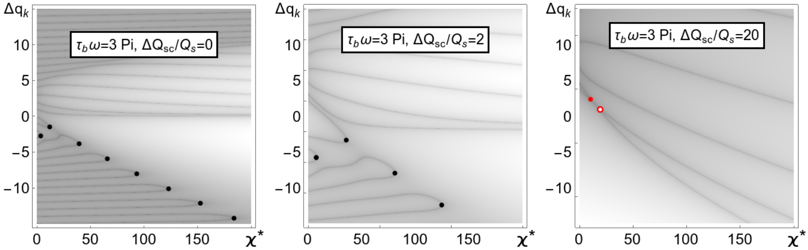


Figure: Spectra for ABS model and sin-wake ($\Delta Q_{sc}/Q_s = 0, 2, 20$).

- 1. When SC is zero, there is no instability for the positive modes.
- 2. When SC is increased, a cascade of mode couplings and decouplings appears.

4.2 TMCI threshold as a function of SC (ABS)

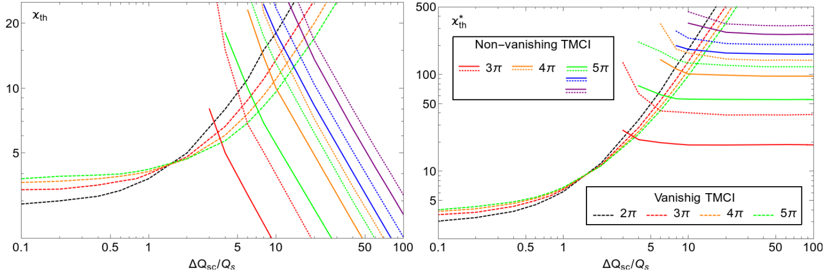
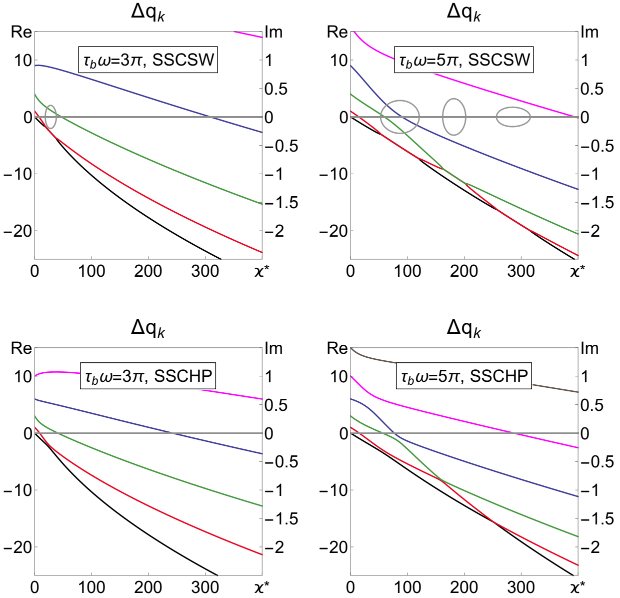


Figure: TMCI threshold in terms of the usual (left) and normalized (right) wake parameters versus the SC parameter for the ABS model and sine wake. The dashed lines correspond to the vanishing TMCI in a negative part of the spectra. The solid and dotted lines represent first couplings and first decouplings for positive modes.

4.3 sin-wake at SSC (SSCSW vs SSCHP$_0$)



4.4 TMCI threshold as a function of $\omega\,\tau_b$ (SSCSW)

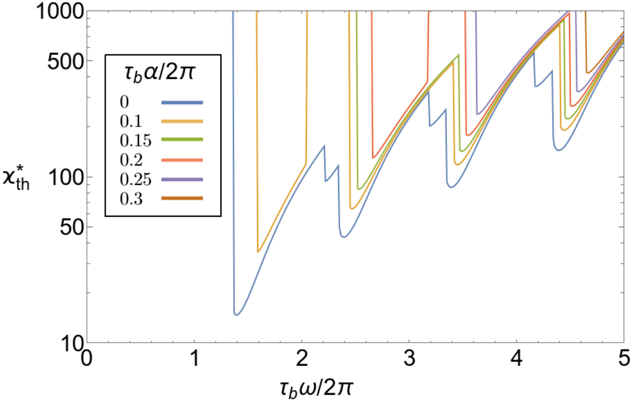


Figure: TMCI threshold for SSCSW model with resonator wake as a function of wake phase advance $\omega\,\tau_b$. The different curves correspond to the different values of the wake decay rate, $\alpha\,\tau_b$. The blue curve with $\alpha\,\tau_b = 0$ is for the sine wake. The vertical lines reflect an absolute threshold with respect to $\omega\,\tau_b$.

4.5 Convergence for SSCG model

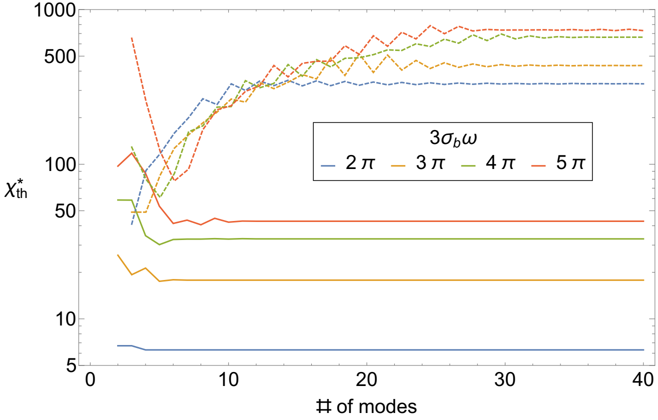


Figure: Instability threshold as a function of the number of modes taken into account for SSCG model with cosine (solid lines) and sine (dashed lines) wakes.

4.6 Bunch spectra for SSCG

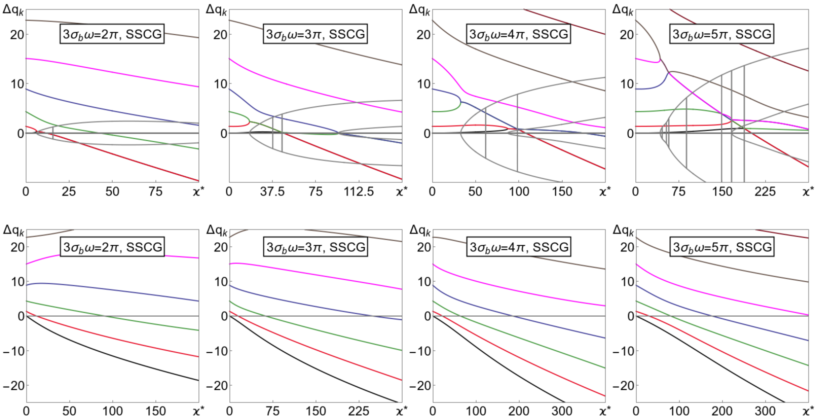


Figure: Bunch spectra for SSCG with the cos- (top) and sin-wakes (bottom) . The columns correspond to the different values of $3\,\omega\,\sigma_b$.

5.1 Comparison with simulations

Proceedings of IPAC'10, Kyoto, Japan TUPD046

EFFECTS OF DIRECT SPACE CHARGE ON THE TRANSVERSE MODE COUPLING INSTABILITY

D. Quatraro, G. Rumolo
European Organization for Nuclear Research (CERN), CH-1211 Genève 23, Switzerland

Proceedings of IPAC2012, New Orleans, Louisiana, USA WEPPR097

COMPARING NEW MODELS OF TRANSVERSE INSTABILITY WITH SIMULATIONS*

M. Blaskiewicz†, BNL, Upton, NY 11973, USA

5.2 CERN SPS

Broadband resonator wake model:

$$W(\tau) = Z_t \frac{\omega_r^2}{\hat{\omega}\, Q_r} \sin(\hat{\omega}\,\tau)\, e^{\omega_r \tau/(2\,Q_r)}, \qquad \hat{\omega} = \sqrt{\omega_r^2 - \omega_r^2/(2\,Q_r)^2},$$

with $\omega_r = 1.3 \times 2\pi$ GHz, $Q_r = 1$ and $Z_t = 5$ MΩ/m.

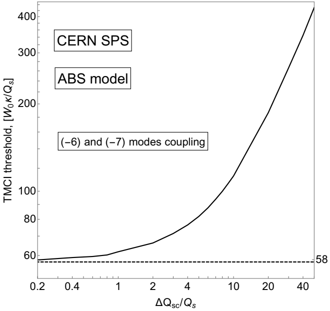


Figure: The lowest TMCI threshold as a function of space charge for CERN SPS ring (ABS model). Dashed line shows the value of threshold at zero $\Delta Q_{\rm sc}$.

5.3 Comparison with M. Blaskiewicz

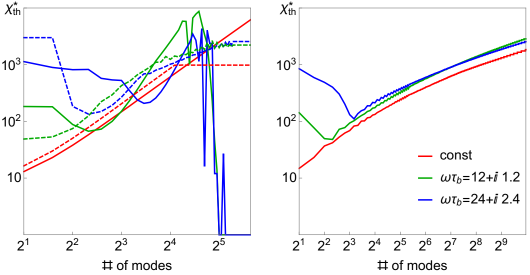


Figure: TMCI threshold as a function of the number of modes taken into account for SSCHP_0 (solid lines) and $\text{SSCHP}_{7/2}$ (dashed lines) models with wake functions from M.B.. The saturation with number of modes for $\text{SSCHP}_{7/2}$ case is determined by precision in calculation of matrix elements. For SSCHP_0 with resonator wakes there is the numerical instability of threshold when the number of modes ≈ 20. Right plot is for SSCHP_0 with the same wakes but using expansion in SSCSW basis: no saturation nor numerical instability is observed.

Summary

TMCI Theorem: there are only 2 types of TMCI with respect to SC

Vanishing TMCI

The threshold value of the wake tune shift κW_0 grows linearly with the SC tune shift when the latter is high enough; this sort of instability cannot be seen in the SSC approximation, and that is why we call it vanishing. For the vanishing TMCI, there is an absolute threshold of the wake amplitude, such that the beam is stable for any number of particles, provided that the wake amplitude is below its absolute threshold value. Note that the latter is inversely proportional to the transverse emittance and beta-function.

SSC TMCI

The threshold of the wake tune shift is asymptotically inversely proportional to the SC tune shift; in fact, it is proportional to $Q_s^2/Q_{\text{eff}}(0)$.

Summary

Studies of particular examples

Vanishing TMCI

Vanishing type of TMCI was always observed for negative and effectively negative wakes, which agrees with similar results of M. Blaskiewicz and V. Balbekov. The same sort of TMCI has been found for resonator wakes with arbitrary decay rates, for all HP and Gaussian bunches inside a parabolic potential well.

SSC TMCI

The alternative type of TMCI, the SSC one, takes place for the cosine wakes without respect to the potential well and bunch distribution; it is also the case for the sine wakes at square potential wells, provided that phase advances of the oscillatory wakes are sufficiently large.

Thank you for your attention!

Questions?