Modelling Curvilinear Fields for Beam Dynamics in the Muon *g*–2 Storage Ring

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1 – Introduction

The E989 muon *g*-2 storage ring stores antimuons in a constant, uniform dipole field by use of a single ring magnet of radius $\rho = 7.112\,\mathrm{m}$. Straddling the beam trajectory are 4 pairs of charged-plate **electrostatic quadrupoles**, used to focus the beam.

For beam dynamics in such a ring, it makes sense to use a **curvilinear frame**. Multipole field descriptions (used for efficient beam simulations) in such a frame already exist[1,2,3], but cannot be fit accurately to a given field map (obtainable from simulation or measurement). Of further concern are the fringe fields of the electrostatic quadrupoles, and variation in the magnetic field.

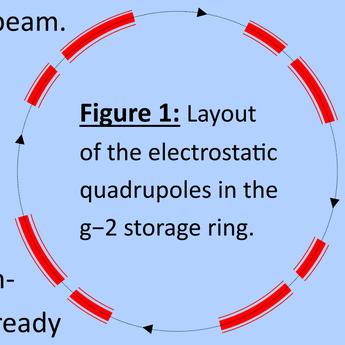

Figure 1: Layout of the electrostatic quadrupoles in the g–2 storage ring.

2 – Solutions of the Toroidal Laplacian

The toroidal coordinates (u, v) are defined such that:

$$x = \frac{\rho \sinh(u)}{\cosh(u) - \cos(v)} - \rho \quad \text{and} \quad y = \frac{\rho \sin(v)}{\cosh(u) - \cos(v)}$$

where ρ is the ring radius and x and y are the relative radial and vertical coordinates with respect to the beam axis, respectively.

A solution to the Laplacian in these coordinates yields the scalar potential ϕ for a source-free region:

$$\phi = \sum_{m,n=-\infty}^{\infty} (-i)^m f_{m,n} \sqrt{\frac{\rho}{x+\rho}}\, P_{n-\frac{1}{2}}^{-|m|}\left(\frac{1}{\tanh(u)}\right) e^{imv} e^{in\theta}$$

where $f_{m,n}$ are a set of coefficients to be fit, $P_a^b(x)$ are the Associated Legendre Polynomials of degree a and order b, and θ is the azimuthal ("longitudinal") location within the ring (along the beam axis).

3 – Quadrupole Field Maps

As well as following the ring curvature (as in Fig.1), the design of the quadrupoles incorporates a flat-plate cross-section, as in Fig.2. Using this design, field maps were produced in Opera3D[4] to mimic those produced by the real plates.

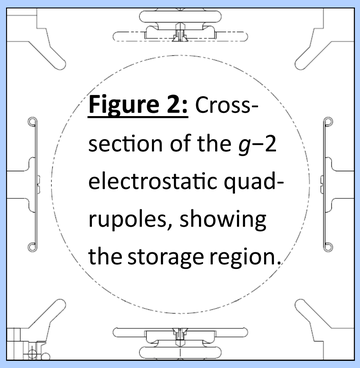

Figure 2: Cross-section of the *g*–2 electrostatic quadrupoles, showing the storage region.

The output potential results in electric fields which peak at the quadrupole ends — a feature not exhibited by standard beamline magnetic fields. We can also expect the cross-section to contain a sextupole component, due to the slight asymmetry between the top/bottom and side plates.

4 – Coefficient Fitting

Similar to fitting generalised gradients[5]; we take the transform of the fitting function and fit this to the potential field, using data on a surface generated by fixing one coordinate. This value (in this case u) is chosen such that the surface extends to near the plates, for greatest accuracy.

The difference between the potentials on the surface is shown in Fig.3. Differences peak at 5% in the fringe, with the difference within the bulk of the quadrupole settling at about 2%.

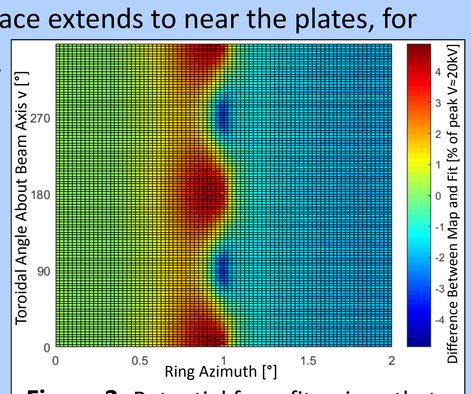

Figure 3: Potential from fit, minus that from map. Plates extend in ring azimuth from 1° onwards.

5 – Tracking Comparisons

A 10,000 particle bunch was used to analyse the phase space differences caused by using the curvilinear model fringes, rather than the field map. The bunch was tracked through an upstream fringe, a multipole (for the quadrupole bulk), and a down-stream fringe.

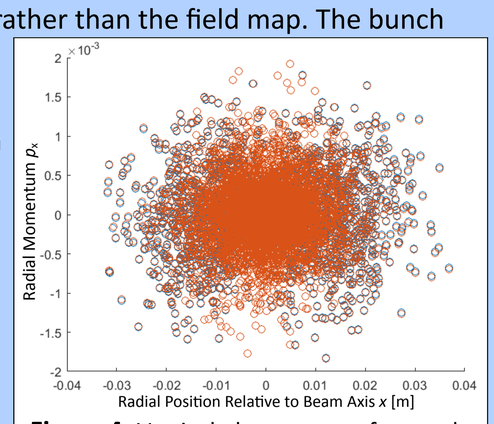

Figure 4: Vertical phase space after tracking with field map fringes (blue) and fitted curvilinear fringes (orange).

Only small differences were found (in Fig.4, many points overlap perfectly). However, only 2 fringe field regions were tracked through.

6 – Conclusions

The results in Fig.4 look promising. However, two potential improvements are suggested:

- Using a better integration method to fulfil the integrations required for calculating the coefficients. It is hoped that this will reduce the peak values and remove the uniform 2% discrepancy on the right in Fig.3.
- Tracking through more fringe fields for the analysis. This should give a clearer picture as to the accuracy of the new method. This was not performed here due to the current implementation being very inefficient.

This technique can also be used to model the bulk field regions, for which we may also include the *g*–2 magnetic dipole field. However, the efficiency of the calculations during tracking must be vastly improved from the current setup for this method to be useful.

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[3] T. Zolkin, "Sector magnets or transverse electromagnetic fields in cylindrical coordinates", *Phys Rev Accel and Beams*, vol. 20, p. 043501, Apr. 2017

[4] W. Wu of University of Mississippi, Private Communication, November 2016

[5] M. Venturini, D. Abell, and A. J. Dragt, in *Proc. ICAP'98*, pp. 184–188.