SINGLE-STATE METHOD WITHIN THE HORSE (J-MATRIX) FORMALISM

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GENERAL IDEA:

- NCSM + HORSE = continuum spectrum

No-core Shell Model

- NCSM is a standard tool in *ab initio* nuclear structure theory
- NCSM: antisymmetrized function of all nucleons
- Wave function: $\Psi = \mathcal{A} \prod_i \phi_i(r_i)$
- Traditionally single-particle functions $\phi_i(r_i)$ are harmonic oscillator wave functions
- N_{max} truncation makes it possible to separate c.m. motion

No-core Shell Model

- NCSM is a bound state technique, no continuum spectrum; not clear how to interpret states in continuum above thresholds − how to extract resonance widths or scattering phase shifts
- HORSE (*J*-matrix) formalism can be used for this purpose
- Other possible approaches: NCSM+RGM; Gamov SM; Continuum SM; SM+Complex Scaling; …
- All of them make the SM much more complicated. Our goal is to interpret directly the SM results above thresholds obtained in a usual way without additional complexities and to extract from them resonant parameters and phase shifts at low energies.
- **I will discuss a more general interpretation of SM results**

J-matrix (Jacobi matrix) formalism in scattering theory

- Two types of L^2 basises:
- Laguerre basis (atomic hydrogen-like states) — atomic applications
- Oscillator basis — nuclear applications
- Other titles in case of oscillator basis:

HORSE (harmonic oscillator representation of scattering equations),

Algebraic version of RGM

J-matrix formalism

- Initially suggested in atomic physics (E. Heller, H. Yamani, L. Fishman, J. Broad, W. Reinhardt) :

  H.A.Yamani and L.Fishman, J. Math. Phys **16**, 410 (1975). Laguerre and oscillator basis.
- Rediscovered independently in nuclear physics (G. Filippov, I. Okhrimenko, Yu. Smirnov):

  G.F.Filippov and I.P.Okhrimenko, Sov. J. Nucl. Phys. **32**, 480 (1980). Oscillator basis.

HORSE:

- Schrödinger equation:

$$H^l \Psi_{lm}(E,r) = E\Psi_{lm}(E,r)$$

- Wave function is expanded in oscillator functions:

$$\Psi_{lm}(E,\boldsymbol{r}) = \frac{1}{r}\, u_l(E,r)\, Y_{lm}(\hat{\boldsymbol{r}}),$$

$$u_l(E,r) = \sum_{n=0}^{\infty} a_{nl}(E)\, R_{nl}(r),$$

- Schrödinger equation is an infinite set of algebraic equations:

$$\sum_{n'=0}^{\infty} (H^l_{nn'} - \delta_{nn'}) a_{nn'}(E) = 0.$$

where *H*=*T*+*V*,

T — kinetic energy operator,

V — potential energy

HORSE:

- Potential energy matrix elements:

$$|nlm\rangle \equiv \phi_{nlm}(\boldsymbol{r}) = \frac{1}{r} R_{nl}(r)\, Y_{lm}(\hat{\boldsymbol{r}}),$$

$$V^{ll'}_{nn'} \equiv \langle nlm|V|n'l'm'\rangle = \int \phi_{nlm}(\boldsymbol{r})\, V\, \phi_{n'l'm'}(\boldsymbol{r})\, d^3\boldsymbol{r}$$

- For central potentials:

$$V^{ll'}_{nn'} = V^{l}_{nn'} = \delta_{mm'}\delta_{ll'} \int R_{nl}(r)\, V\, R_{n'l}(r)\, dr$$

- Note! Potential energy tends to zero as *n* and/or *n'* increases:

$$V^{ll'}_{nn'} \to 0, \quad n, n' \to \infty$$

- Therefore for large *n* or *n'*:

$$T^{l}_{nn'} \gg V^{ll'}_{nn'}, \quad n \text{ or/and } n' \gg 1$$

A reasonable approximation when *n* or *n'* are large

$$H^{l}_{nn'} = T^{l}_{nn'} + V^{l}_{nn'} \approx T^{l}_{nn'}, \quad n \text{ or/and } n' \gg 1.$$

HORSE:

- In other words, it is natural to truncate the potential energy:

$$\widetilde{V}^l_{nn'} = \begin{cases} V^l_{nn'} & \text{if } n \text{ and } n' \leq N; \\ 0 & \text{if } n \text{ or } n' > N. \end{cases}$$

- This is equivalent to writing the potential energy operator as

$$V = \sum_{n=0}^{N} \sum_{n'=0}^{N} \sum_{l,l',m,m'} |nlm\rangle \, V^{ll'}_{nn'} \, \langle n'l'm'|$$

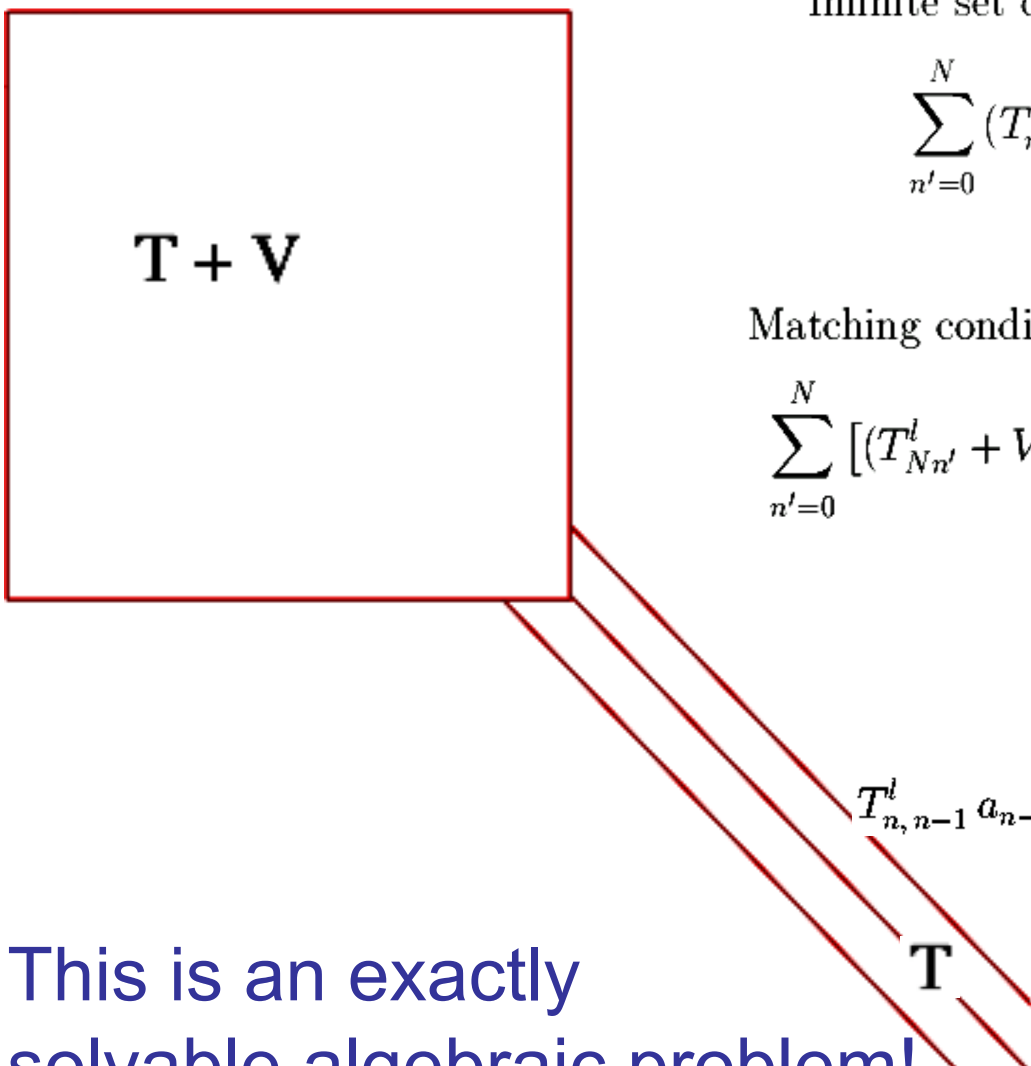
- For large *n*, the Schrödinger equation

$$\sum_{n'=0}^{\infty} (H^l_{nn'} - \delta_{nn'} E) \, a_{n'l}(E) = 0$$

takes the form

$$\sum_{n'=0}^{\infty} (T^l_{nn'} - \delta_{nn'} E) a_{n'l}(E) = 0, \qquad n \geq N+1$$

General idea of the HORSE formalism



Infinite set of algebraic equations

$$\sum_{n'=0}^{N} \left(T^l_{nn'} + V^l_{nn'} - \delta_{nn'} E\right) a_{n'l}(E) = 0, \quad n \leq N-1$$

Matching condition at $n = N$:

$$\sum_{n'=0}^{N} \left[\left(T^l_{Nn'} + V^l_{Nn'} - \delta_{Nn'} E\right) a_{n'l}(E)\right] + T^l_{N,N+1} a_{N+1,l}(E) = 0$$

$$\sum_{n'=0}^{\infty} \left(T^l_{nn'} - \delta_{nn'} E\right) a_{n'l}(E) = 0, \quad n \geq N+1$$

$$T^l_{n,\,n-1}\, a_{n-1,\,l}(E) + \left(T^l_{nn} - E\right) a_{nl}(E) + T^l_{n,\,n+1}\, a_{n+1,\,l}(E) = 0$$

And this looks like a natural extension of SM where both potential and kinetic energies are truncated

This is an exactly solvable algebraic problem!

Asymptotic region $n \geq N$

- Schrödinger equation takes the form of three-term recurrent relation:

$$T^l_{n,n-1}\, a_{n-1,l}(E) + (T^l_{nn} - E)\, a_{nl}(E) + T^l_{n,n+1}\, a_{n+1,l}(E) = 0$$

- This is a second order finite-difference equation. It has two independent solutions:

$$S_{nl}(E) = \sqrt{\frac{\pi\, r_0\, n!}{\Gamma(n+l+3/2)}}\, q^{l+1} \exp\left(-\frac{q^2}{2}\right)\, L_n^{l+\frac{1}{2}}(q^2),$$

$$C_{nl}(E) = (-1)^l \sqrt{\frac{\pi\, r_0\, n!}{\Gamma(n+l+3/2)}}\, \frac{q^{-l}}{\Gamma(-l+1/2)} \exp\left(-\frac{q^2}{2}\right) \times \Phi(-n-l-1/2,\, -l+1/2;\, q^2)$$

where dimensionless momentum $q = \sqrt{\frac{2E}{\hbar\omega}}$

For derivation, see S.A.Zaytsev, Yu.F.Smirnov, and A.M.Shirokov, Theor. Math. Phys. **117**, 1291 (1998)

Asymptotic region $n \geq N$

- Schrödinger equation:

$$T^l_{n,n-1}\, a_{n-1,l}(E) + (T^l_{nn} - E)\, a_{nl}(E) + T^l_{n,n+1}\, a_{n+1,l}(E) = 0$$

- Arbitrary solution $a_{nl}(E)$ of this equation can be expressed as a superposition of the solutions $S_{nl}(E)$ and $C_{nl}(E)$, e.g.:

$$a_{nl}(E) = \cos\delta(E)\; S_{nl}(E) + \sin\delta(E)\; C_{nl}(E)$$

- Note that

$$\sum_{n=M}^{\infty} S_{Nl}(E) R_{nl}(r) \underset{r\to\infty}{\longrightarrow} j_l(qr) \sim \sin\left(qr - \frac{\pi l}{2}\right),$$

$$\sum_{n=M}^{\infty} C_{Nl}(E) R_{nl}(r) \underset{r\to\infty}{\longrightarrow} -n_l(qr) \sim \cos\left(qr - \frac{\pi l}{2}\right)$$

Asymptotic region $n \geq N$

- Therefore our wave function

$$u_l(E,r) = \sum_{n=0}^{\infty} a_{nl}(E)\, R_{nl}(r) \underset{r\to\infty}{\longrightarrow} \sin\left(qr + \delta - \frac{\pi l}{2}\right)$$

- Reminder: the ideas of quantum scattering theory.
- Cross section

$$\sigma \sim \sin^2 \delta$$

- Wave function

$$\Psi \underset{r\to\infty}{\longrightarrow} \sin\left(qr + \delta - \frac{\pi l}{2}\right)$$

- δ in the HORSE approach is the phase shift!

HORSE solutions

- Schrödinger equation

$$\sum_{n'=0}^{N} H^{l}_{nn'} \langle n' | \lambda \rangle = E_{\lambda} \langle n | \lambda \rangle, \qquad n \leq N$$

- Inverse Hamiltonian matrix:

$$(H - E)^{-1}_{nn'} \equiv -\mathscr{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n | \lambda' \rangle \langle \lambda' | n' \rangle}{E_{\lambda'} - E}$$

- Phase shifts:

$$\tan \delta(E) = -\frac{S_{Nl}(E) - \mathscr{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathscr{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,l}(E)}$$

- $S_{Nl}(E)$ and $C_{Nl}(E)$ are the functions which can be expressed analytically

J-matrix, *P*-matrix, *R*-matrix

- HORSE is a discrete analogue of the *P*-matrix approach, $P = R^{-1}$.
- Oscillator expansion: $\Psi = \sum_N a_N \varphi_N$.
- At large quanta N, the oscillator function φ_N is a high-oscillating function at distances up to the classical turning point $b^{cl} = r_0\sqrt{2N+3}$ and rapidly decreases at $r > b^{cl}$; hence only the vicinity of b^{cl} contributes to the integral $\int \varphi_N(r) f(r) dr$ and
- $$\varphi_N(r) \xrightarrow[N\to\infty]{} A_N \delta(r - r_0\sqrt{2N+3}).$$
- Truncating potential matrix within HORSE at very large N is equivalent to *P*-matrix formalism with channel radius $b = r_0\sqrt{2N+7}$. If N is not extremely large, HORSE is a discrete analogue of the *P*-matrix formalism with a natural channel radius $b = r_0\sqrt{2N+7}$; the oscillator function φ_N differs essentially from the δ-function, but the matching to free solutions is defined not in the coordinate space but in the discrete space of oscillator functions that seems to be more natural for RGM, shell model and other approaches utilizing oscillator basis

  (see details in Bang, Mazur, AMS, Smirnov, Zaytsev, Ann. Phys. (NY), **280**, 299 (2000))

HORSE: charged particle scattering

J. M. Bang, A. I. Mazur, A. M. Shirokov, Yu. F. Smirnov and S. A. Zaytsev, Ann. Phys. (N.Y.) 280, 299 (2000

Auxiliary short-range potential

$$V^{sh} = \begin{cases} V^{nucl} + V^{Cl}, \; r \le R', & R' > R^{nucl} \\ 0, \; r \le R' \end{cases}$$

Within HORSE one can calculate phase shift δ_l^{sh} .

$$\tan\delta^{sh} = -\frac{S_{Nl} - G_{NN} T^l_{N,N+1} S_{N+1,l}}{C_{Nl} - G_{NN} T^l_{N,N+1} C_{N+1,l}}$$

Extension HORSE

$$\tan\delta_l = -\frac{W_{R'}(j_l, F_l) - W_{R'}(n_l, F_l)\tan\delta_l^{sh}}{W_{R'}(j_l, G_l) - W_{R'}(n_l, G_l)\tan\delta_l^{sh}}$$

Quasi-Wronskian

$$W_{R'}(j_l, F_l) = \left(\frac{d}{dr}[j_l(kr)] F_l(\eta, kr) - j_l(kr) \frac{d}{dr}[F_l(\eta, kr)] \right)\Bigg|_{r=R'}$$

$j_l(kr), \quad n_l(kr)$ — Bessel and Neumann functions;

$F_l(\eta, kr), \quad G_l(\eta, kr)$ — Coulomb functions;

$\eta = \dfrac{\mu Z_1 Z_2 e^2}{k}$ — Sommerfeld parameter.

Natural channel radius $R' = b^{cl} = r_0\sqrt{2N+7}$ is the optimal choice for R'.

HORSE applicability

- HORSE was successfully used within RGM
- HORSE was successfully used in various cluster models, e.g., ^{11}Li disintegration
- Coulomb interaction can be accounted for within HORSE
- Inverse scattering HORSE theory has been developed and used, e.g., for constructing JISP16 *NN* interaction
- However there are problems with a direct HORSE extension of modern shell model calculations

Problems with direct HORSE application to NCSM

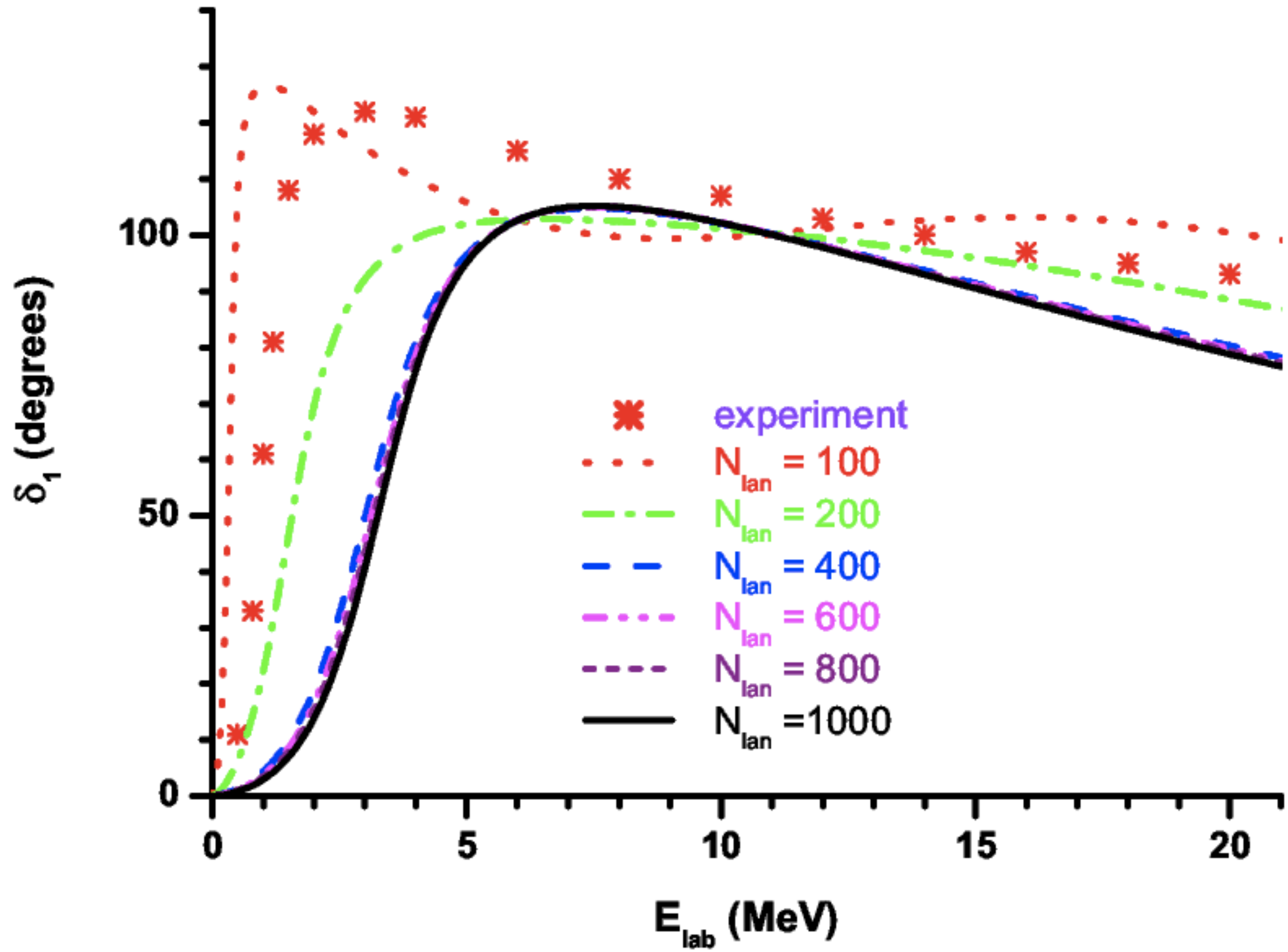
$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\,T^{l}_{N,\,N+1}\,S_{N+1,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\,T^{l}_{N,\,N+1}\,C_{N+1,l}(E)}$$

$$(H-E)^{-1}_{nn'} \equiv -\mathcal{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'}-E}$$

$$\sum_{n'=0}^{N} H^{l}_{nn'}\langle n'|\lambda\rangle = E_{\lambda}\langle n|\lambda\rangle, \qquad n \le N$$

- A lot of E_λ eigenstates needed while SM codes usually calculate few lowest states only
- One needs highly excited states and to get rid from CM excited states.
- $\langle n'|\lambda\rangle$ are normalized for all states including the CM excited ones, hence renormalization is needed.
- We need $\langle n'|\lambda\rangle$ for the relative *n*-nucleus coordinate r_{nA} but NCSM provides $\langle n'|\lambda\rangle$ for the *n* coordinate r_n relative to the nucleus CM. Hence we need to perform Talmi-Moshinsky transformations for all states to obtain $\langle n'|\lambda\rangle$ in relative *n*-nucleus coordinates.
- Concluding, the direct application of the HORSE formalism in *n*-nucleus scattering is unpractical.

Example: $n\alpha$ scattering



Single-state HORSE (SS-HORSE)

$$\sum_{n'=0}^{N} H^{l}_{nn'} \langle n'|\lambda\rangle = E_\lambda \langle n|\lambda\rangle, \qquad n \le N$$

$$(H-E)^{-1}_{nn'} \equiv -\mathscr{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'}-E}$$

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathscr{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathscr{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,l}(E)}$$

Suppose $E = E_\lambda$:

$$\tan\delta(E_\lambda) = \frac{S_{N+1,l}(E_\lambda)}{C_{N+1,l}(E_\lambda)}$$

Calculating a set of E_λ eigenstates with different $\hbar\Omega$ and N_{max} within SM, we obtain a set of $\delta(E_\lambda)$ values which we can approximate by a smooth curve at low energies.

Single-state HORSE (SS-HORSE)

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Suppose $E = E_\lambda$:

$$\tan\delta(E_\lambda) = \frac{S_{N+1,l}(E_\lambda)}{C_{N+1,l}(E_\lambda)}$$

Note, information about wave function disappeared in this formula, any channel can be treated

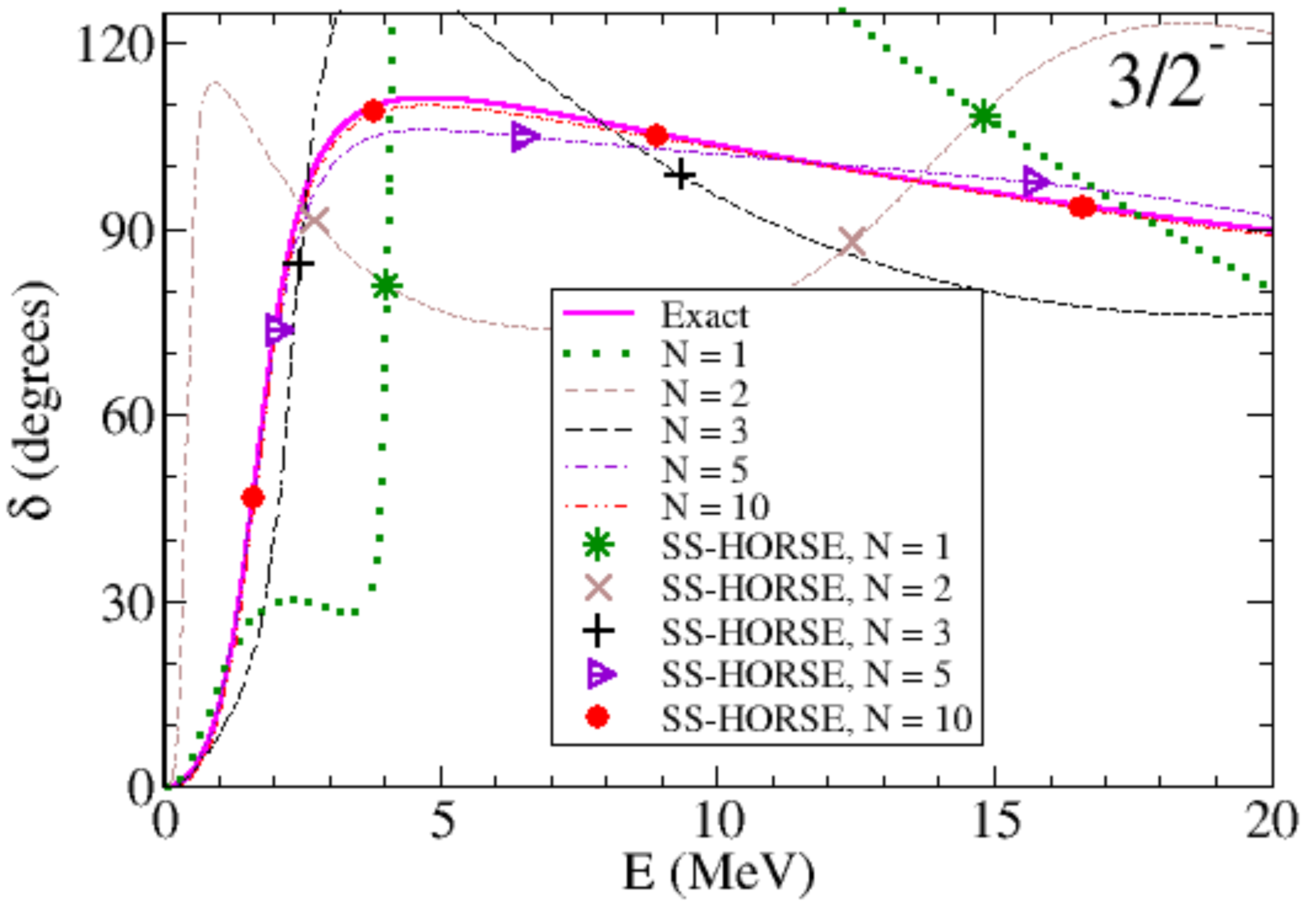
Calculating a set of E_λ eigenstates with different $\hbar\Omega$ and N_{max} within SM, we obtain a set of $\delta(E_\lambda)$ values which we can approximate by a smooth curve at low energies.

Convergence: model problem

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,l}(E)}$$

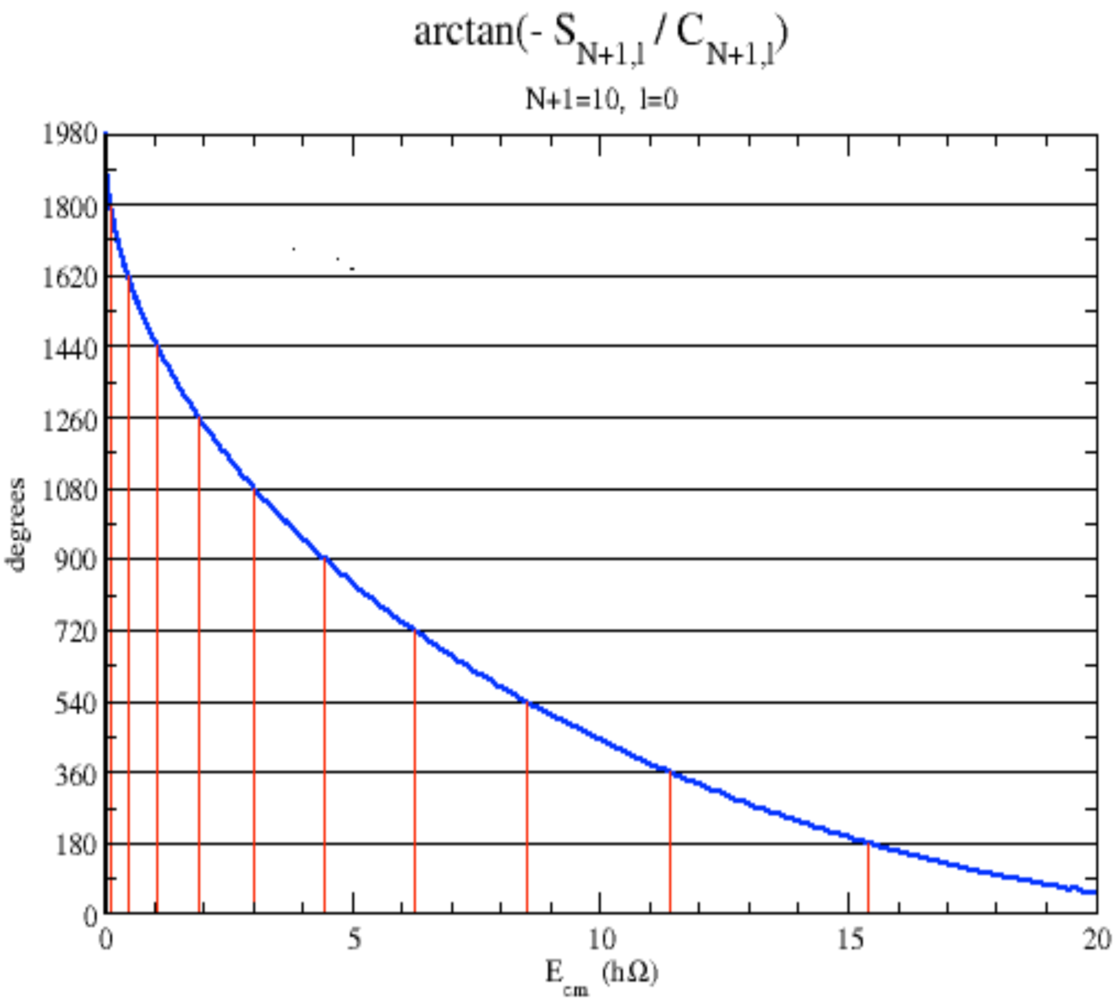
$$(H-E)^{-1}_{nn'} \equiv -\mathcal{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'} - E}$$

$$\sum_{n'=0}^{N} H^{l}_{nn'}\langle n'|\lambda\rangle = E_{\lambda}\langle n|\lambda\rangle, \qquad n \leq N$$



Universal function

$$f_{nl}(E) = \arctan\left(-\frac{S_{nl}(E)}{C_{nl}(E)}\right)$$



$$f_{nl}(E) = \arctan\left(-\frac{S_{nl}(E)}{C_{nl}(E)}\right)$$ scaling property

Limit $n \to \infty$: $n \gg \sqrt{\frac{2E}{\hbar\Omega}}$

$$S_{nl}(q) \approx q\sqrt{r_0}\,(n+l/2+3/4)^{\frac{1}{4}}\, j_l(2q\sqrt{n+l/2+3/4})$$
$$\approx \sqrt{r_0}\,(n+l/2+3/4)^{-\frac{1}{4}}\,\sin[2q\,\sqrt{n+l/2+3/4}-\pi l/2]$$

$$C_{nl}(q) \approx -q\sqrt{r_0}\,(n+l/2+3/4)^{\frac{1}{4}}\, n_l(2q\sqrt{n+l/2+3/4})$$
$$\approx \sqrt{r_0}\,(n+l/2+3/4)^{-\frac{1}{4}}\,\cos[2q\,\sqrt{n+l/2+3/4}-\pi l/2]$$

$$q = \sqrt{\frac{2E}{\hbar\Omega}}$$

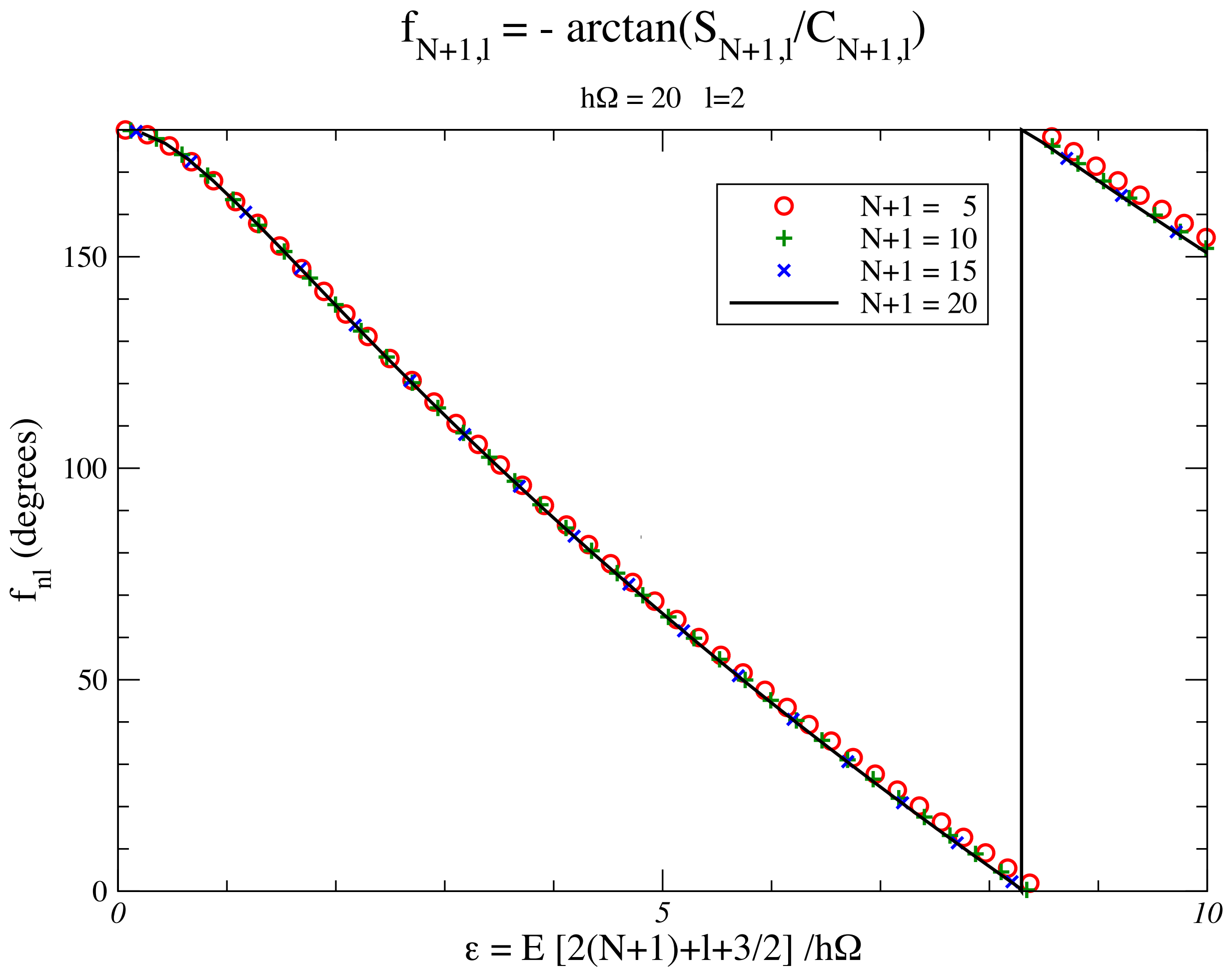
So $f_{n+1,l}(E) = \arctan\left(\frac{S_{n+1,l}(E)}{C_{n+1,l}(E)}\right)$ is a function of $\varepsilon = \frac{E_{cm}}{s}$,

where scaling parameter $s = \frac{\hbar\Omega}{2n+l+{}^{7}/_{2}} = \frac{\hbar\Omega}{N+{}^{7}/_{2}}$

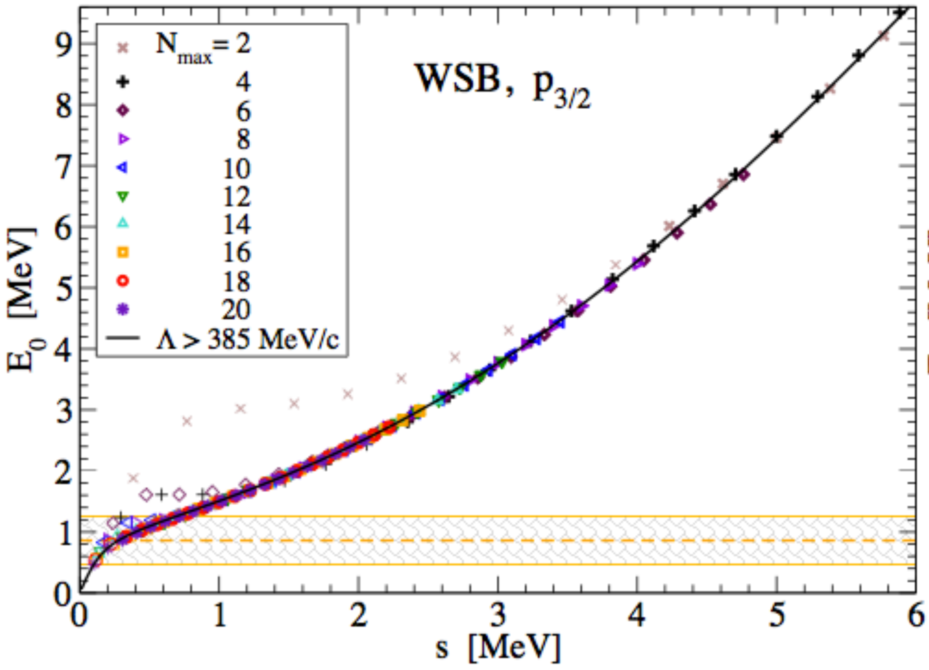
Universal function scaling

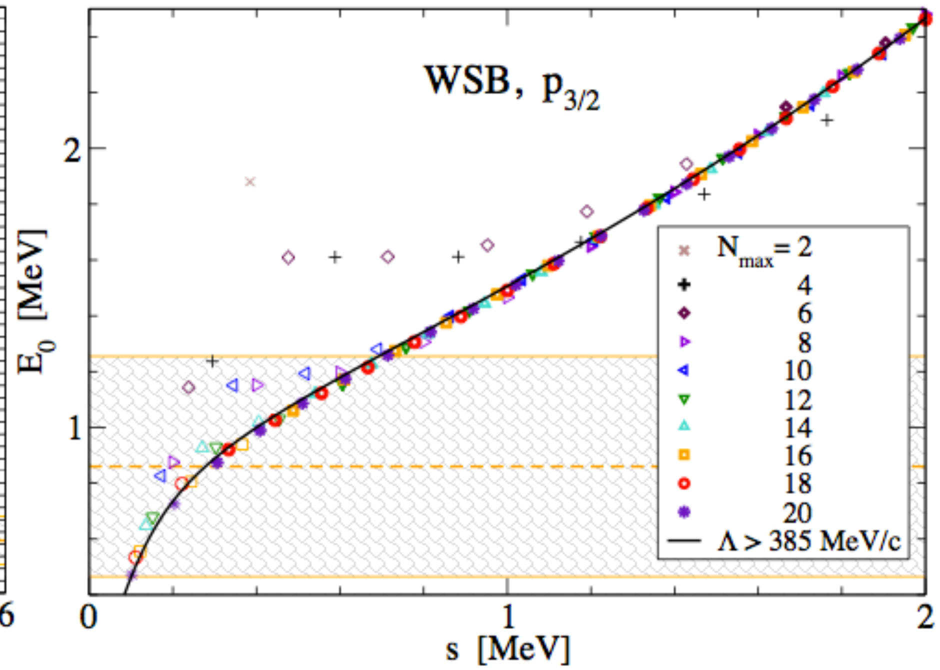
$$E_{cm}(\text{MeV}) \Rightarrow \varepsilon = \frac{E_{cm}[2(N+1)+l+3/2]}{\hbar\Omega}$$

l=2



Eigenstate behavior in the presence of resonance





$$s = \frac{\hbar\Omega}{2n + l + {}^{7}/_{2}} = \frac{\hbar\Omega}{N + {}^{7}/_{2}}$$

$$\tan\delta(E_\lambda) = \frac{S_{N+1,l}(E_\lambda)}{C_{N+1,l}(E_\lambda)}$$

S-matrix at low energies

Symmetry property:
$$S(-k) = \frac{1}{S(k)}$$
$$S(k) = \exp 2i\delta$$

Hence $\delta(-k) = -\delta(k), \qquad k \sim \sqrt{E},$

$$\delta \simeq C\sqrt{E} + D(\sqrt{E})^3 + F(\sqrt{E})^5 + \ldots$$

As $k \to 0: \quad \delta_\ell \sim k^{2\ell+1} \sim (\sqrt{E})^{2\ell+1}$

Bound state: $S_b^{(i)}(k) = \dfrac{k + ik_b^{(i)}}{k - ik_b^{(i)}},$

$$\delta_0 \simeq \pi - \arctan\sqrt{\frac{E}{|E_b|}} + c\sqrt{E} + d(\sqrt{E})^3 + f(\sqrt{E})^5 \ldots$$

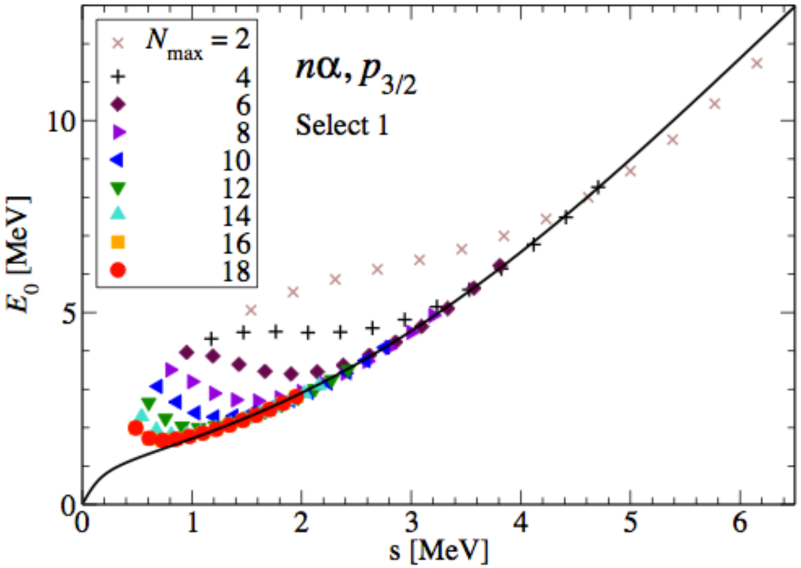
Resonance: $S_r^{(i)}(k) = \dfrac{(k + \kappa_r^{(i)})(k - \kappa_r^{(i)*})}{(k - \kappa_r^{(i)})(k + \kappa_r^{(i)*})}$

$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E - b^2} + c\sqrt{E} + d(\sqrt{E})^3 + \ldots, \qquad c = -\frac{a}{b^2}.$$

How it works

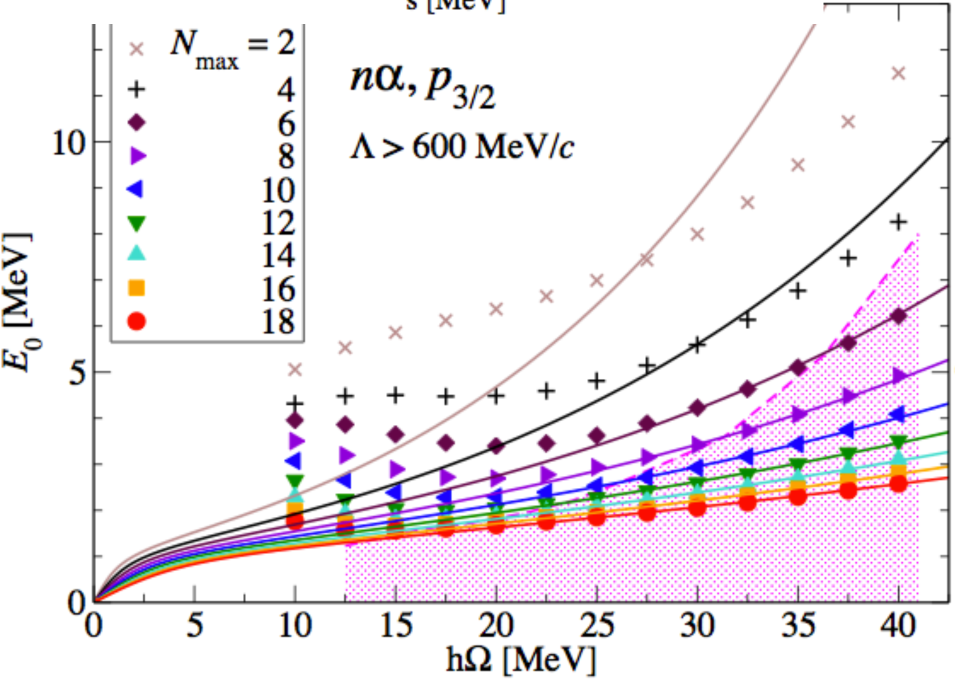
$n\alpha$ scattering: NCSM, JISP16

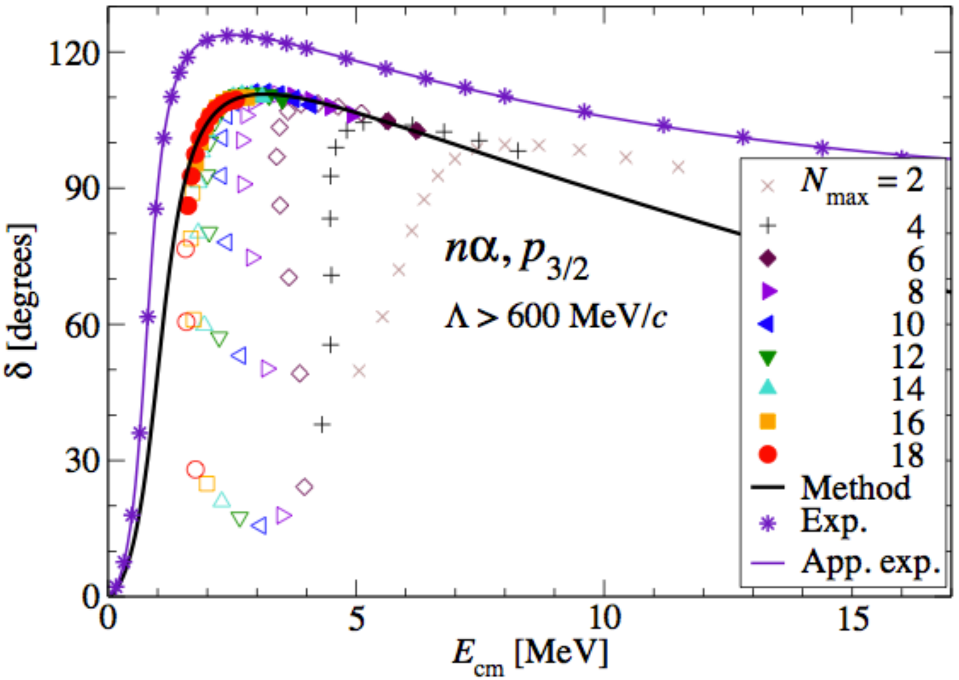
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



$$s = \frac{\hbar\Omega}{(N_{\max} + 2 + \ell + 3/2)}.$$

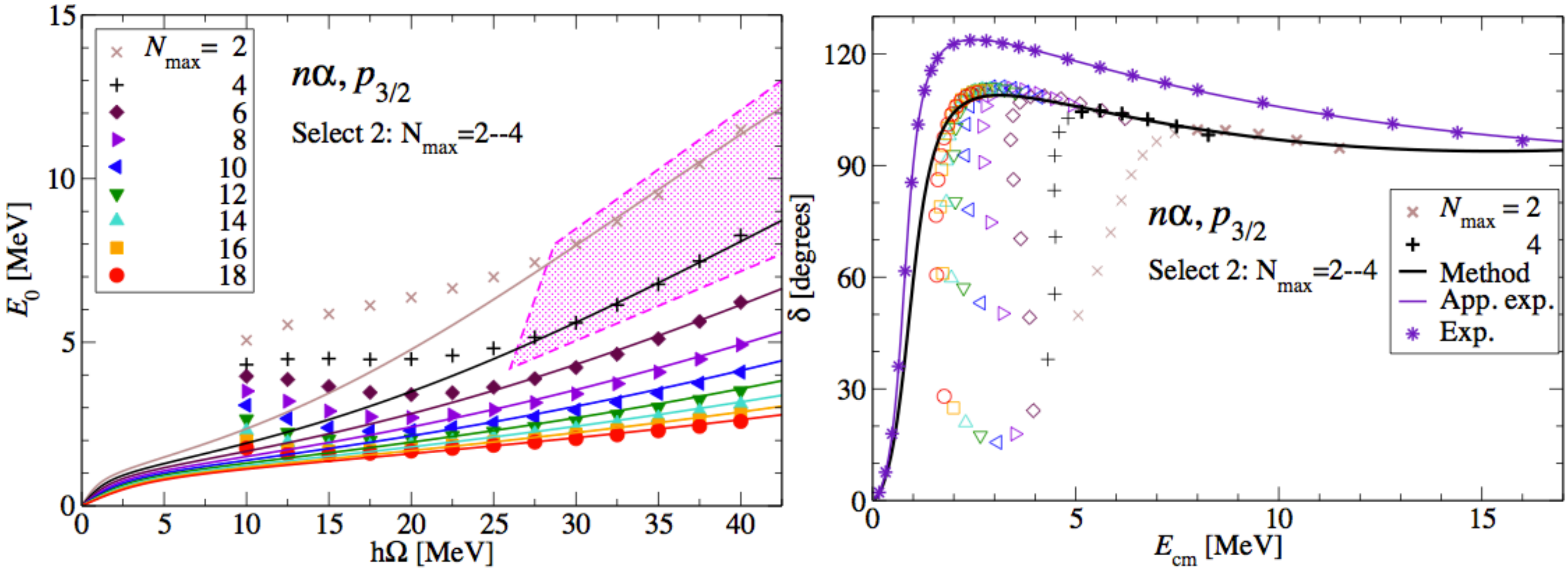
$$\tan\delta(E_\lambda) = \frac{S_{N+1,l}(E_\lambda)}{C_{N+1,l}(E_\lambda)}$$





*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



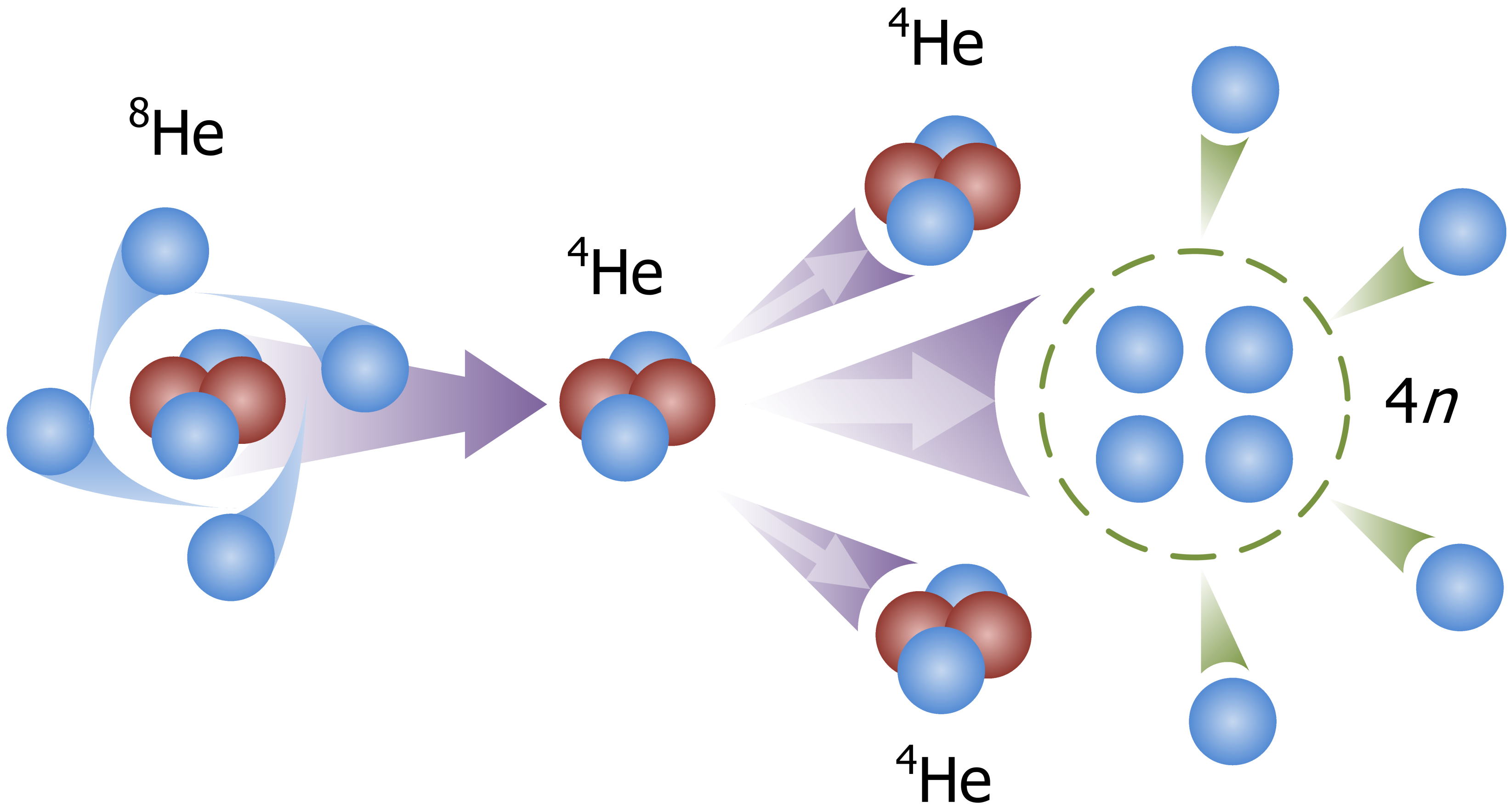
*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$

$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E-b^2} + c\sqrt{E} + d(\sqrt{E})^3 + \ldots, \quad c = -\frac{a}{b^2}.$$

$3/2^-$, n–^{4}He	a, MeV$^{\frac{1}{2}}$	b^2, MeV	d, MeV$^{-\frac{3}{2}}$	E_r, MeV	Γ, MeV	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 600$, $N_{max} = 6 \div 18$	0.505	1.135	−0.00009	1.008	1.046	0.031	46
Select 1	0.506	1.054	+0.00647	0.926	1.008	0.053	63
Select 1 + ($N_{max} = 2$)	0.506	1.019	+0.00932	0.891	0.989	0.070	68
Select 2: $N_{max} = 2 \div 4$	0.515	1.025	+0.0101	0.892	1.008	0.106	11
Select 2: $N_{max} = 2 \div 6$	0.512	1.022	+0.00988	0.891	1.002	0.097	18
Select 3: $N_{max} = 12$	0.469	1.307	−0.0265	1.197	1.050	0.011	8
R-matrix [3]				0.80	0.65		
J-matrix [4]				0.772	0.644		
Fit with exp. data	0.358	0.839	+0.00559	0.774	0.643	0.21 [deg]	26

Tetraneutron



Experiment: K. Kisamori et al., Phys. Rev. Lett. 116, 052501 (2016):
$E_R = 0.83 \pm 0.63$(statistical) $\mp$ 1.25(systematic) MeV; width $\Gamma \leq 2.6$ MeV

Tetraneutron

- Democratic decay (no bound subsystems)
- Hyperspherical harmonics:

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_A) = \Phi(\rho)\mathcal{Y}_{k\nu}(\Omega), \quad \rho = \sqrt{\sum_{i=1}^{A}(\boldsymbol{r}_i - \boldsymbol{R})^2},$$

$$\Phi_{nK} \equiv \Phi_n^{\mathcal{L}}(\rho) = \rho^{-(3A-4)/2}\varphi_{nK}(\rho), \quad \mathcal{L} = K + \frac{3A-6}{2};$$

$$\frac{\hbar^2}{2m}\left[-\frac{d^2}{d^2\rho} + \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2}\right]\Phi_n^{\mathcal{L}}(\rho) + \sum_{\mathcal{L}'} V_{\mathcal{L},\mathcal{L}'}\Phi_n^{\mathcal{L}'}(\rho) = E\Phi_n^{\mathcal{L}}(\rho).$$

Approximation:

the only open channel is with $\mathcal{L} = \mathcal{L}_{\min} = K_{\min} + 3 = 5$.

All possible $\mathcal{L}$ (K) values are accounted for in diagonalization of the NCSM Hamiltonian

Tetraneutron

S-matrix: $S = \exp 2i\delta^{\mathcal{L}}$

$$\delta^{\mathcal{L}} = C\sqrt{E} + D(\sqrt{E})^3 + F(\sqrt{E})^5 + \ldots$$

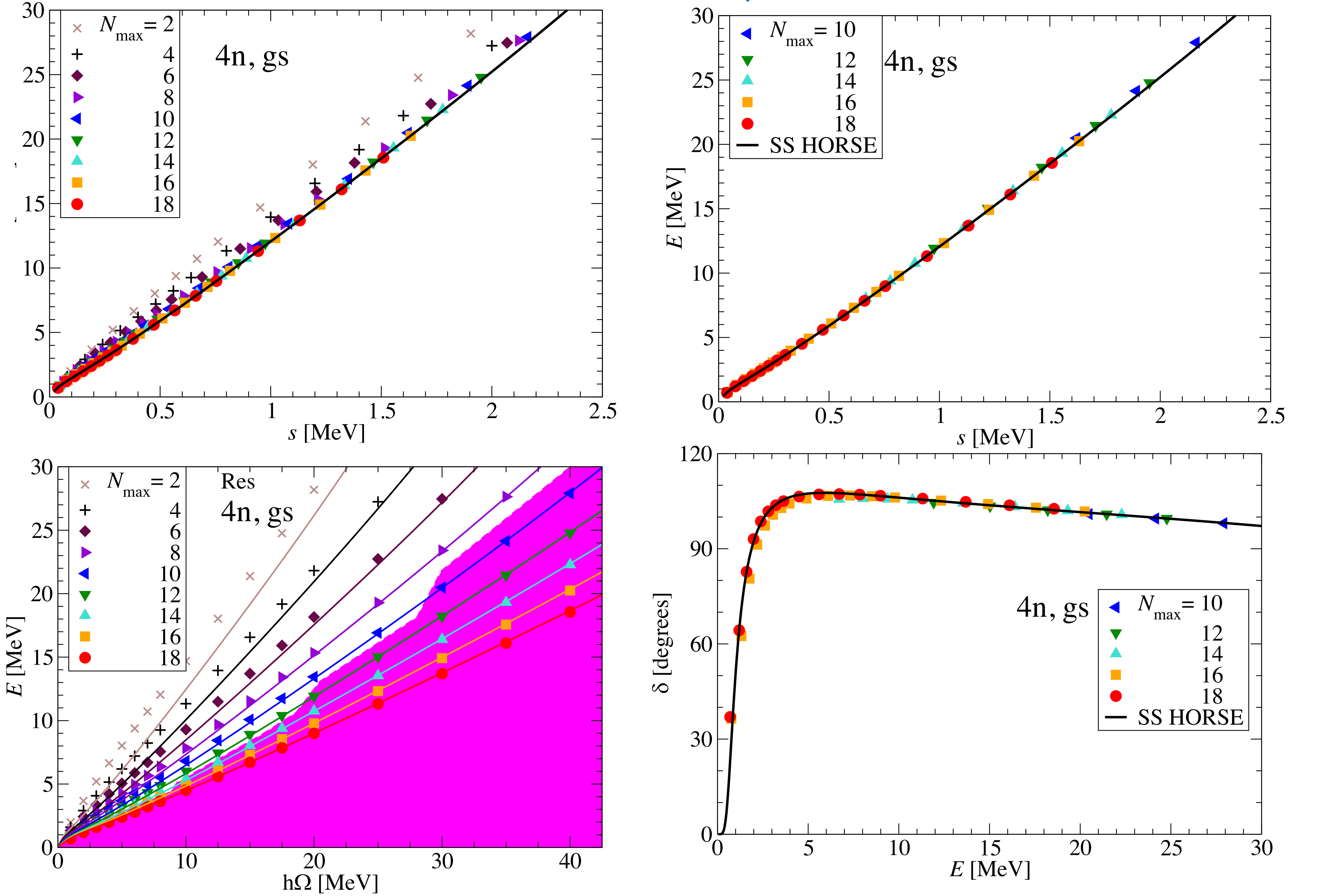
As $E \to 0$: $\delta^{\mathcal{L}} \sim (\sqrt{E})^{2\mathcal{L}+1} \sim (\sqrt{E})^{11}$ – huge power!

$$\delta = -\arctan \frac{a\sqrt{E}}{E - b^2} - \phi_{3,6}(E),$$

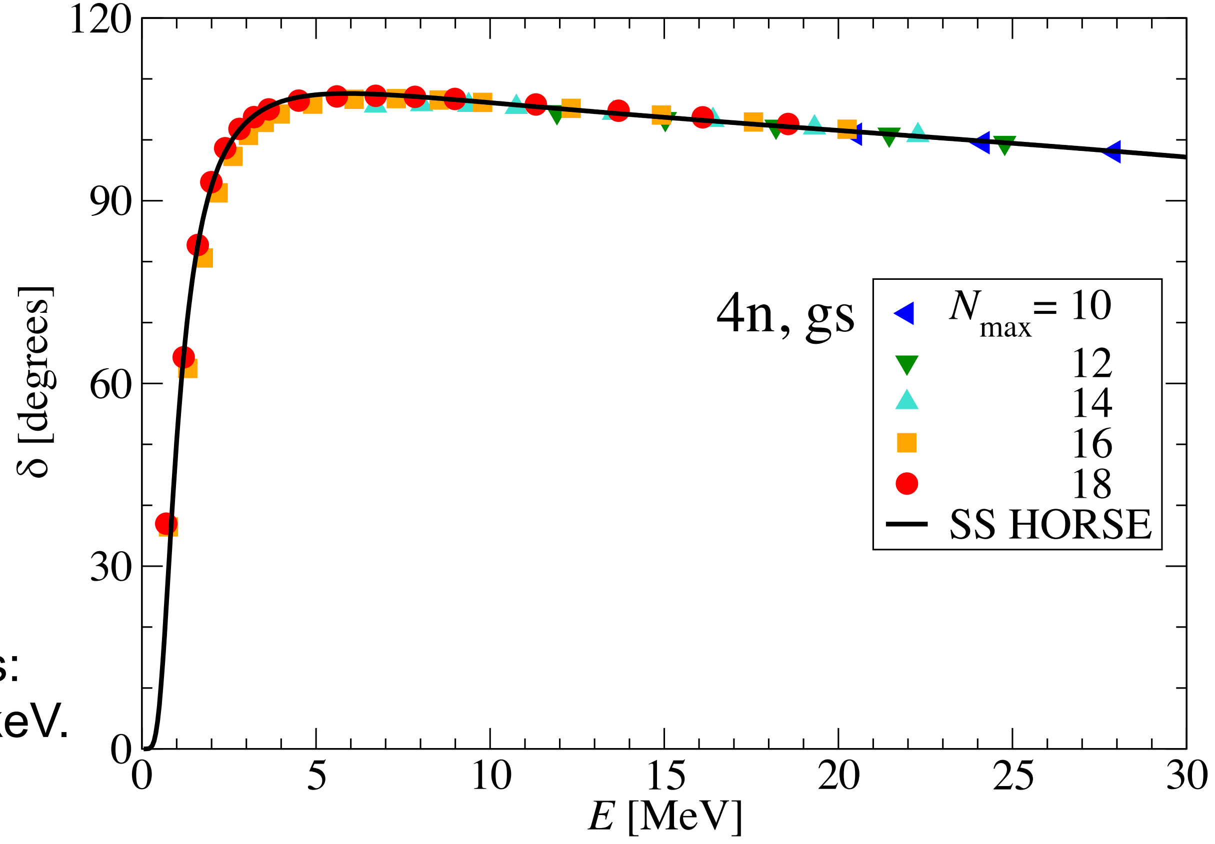
$$\phi_{3,6}(E) = \frac{w_1\sqrt{E} + w_3\left(\sqrt{E}\right)^3 + c\left(\sqrt{E}\right)^5}{1 + w_2 E + w_4 E^2 + w_6 E^3 + dE^4},$$

$$\phi_{3,6}(E) = M_9\left(\sqrt{E}\right) + O\left(\left(\sqrt{E}\right)^{11}\right).$$

Tetraneutron, JISP16



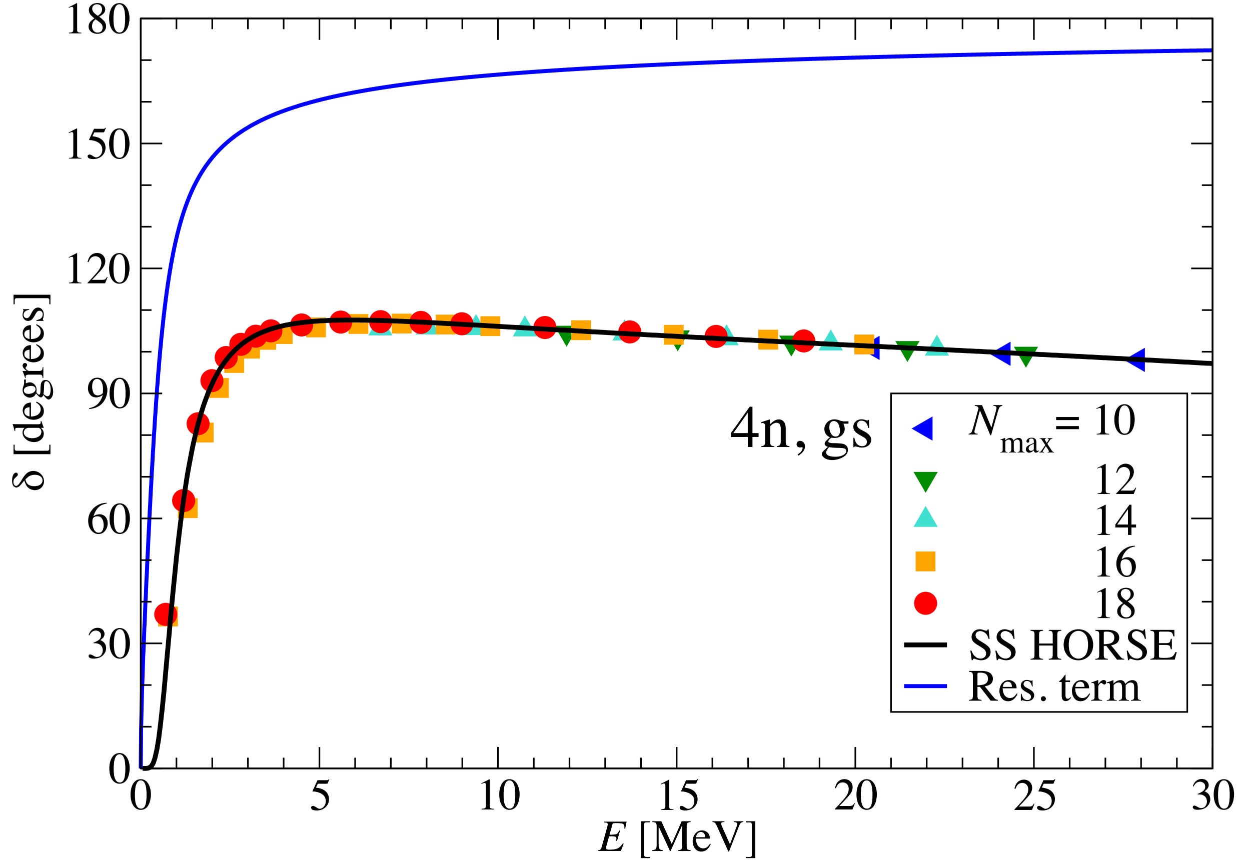
Tetraneutron, JISP16



Resonance parameters:
E_r = 186 keV, Γ = 815 keV.

A resonance around E_r = 850 keV with width around Γ = 1.3 MeV is expected!

Tetraneutron, JISP16



Resonance parameters:
E_r = 186 keV, Γ = 815 keV.

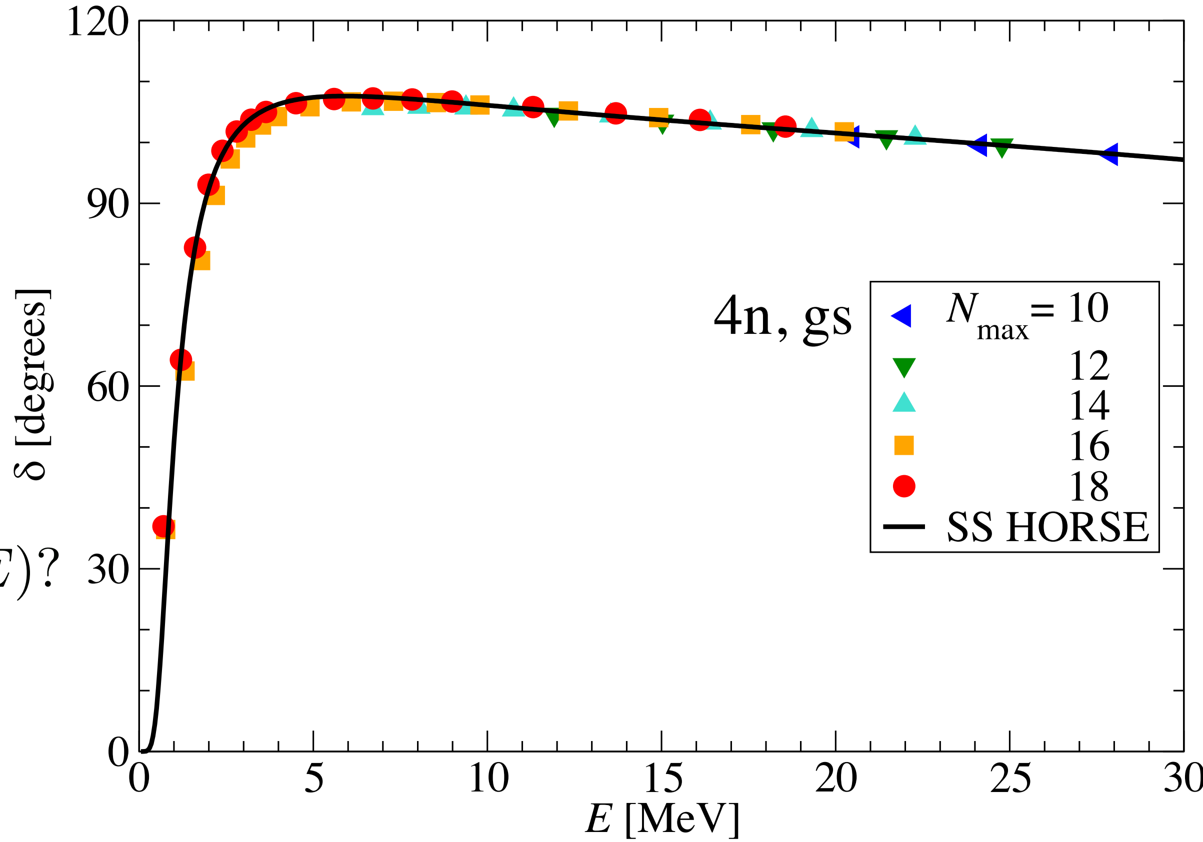
A resonance around E_r = 850 keV with width around Γ = 1.3 MeV is expected!

Can it be a virtual state? No.

Tetraneutron, JISP16

Can it be a combination of a false pole and resonant pole:

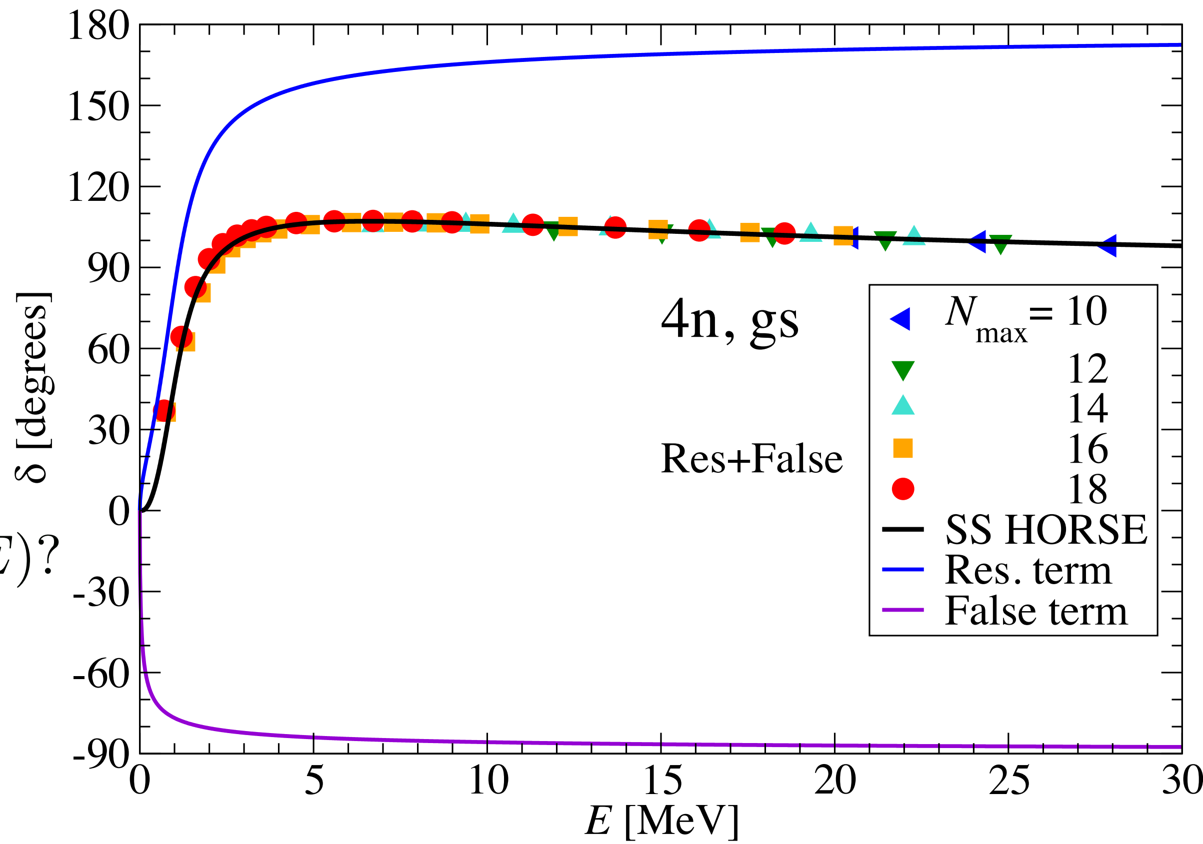
$$\delta = -\arctan\frac{a\sqrt{E}}{E-b^2} - \arctan\sqrt{\frac{E}{|E_f|}} - \phi_{3,6}(E)?$$



Tetraneutron, JISP16

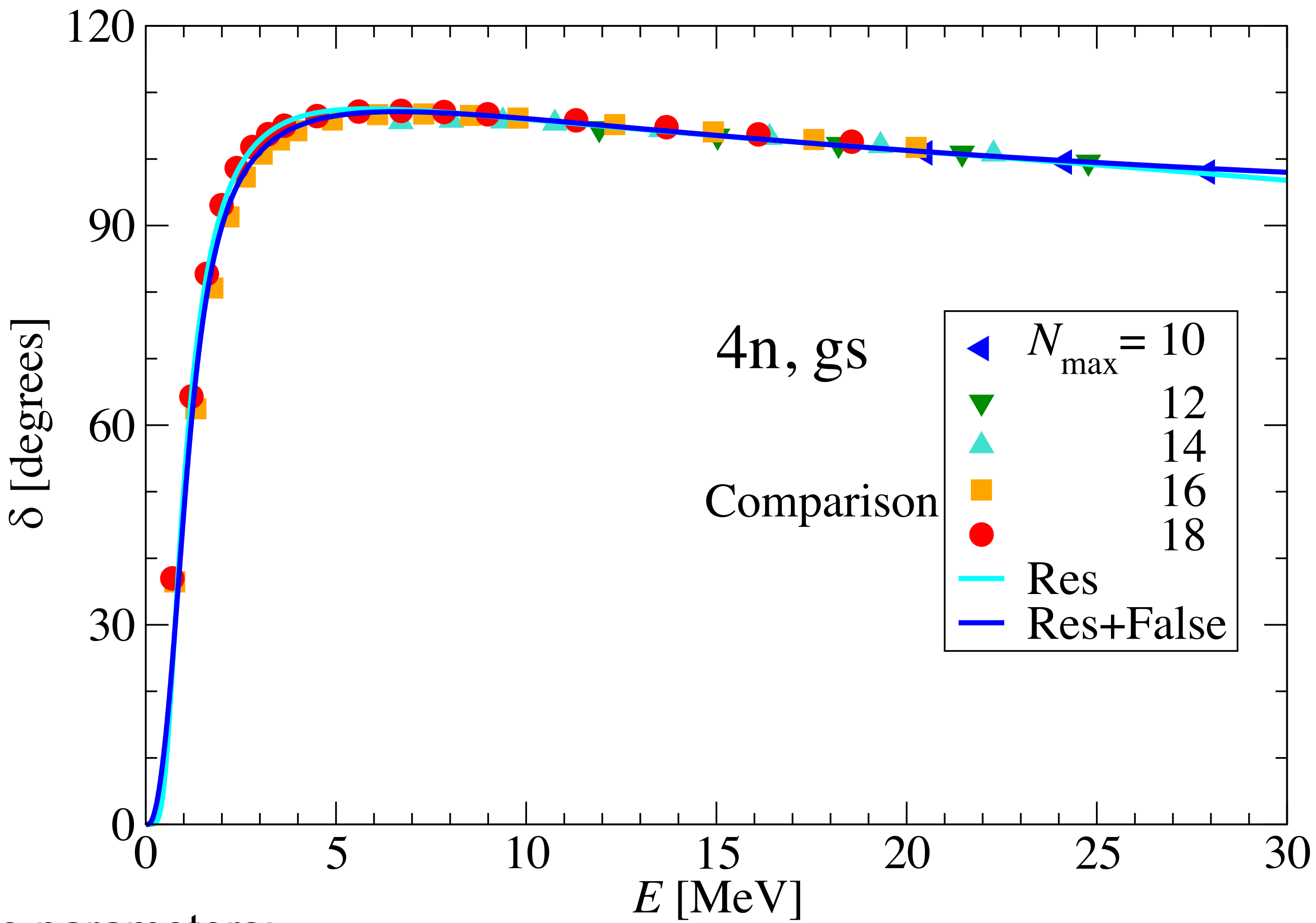
Can it be a combination of a false pole and resonant pole:

$$\delta = -\arctan \frac{a\sqrt{E}}{E - b^2} - \arctan \sqrt{\frac{E}{|E_f|}} - \phi_{3,6}(E)?$$



Yes!
Resonance parameters:
E_r = 844 keV, Γ = 1.378 MeV,
E_{false} = -55 keV.

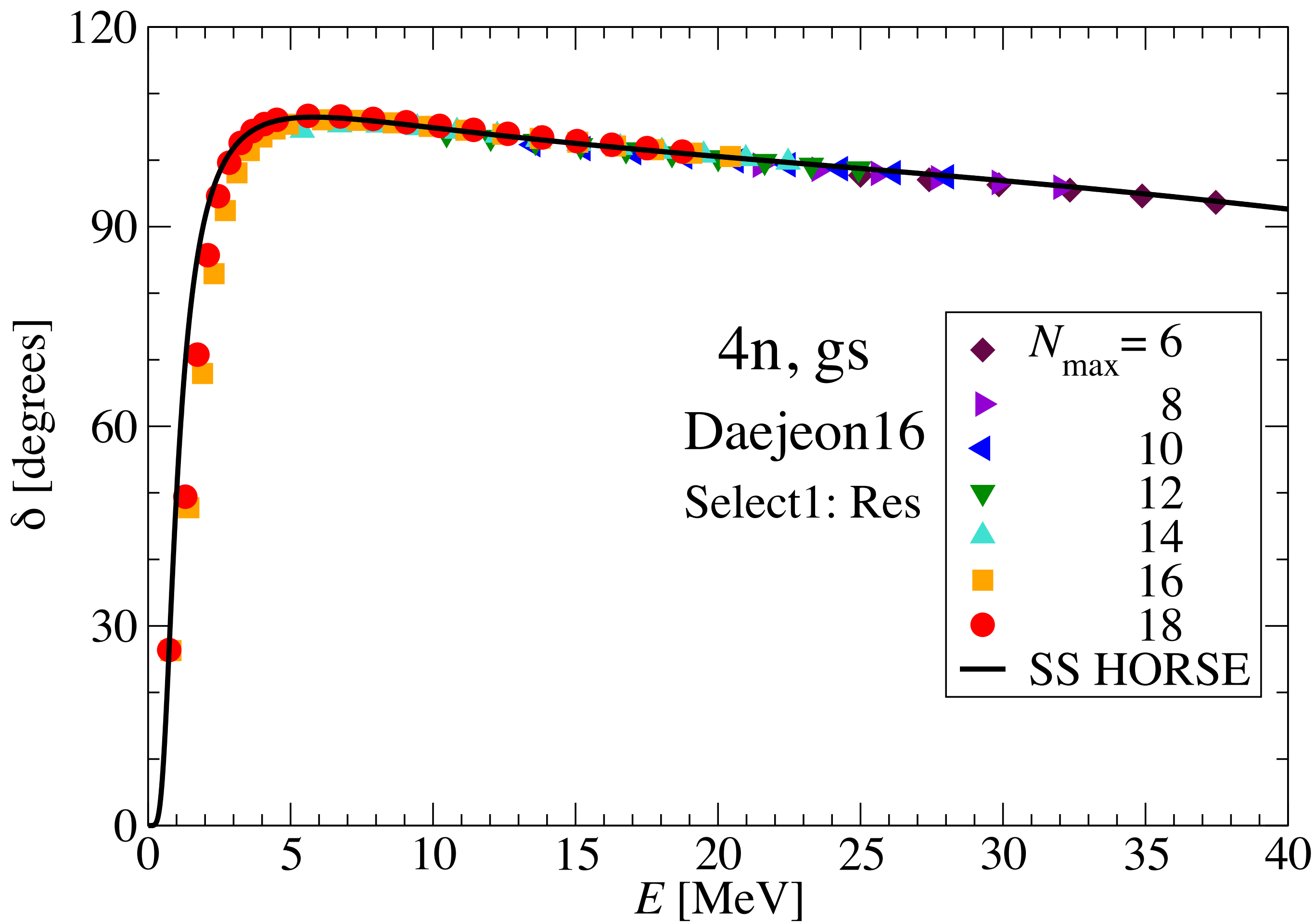
Tetraneutron, JISP16



Options:
Resonance parameters:
E_r = 844 keV, Γ = 1.378 MeV,
E_{false} = -55 keV.

Or
E_r = 186 keV, Γ = 815 keV ???

Tetraneutron, Daejeon16



.

Resonance parameters:

E_r = 0.997 MeV, Γ = 1.60 MeV,

E_{false} = -63.4 keV.

Similar results with SRG-evolved Idaho N3LO

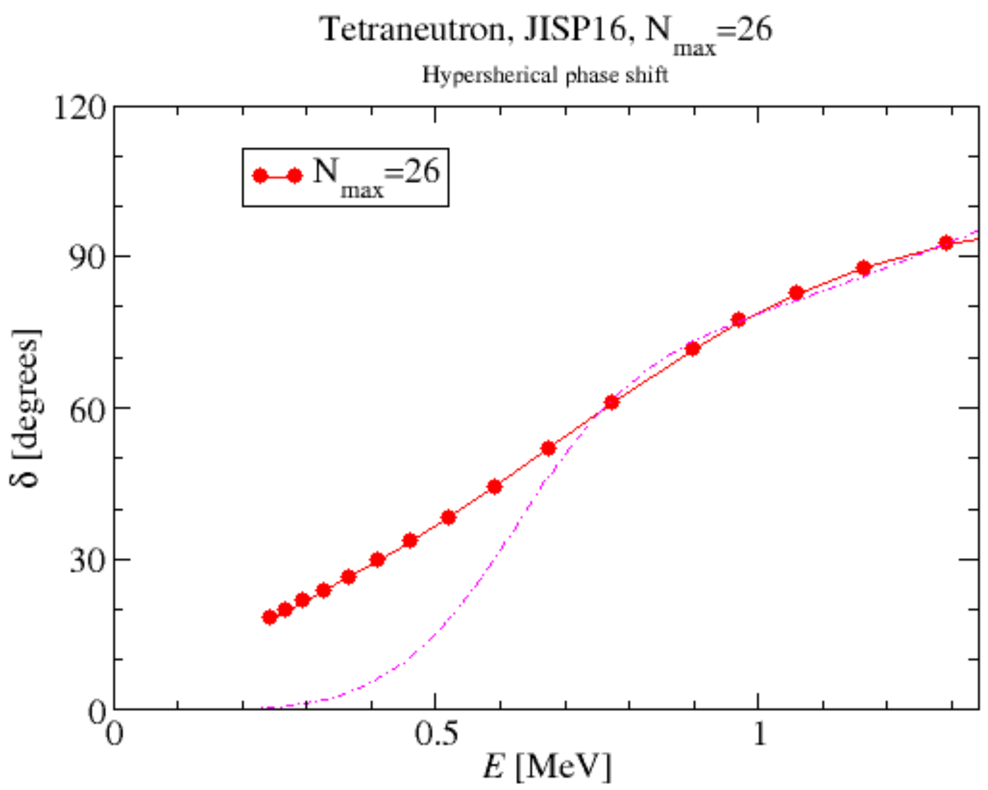
The 2018 development

Larger model spaces (up to $N_{max} = 26$) and smaller $\hbar\Omega$ values:
We get phase shifts at smaller energies and find that it is impossible to fit $\delta \sim k^{11}$ at low energies

Origin:
Hyperspherical potentials are long-ranged: $V \sim \rho^{-3}$ for 3 bodies, for 4 bodies?

Such a slow decrease of the interaction spoils the phase shifts at low energies

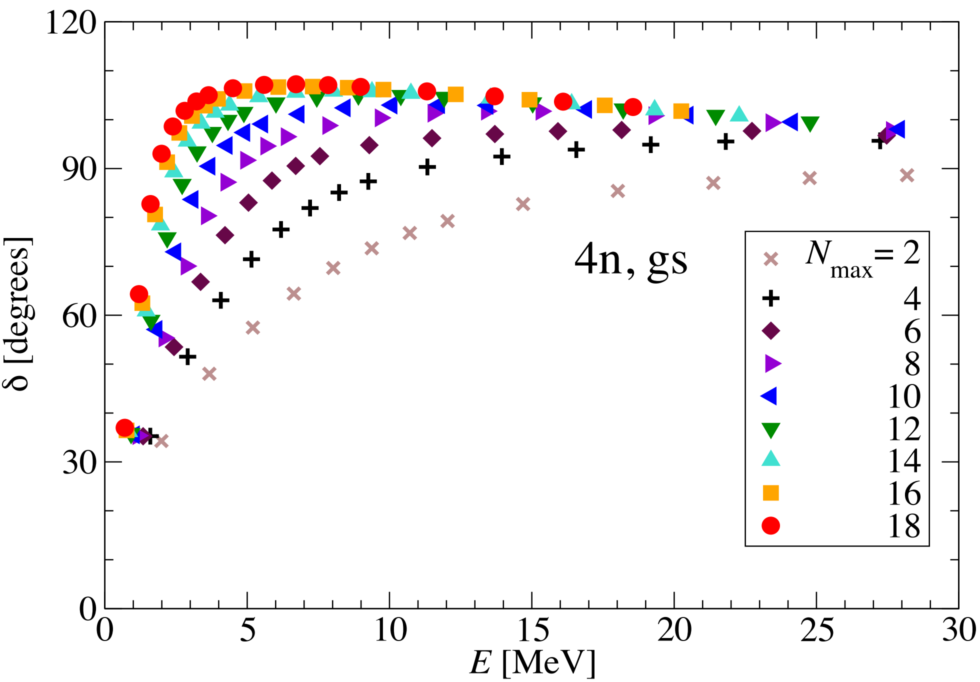
The long-range $V \sim \rho^{-3}$ (?) behavior of hyperspherical potentials spoils the phase shifts at low energies and results in convergence problems at large $N_{\max}$

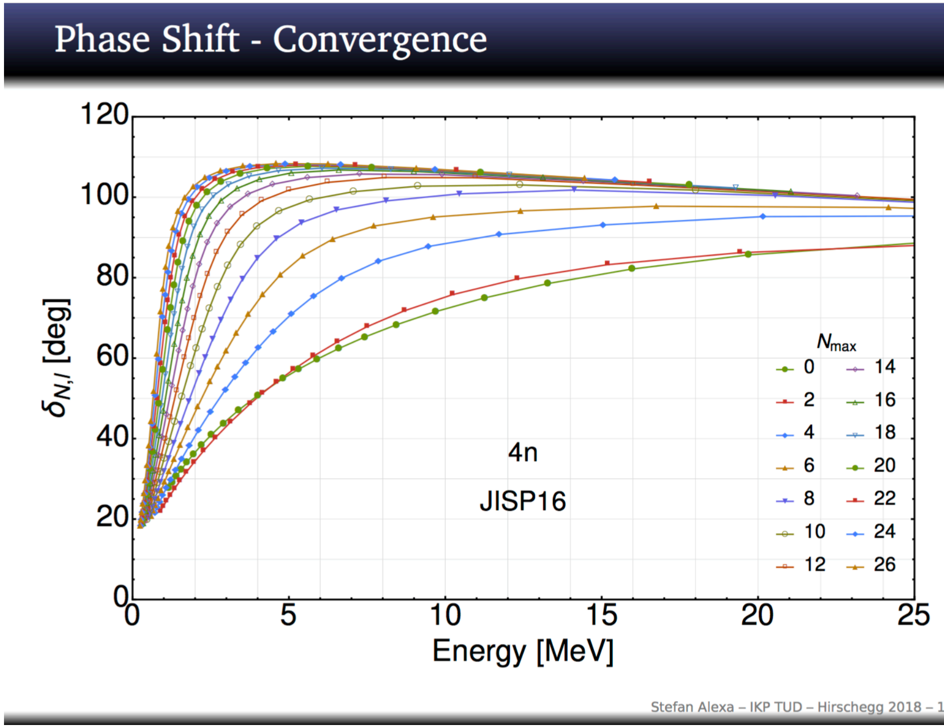


The 2018 results with JISP16

Before 2018: convergence seems to be achieved at $N_{\max} \leq 18$

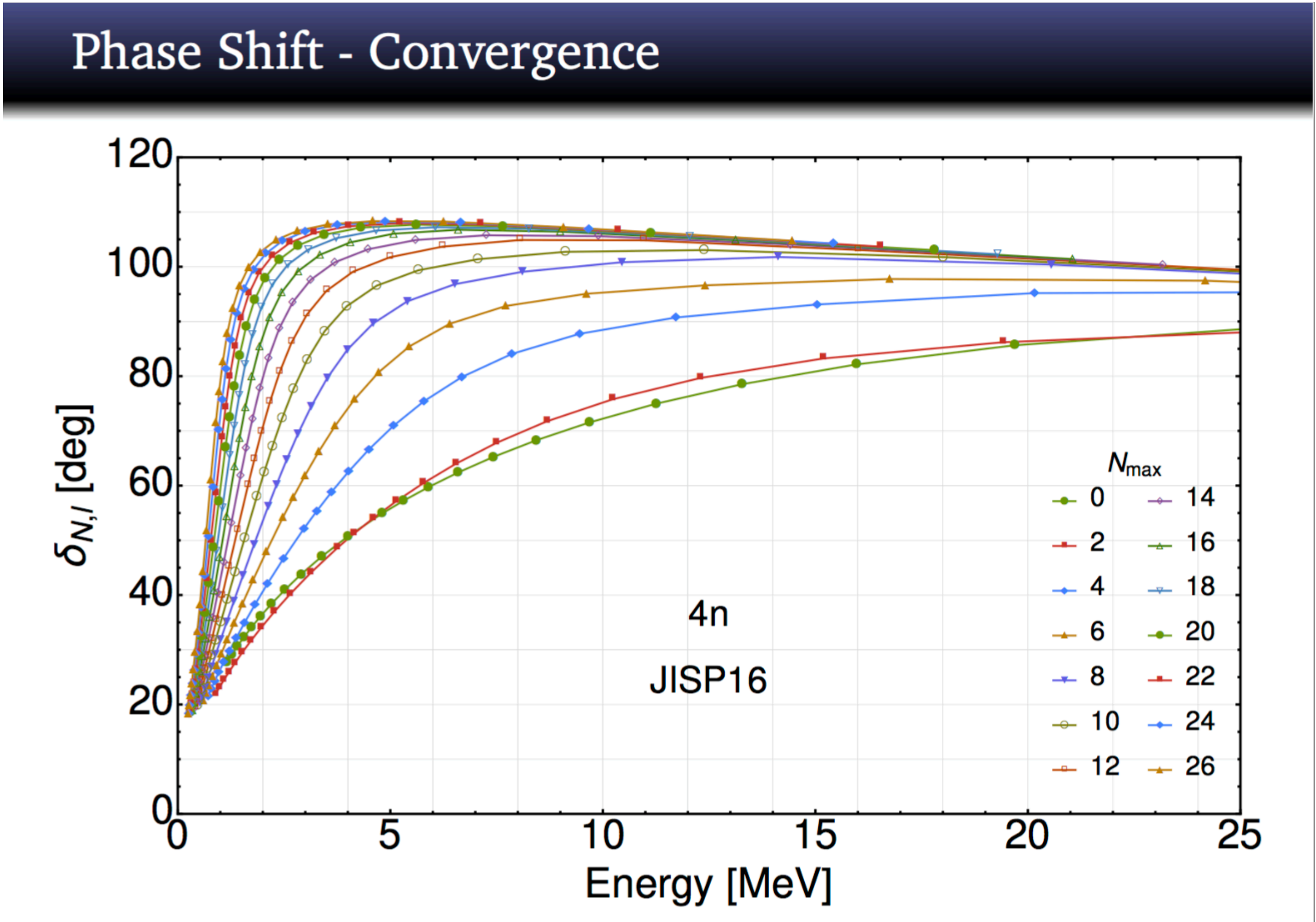
At 2018: convergence seems to be not achieved when larger $N_{\max}$ were calculated





The 2018 results with JISP16

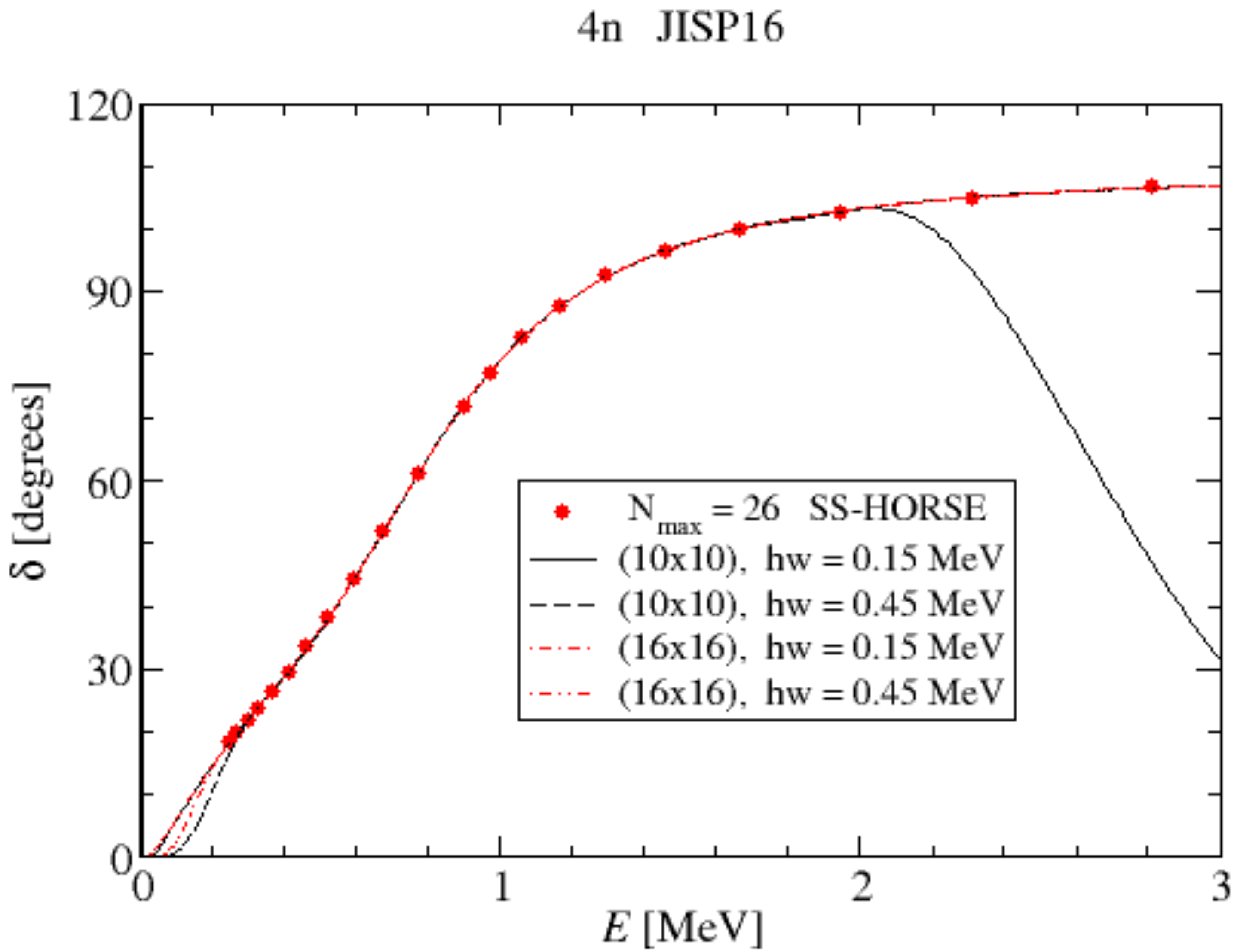
At 2018: however, the convergence seems to be achieved at the smallest energies



The 2018 development

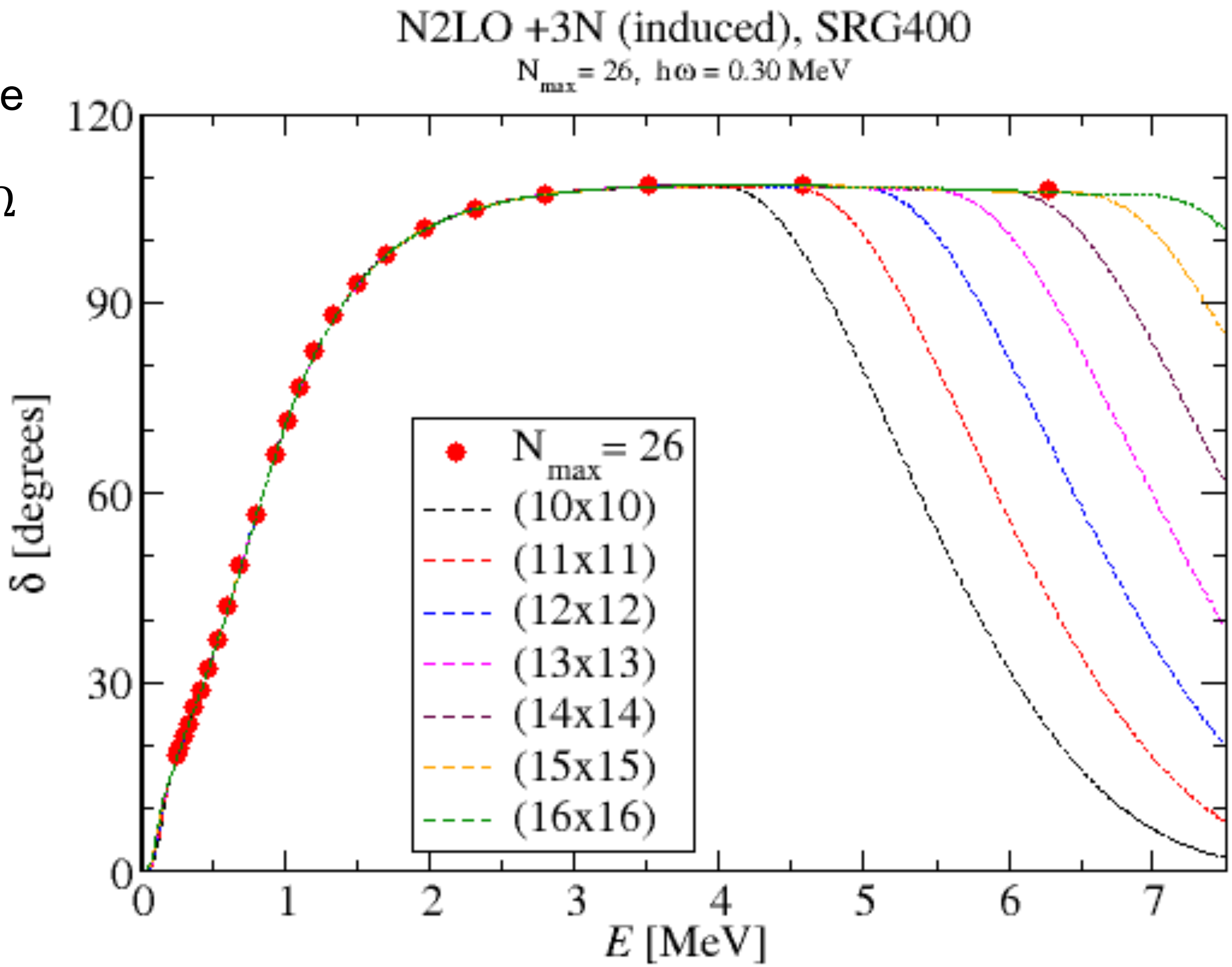
- To resolve this problem we use the *J*-matrix inverse scattering approach (S. A. Zaytsev, Theor. Math. Phys. **115**, 575 (1998); AMS *et al*, PRC **70**, 044005 (2004); PRC **79**, 014610 (2009)); i.e., we construct an interaction as a finite tridiagonal matrix in the oscillator basis describing our SS-HORSE hyperspherical phase shifts obtained with some $N_{\max}$ value and search numerically for the *S*-matrix poles.
- Ideally we need to construct the infinite potential matrix to guarantee the description of the long-range ρ^{-3} interaction tail, but ...
- So, we construct a set of interaction matrices of increasing rank *N*, obtain the poles and extrapolate the resonant energies and widths supposing their exponential convergence with *N.*

The 2018 results: inverse scattering phase shifts

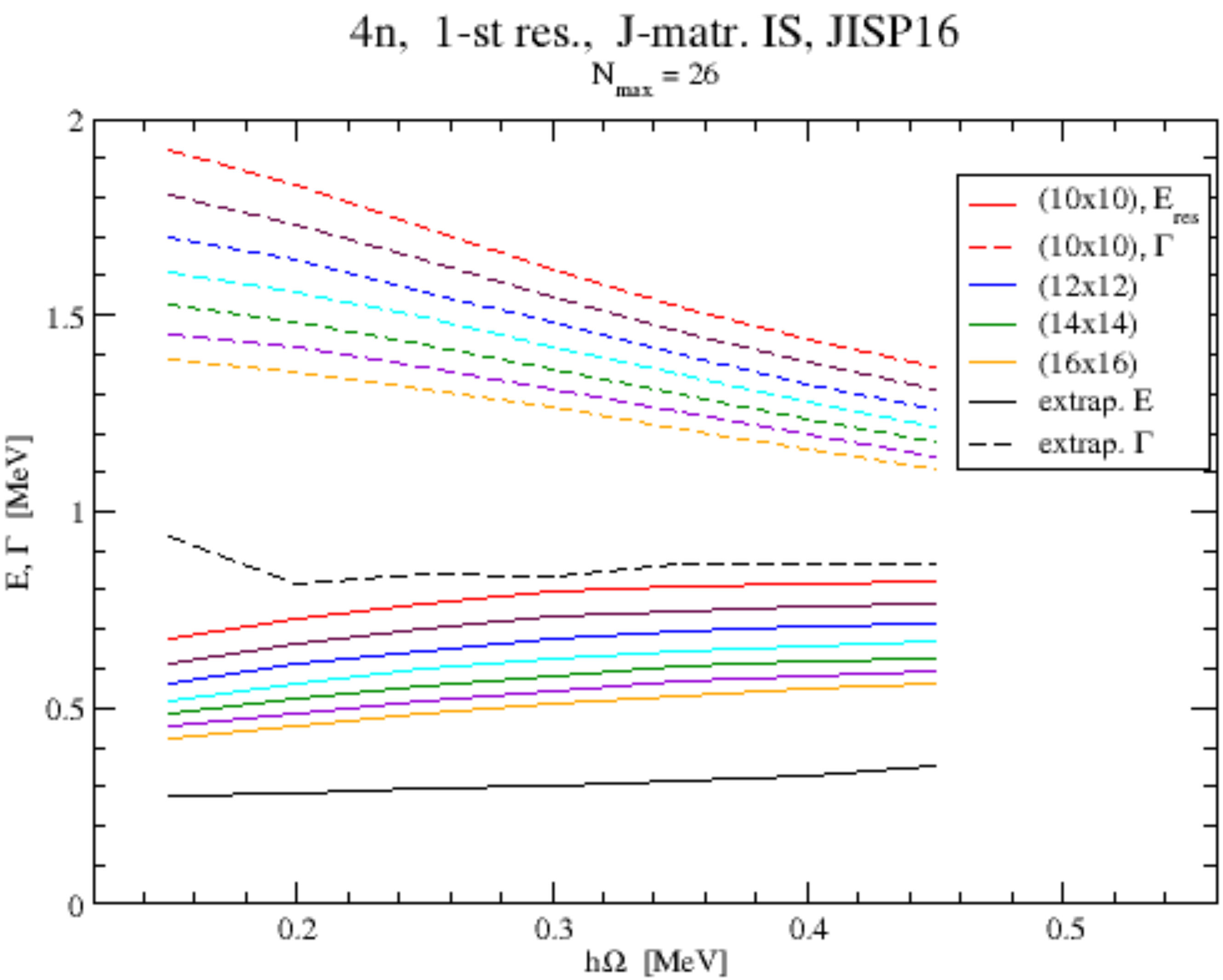


The 2018 results: inverse scattering phase shifts

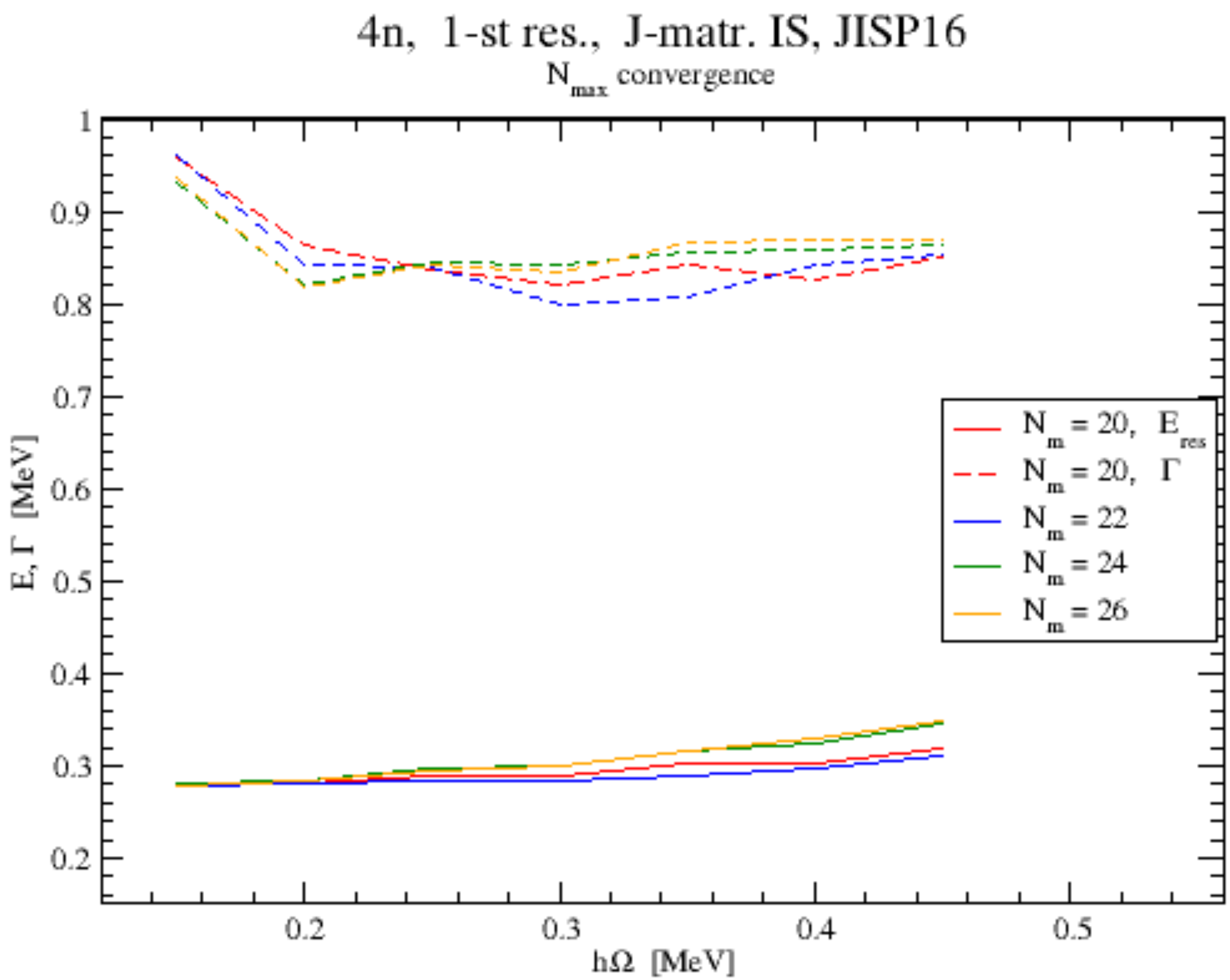
With larger matrix of the inverse scattering potential (and larger $\hbar\Omega$ value) we describe phase shifts in a larger energy interval



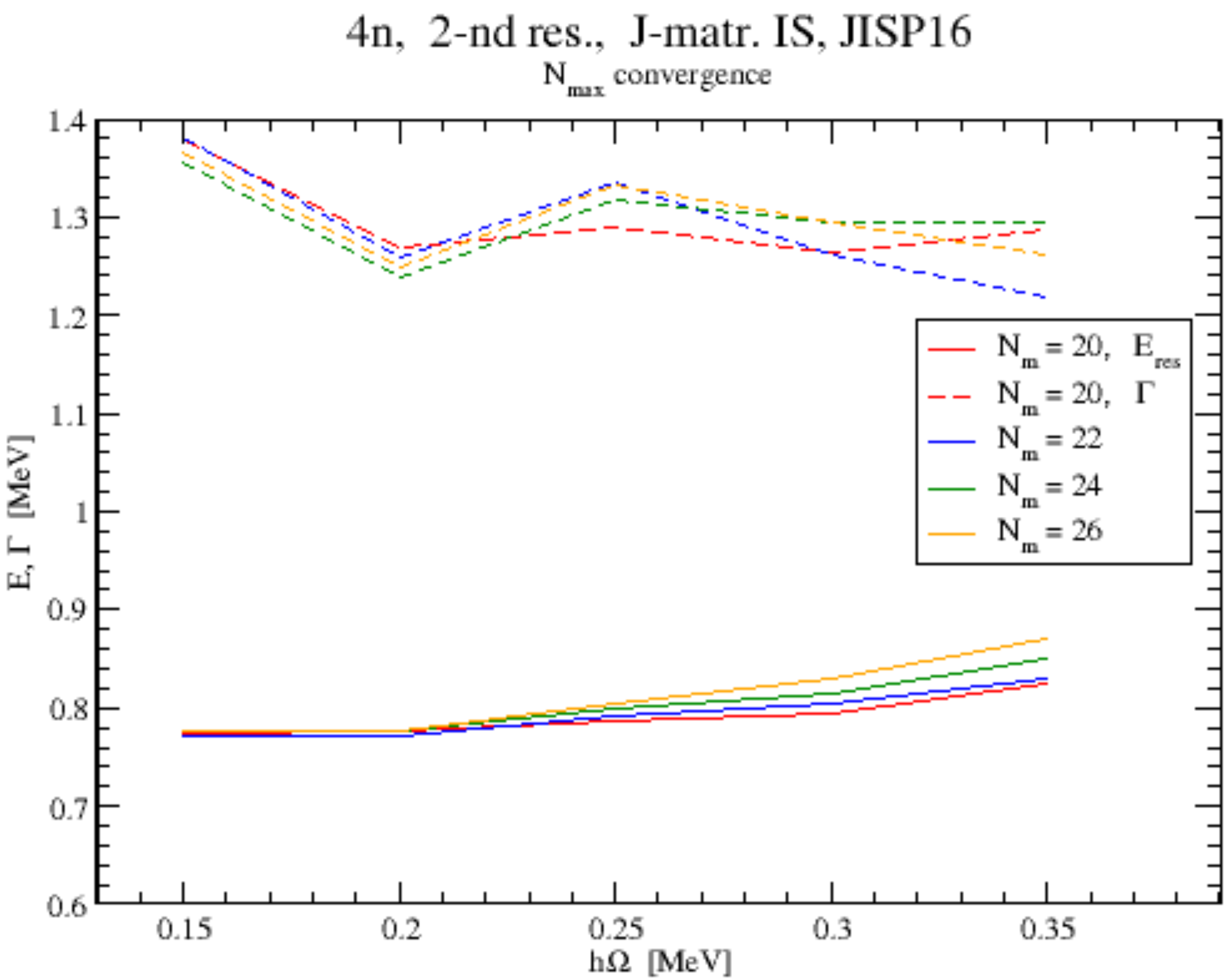
The 2018 results: energy and width for $N_{\max} = 26$



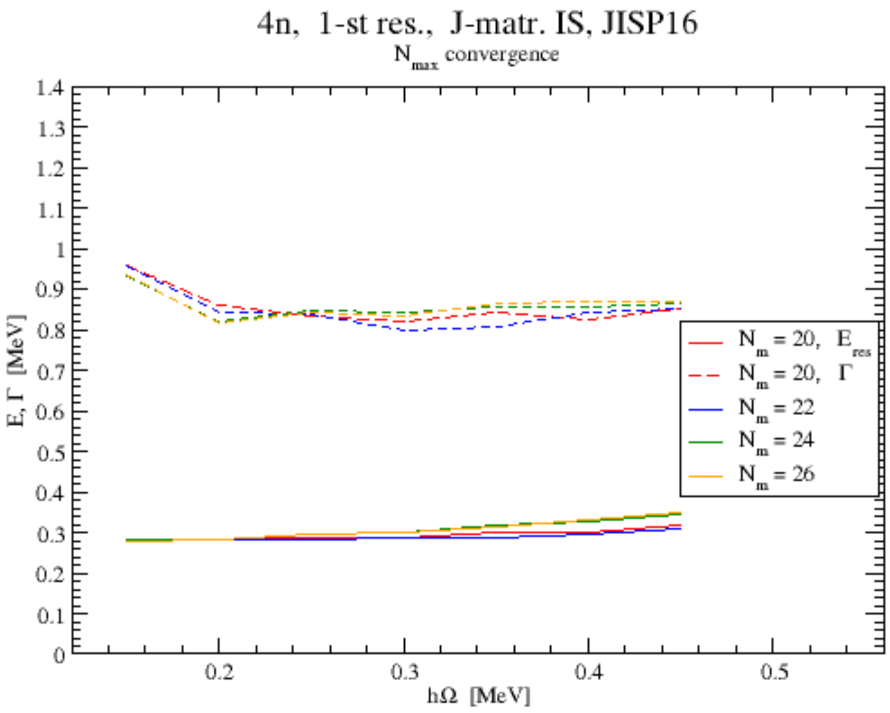
The 2018 results: energy and width for various $N_{\max}$

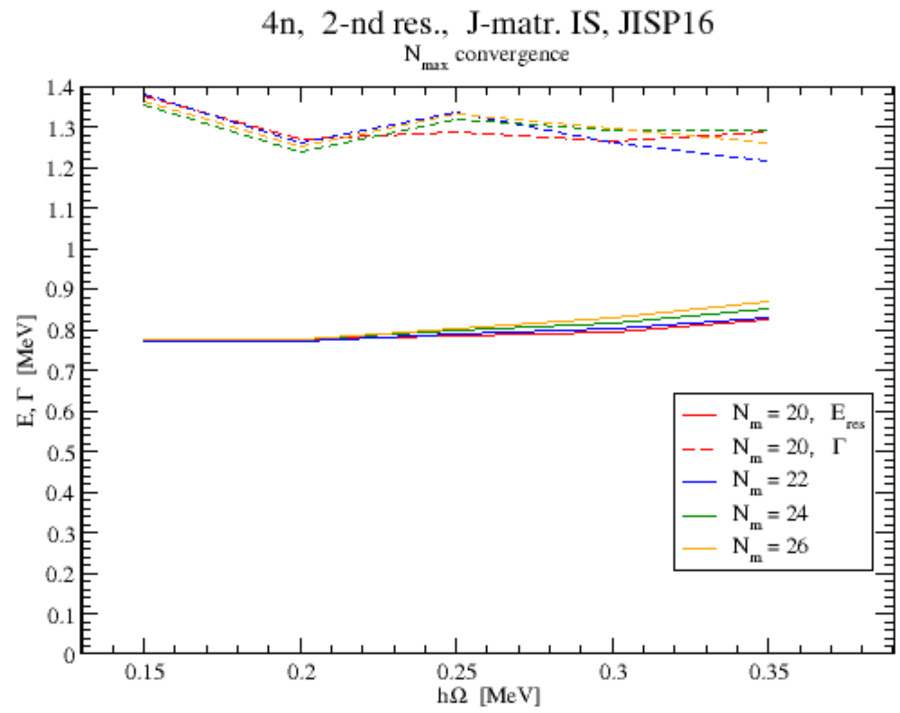


The 2018 results: surprisingly, we have two resonances



The 2018 JISP16 results: extrapolated resonance energies and widths





$E \approx 0.29 \text{ MeV}, \Gamma \approx 0.85 \text{ MeV}$

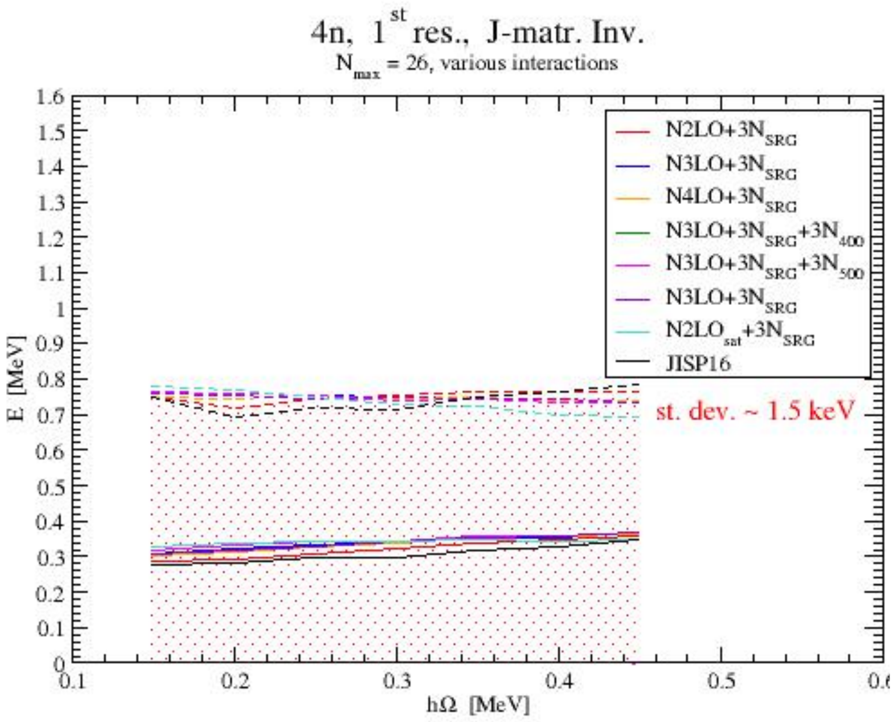
$E \approx 0.8 \text{ MeV}, \Gamma \approx 1.3 \text{ MeV}$

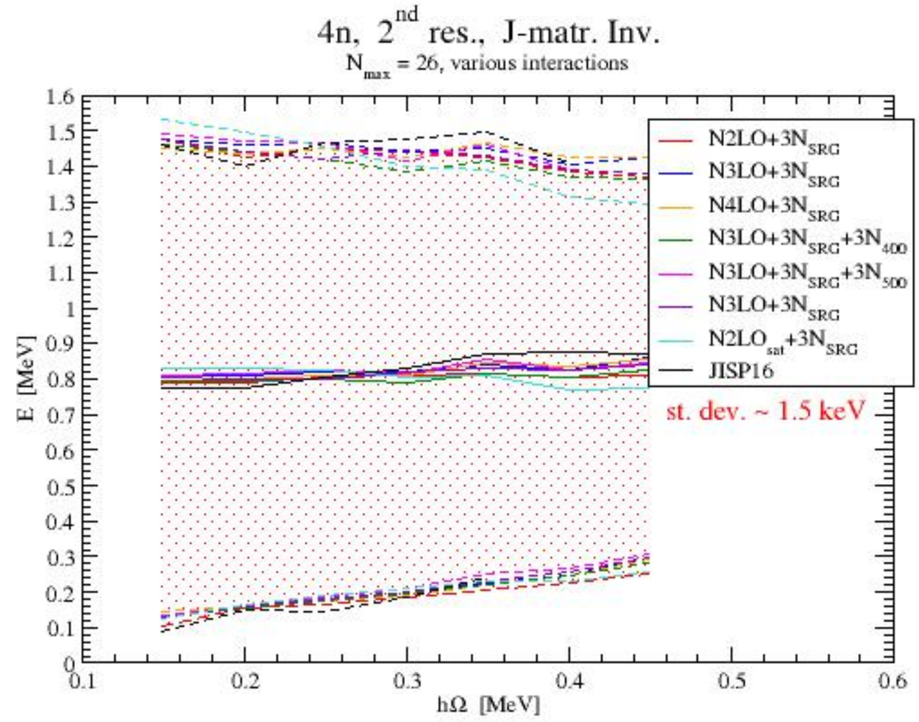
Before we had:

E_r = 186 keV, Γ = 815 keV

E_r = 844 keV, Γ = 1.378 MeV,
E_{false} = -55 keV

The 2018: extrapolated resonance energies and widths with various interactions





$E \approx 0.3$ MeV, $\Gamma \approx 0.85$ MeV

$E \approx 0.8$ MeV, $\Gamma \approx 1.3$ MeV

Before we had:

E_r = 186 keV, Γ = 815 keV

E_r = 844 keV, Γ = 1.378 MeV,
E_{false} = -55 keV

Summary

- HORSE can successfully used within RGM, cluster models, etc.
- Within the NCSM, the SS-HORSE version seems to be more practical.
- For three- or four-body democratic decays one needs additional efforts like inverse-scattering parametrization to allow for long-range interaction.

Workshop questions

- We were discussing what do the people doing reactions need from the people doing the nuclear structure. However structure guys also have some requests for reaction experts.
- Using nuclear structure + scattering theory (but not *reaction theory*) methods we obtained some *S*-matrix resonant poles for tetraneutron. How do they manifest themselves in the cross section of the reaction $^{4}He(^{8}He,^{8}Be)^{4}n$? What is the mechanism of this reaction? Can it be that the increase of the $^{4}He(^{8}He,^{8}Be)^{4}n$ reaction cross section is associated not with the *S*-matrix poles but with the reaction mechanism, e.g., can it be a threshold effect? How to link our *S*-matrix poles, states in the continuum, etc., to the reaction cross sections?

Thank you!