Connecting scattering with structure calculation through Improved Busch formula

Xilin Zhang

University of Washington

FRIB-Theory Alliance workshop: "From bound states to the continuum: Connecting bound state calculations with scattering and reaction theory.", FRIB, East Lansing, MI, June 2018

Outline

- My understanding of the issue
- Busch formula relates two-cluster spectrum in a harmonic trap to the two-cluster scattering
- Improve Busch formula: a toy model and effective field theory (EFT) generalization
- Test the formula and do a proof of principle calculation by studying He-5 system
- Application to NN system
- Summary and outlook

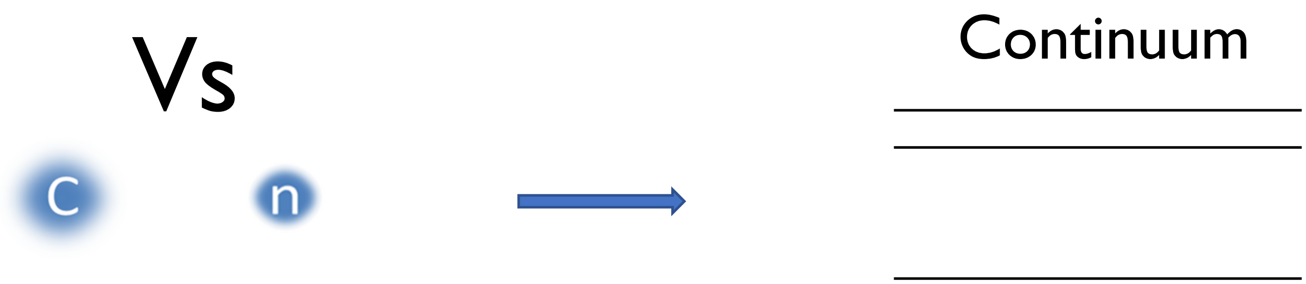
Why are we here: an EFT perspective

- Nuclear structure calculation methods have been developed to study compact system
- When dealing with continuum/resonances, the large distance configuration (DOF) is hard to be included in these methods
- Meanwhile, EFT/cluster-model decrease the "resolution" scale in their descriptions, and focus on the large-distance DOF
- How to combine the two methods? (another way different from RGM)

Busch formula (infrared extrapolation)

Vs

Busch formula (infrared extrapolation)



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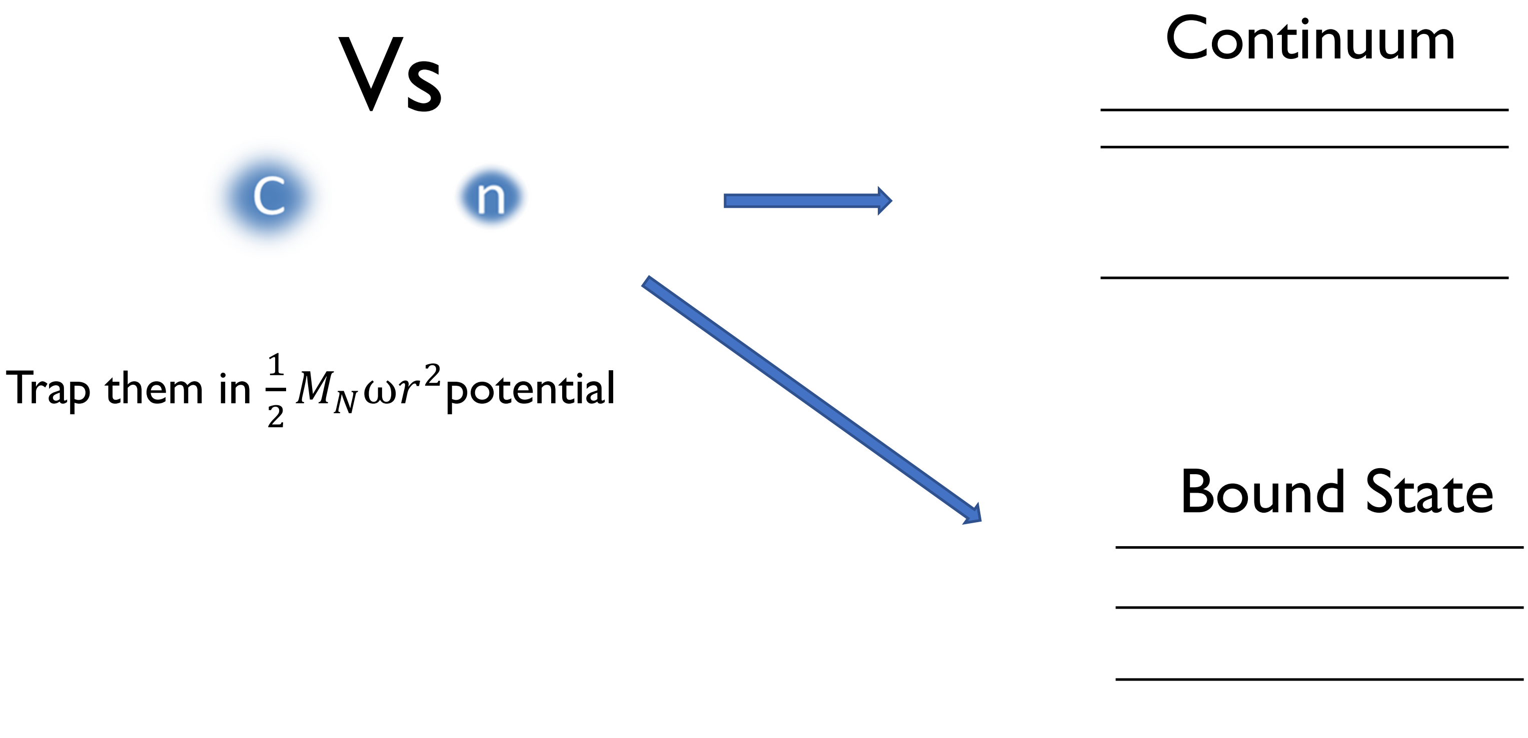

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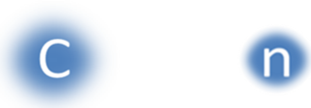

Continuum

Trap them in $\frac{1}{2}M_N\omega r^2$potential

Bound State

Constrain EFT or model on Vs and use it to compute scattering and reaction

Ab initio

calculations

Busch formula

$$p^{2l+1}\cot\delta_l\left(p\right) = (-1)^{l+1}(4M_{\mathrm{R}}\omega)^{l+\frac{1}{2}}\frac{\Gamma\left(\frac{3}{4}+\frac{l}{2}-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{l}{2}-\frac{\epsilon}{2}\right)}$$

$$\text{with}\quad \epsilon\equiv\frac{E}{\omega},\ E\equiv\frac{p^2}{2M_{\mathrm{R}}}$$

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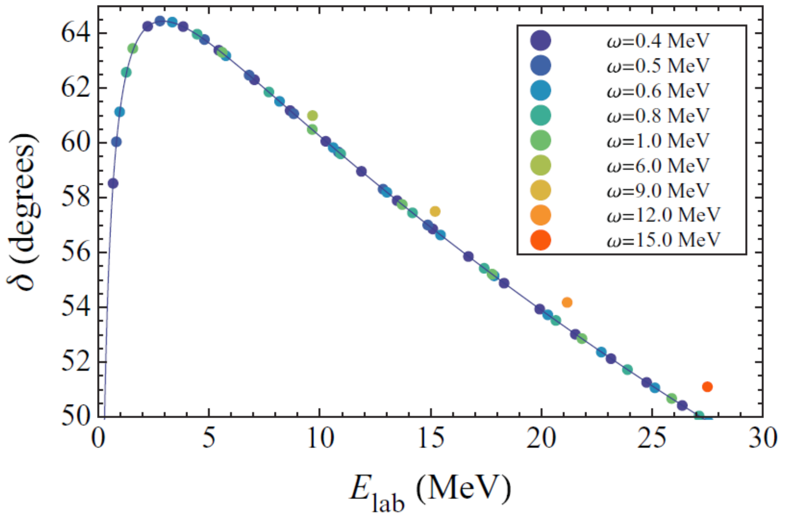

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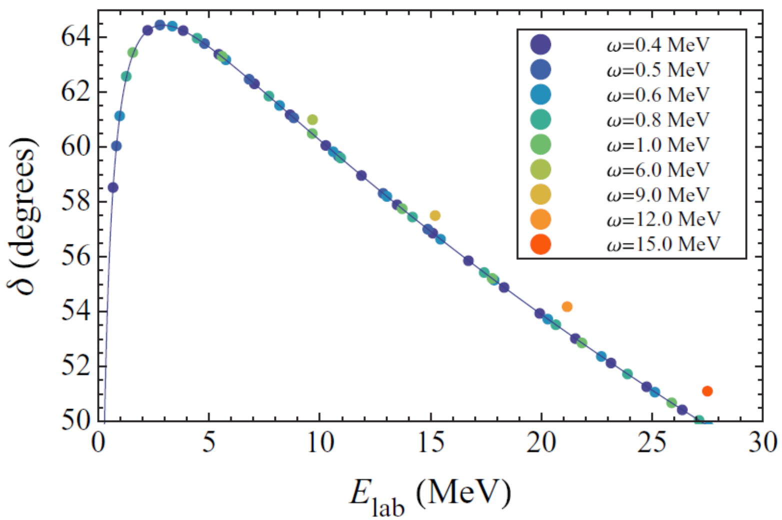

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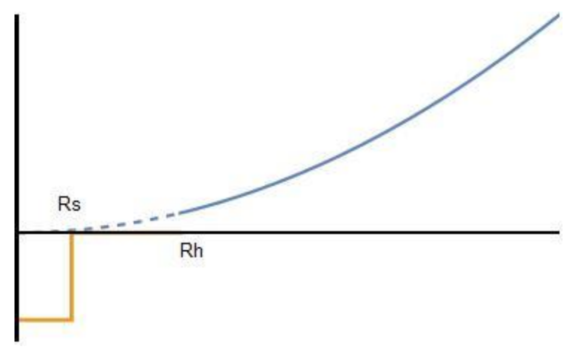

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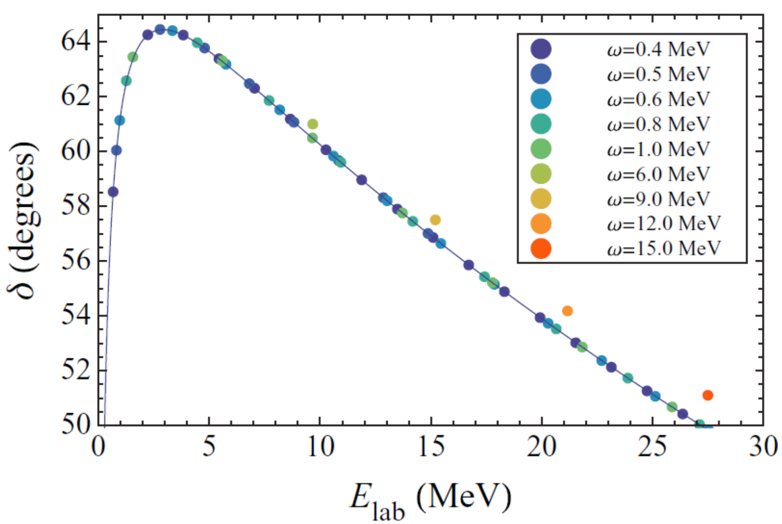

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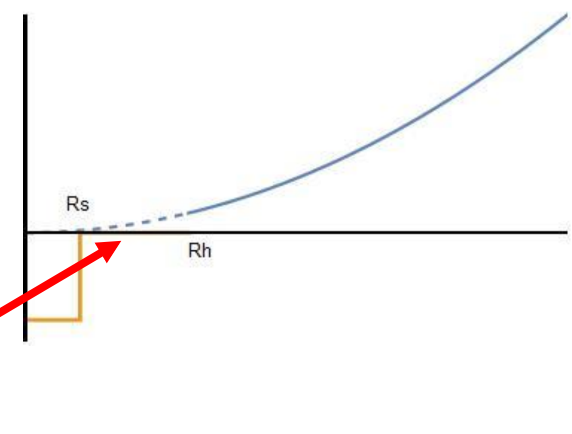

$\mathrm{Cos}\,\delta\, j_l + \mathrm{Sin}\,\delta\, n_l$

Busch formula

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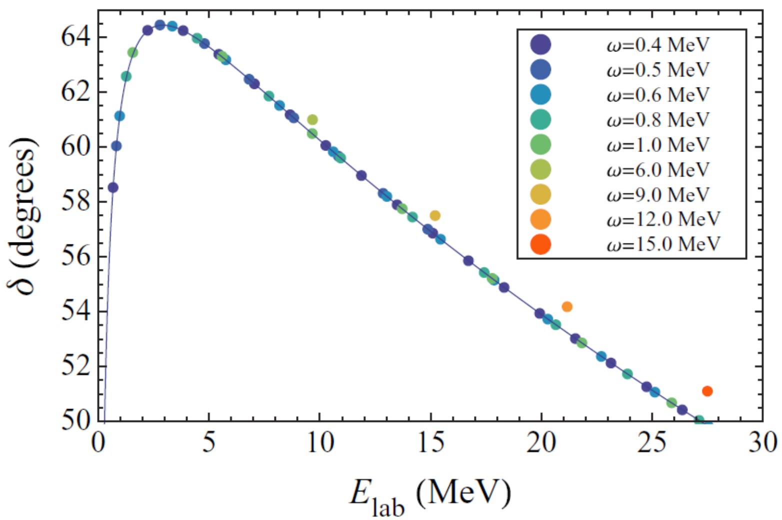

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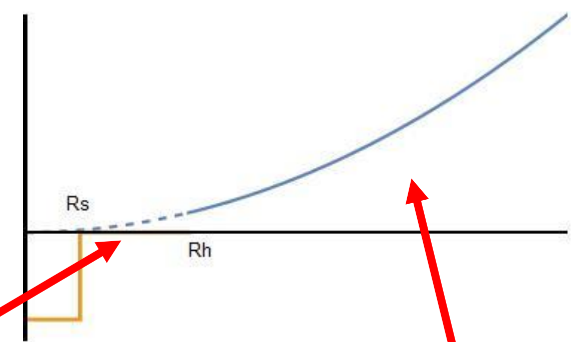

Linear combination of regular and irregular oscillator wave functions

$\cos\delta\, j_l + \sin\delta\, n_l$

Improve Busch Formula: a model

$$V_s(r) = \begin{cases} +\infty & \text{when } r \leq r_c \\ 0 & \text{when } r > r_c \ , \end{cases}$$

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$$= -\frac{(2l+1)!!(2l-1)!!}{r_c^{2l+1}} \left[\frac{(2l+1)\left(\frac{r_c}{b}\right)^4}{2(2l-3)(2l+5)} + \frac{(2l+1)(6l+25)\frac{1}{2}(pr_c)^2 \left(\frac{r_c}{b}\right)^4}{3(2l-5)(2l+3)(2l+5)(2l+7)} + O\left[\left(\frac{r_c}{b}\right)^8, (pr_c)^4 \left(\frac{r_c}{b}\right)^4\right]\right]$$

$$\equiv -L_{a_l} \frac{1}{b^4 r_c^{2l-3}} - L_{r_l} \frac{p^2}{b^4 r_c^{2l-1}} + \ldots$$

Note: $b = \sqrt{\frac{1}{M_R \omega}}$

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Effective range expansion (ERE):

$$p^{2l+1}\cot\delta_l(p) = -\frac{\Lambda^{2l+1}}{a_l} + \frac{1}{2}r_l\Lambda^{2l-1}p^2 + \frac{1}{4}\tilde{r}_l^{(1)}\Lambda^{2l-3}p^4$$

Improve Busch Formula: EFT

$$\mathcal{L}_0 = \begin{pmatrix} c^* & n^* & -\phi^* \end{pmatrix} \mathrm{diag}\left(i\partial_t - \hat{m}_c\psi + \frac{\boldsymbol{\partial}^2}{2M_c}, i\partial_t - \hat{m}_n\psi + \frac{\boldsymbol{\partial}^2}{2M_n}, i\partial_t - \hat{m}_\phi\psi + \frac{\boldsymbol{\partial}^2}{2M_{nc}} + \Delta_0 \right) \begin{pmatrix} c & n & \phi \end{pmatrix}^T$$

$$\mathcal{L}_{I0} = g_0 \phi^* c\, n - \phi^* \left[\sum_{j=2} d_j^{(0)} \left(i\partial_t - \hat{m}_\phi\psi + \frac{\boldsymbol{\partial}^2}{2M_{nc}} \right)^j \right] \phi + \mathrm{C.C}\ .$$

Note: $\psi = \frac{1}{2}\, m_N\, \omega^2 r^2$

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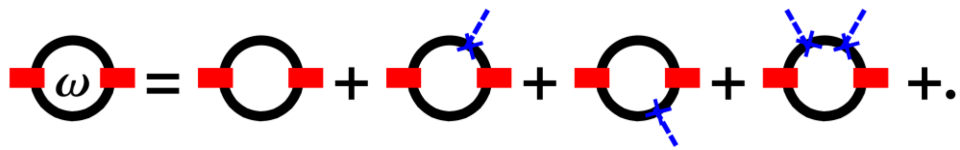
Self-energy bubble:



Dimer-field propagator:

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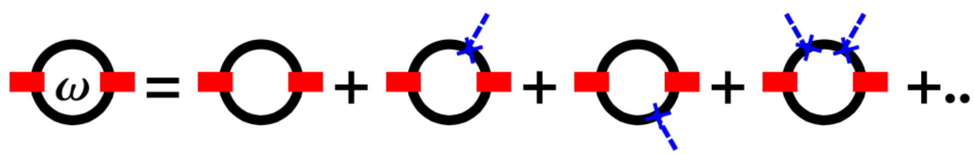

Dimer-field propagator:

$$p_{\tilde{E}} \cot\delta_0(\tilde{E}) = -\frac{2\pi}{g_0^2 M_\mathrm{R}} \left(\Sigma_\omega(\tilde{E}) - \Sigma(\tilde{E}) \right)$$ reproduces the Busch formula.

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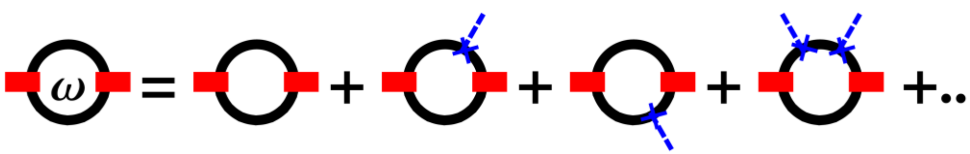

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Then what went wrong?

Improve Busch Formula: EFT

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$$-\phi^* \left[\sum_{j=0} \sum_{k=1} d_{j,k}^{(0)} \left(i\partial_t - \hat{m}_\phi \psi + \frac{\boldsymbol{\partial}^2}{2M_{nc}} \right)^j \left(\frac{M_{\mathrm{R}}^2}{3m} \boldsymbol{\partial}^2 \psi \right)^k \right] \phi$$

Improve Busch Formula: EFT

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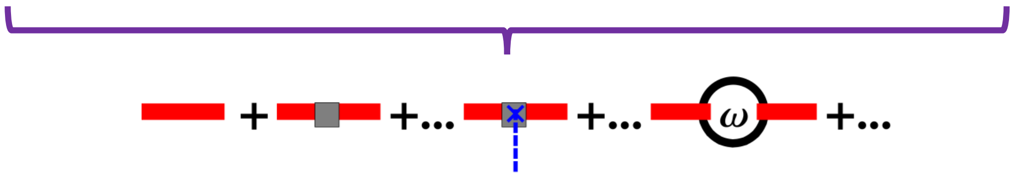

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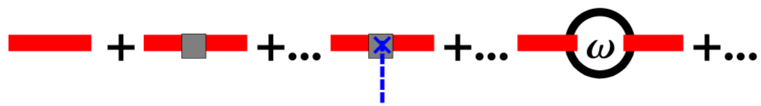

The factorizability of CM motion severely constrains two-body current like couplings.

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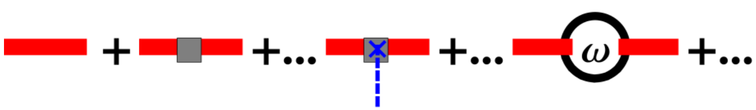

The factorizability of CM motion severely constrains two-body current like couplings.

$$\Sigma_\omega \to \Sigma_\omega + \sum_{j=0} \sum_{k=1} d_{j,k}^{(0)} \tilde{E}^j b^{-4k}$$

$$p \cot \delta_0 (p) \to p \cot \delta_0 (p) + L_{a_0} \frac{1}{b^4 \Lambda^3} + L_{r_0} \frac{p^2}{b^4 \Lambda^5} + L_{\tilde{r}_0^{(1)}} \frac{p^4}{b^4 \Lambda^7} + \ldots$$

$$= -\frac{\Lambda}{a_0} + \frac{1}{2} \frac{r_0}{\Lambda} p^2 + \frac{1}{4} \frac{\tilde{r}_0^{(1)}}{\Lambda^3} p^4 + L_{a_0} \frac{1}{b^4 \Lambda^3} + L_{r_0} \frac{p^2}{b^4 \Lambda^5} + L_{\tilde{r}_0^{(1)}} \frac{p^4}{b^4 \Lambda^7}$$

Test: $\mathrm{n}-\alpha$ system

$$V_s(r) = \begin{cases} V_0(1+\beta \boldsymbol{L}\cdot\boldsymbol{\sigma}) & \text{when } r < r_c \\ 0 & \text{when } r > r_c \end{cases}$$

$V_0 = 33\ \mathrm{MeV}$
$\beta = 0.103,$
$r_c = 2.55\mathrm{fm}$

Test: n $-$ α system

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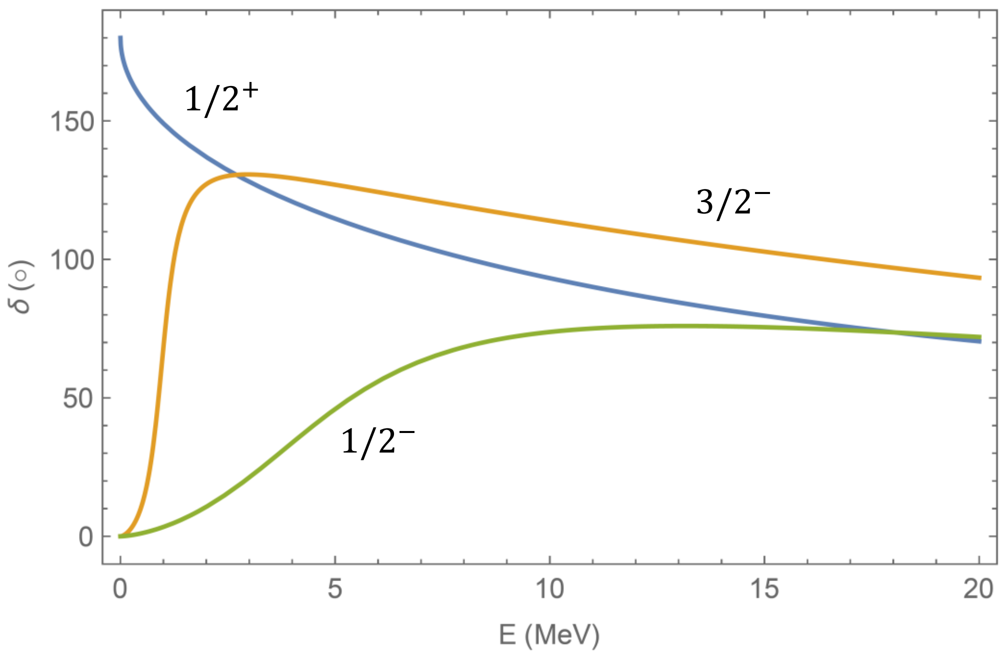
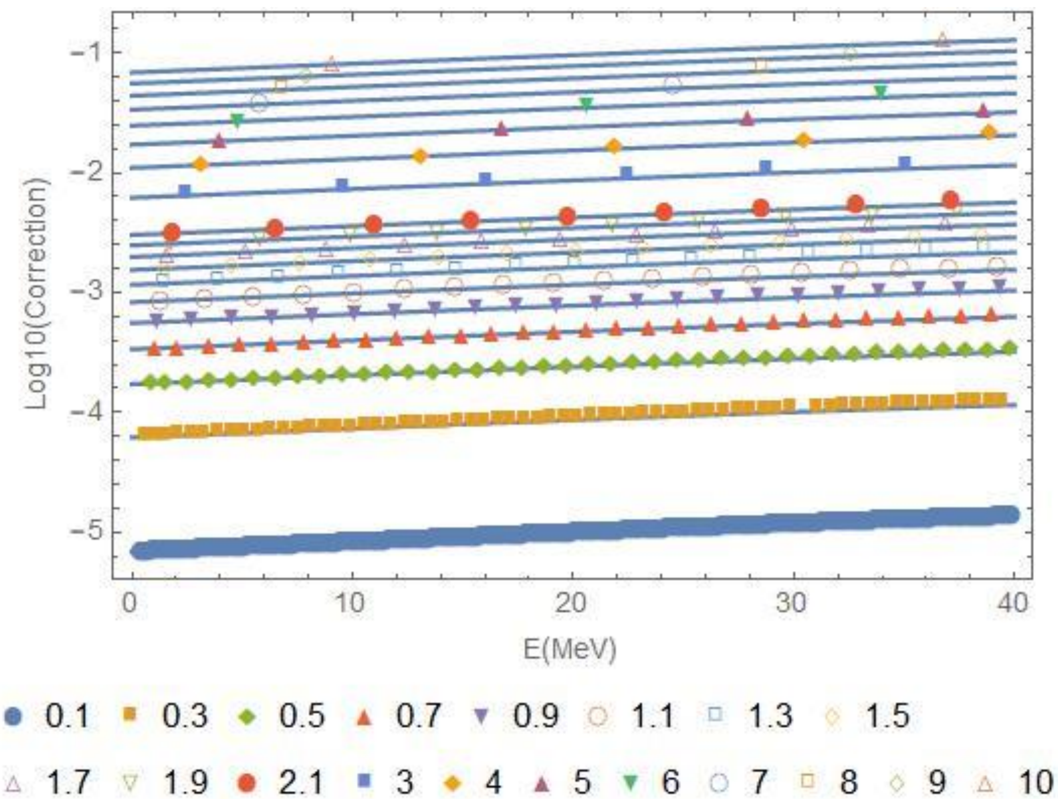


S. Ali et.al., RMP **57,** 923 (1985)

Test: $\mathrm{n}-\alpha$ system in p-wave

$$p^{2l+1}\cot\delta_l\left(p\right)-(-1)^{l+1}(4M_{\mathrm{R}}\omega)^{l+\frac{1}{2}}\frac{\Gamma\left(\frac{3}{4}+\frac{l}{2}-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{l}{2}-\frac{\epsilon}{2}\right)}$$

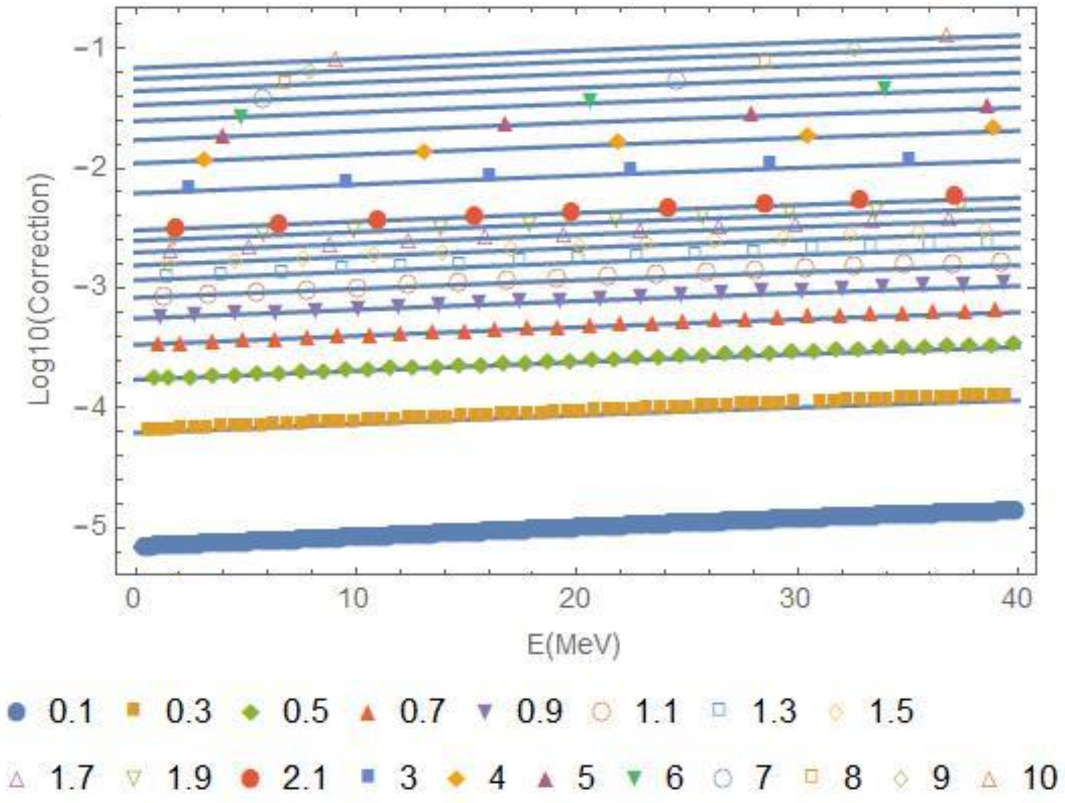


Test: $\mathrm{n} - \alpha$ system in p-wave

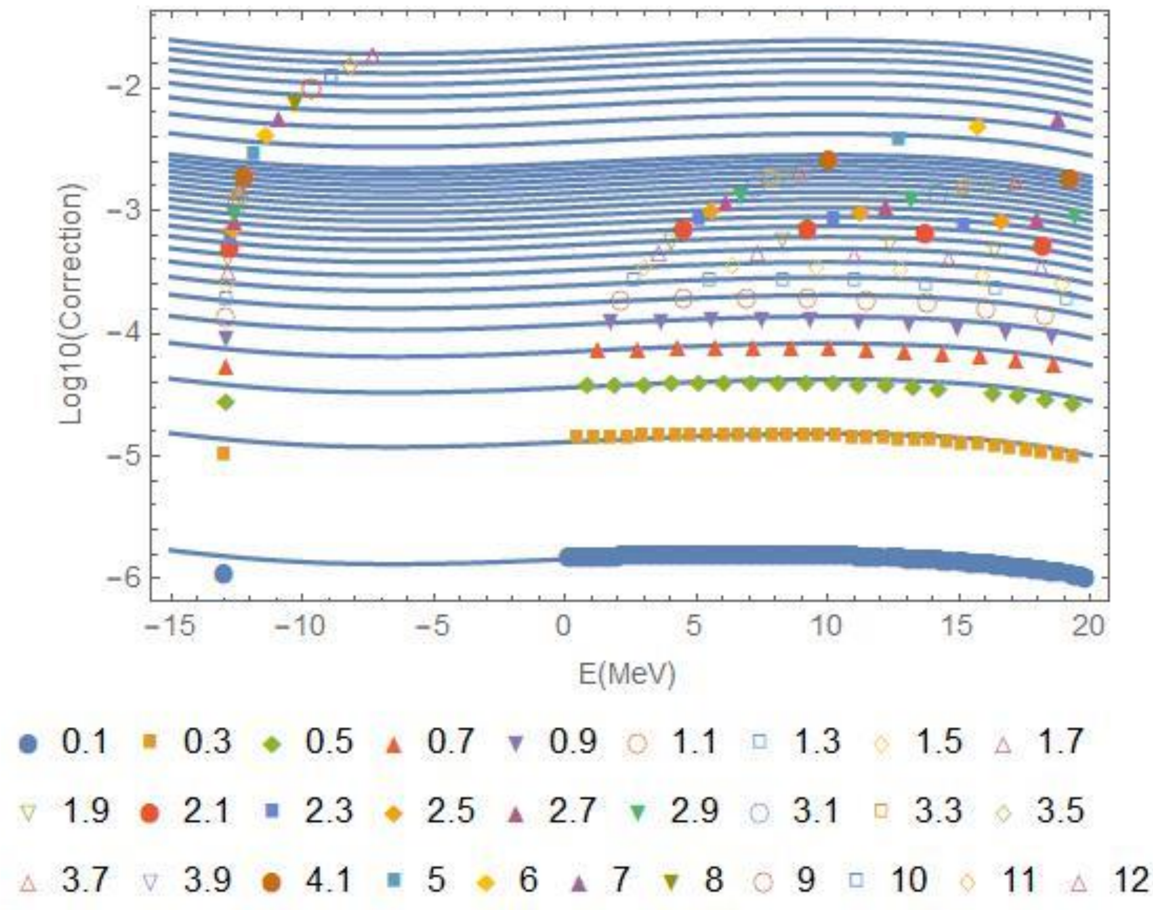
$$p^{2l+1}\cot\delta_l(p) - (-1)^{l+1}(4M_{\mathrm{R}}\omega)^{l+\frac{1}{2}}\frac{\Gamma\left(\frac{3}{4}+\frac{l}{2}-\frac{\epsilon}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{l}{2}-\frac{\epsilon}{2}\right)}$$

1) Corrections $\propto \omega^2$

2) At $\omega = 10$ MeV, 10% corrections



Test: $\mathrm{n}-\alpha$ system in s-wave



A digression to Bayesian inference

$$\mathrm{pr}\left(\boldsymbol{g}, \{\xi_i\} | D; T; I\right) = \mathrm{pr}\left(D | \boldsymbol{g}, \{\xi_i\}; T; I\right) \mathrm{pr}\left(\boldsymbol{g}, \{\xi_i\} | I\right)$$

Posterior distribution

Likelihood function
Here they are delta functions, for exact input data D

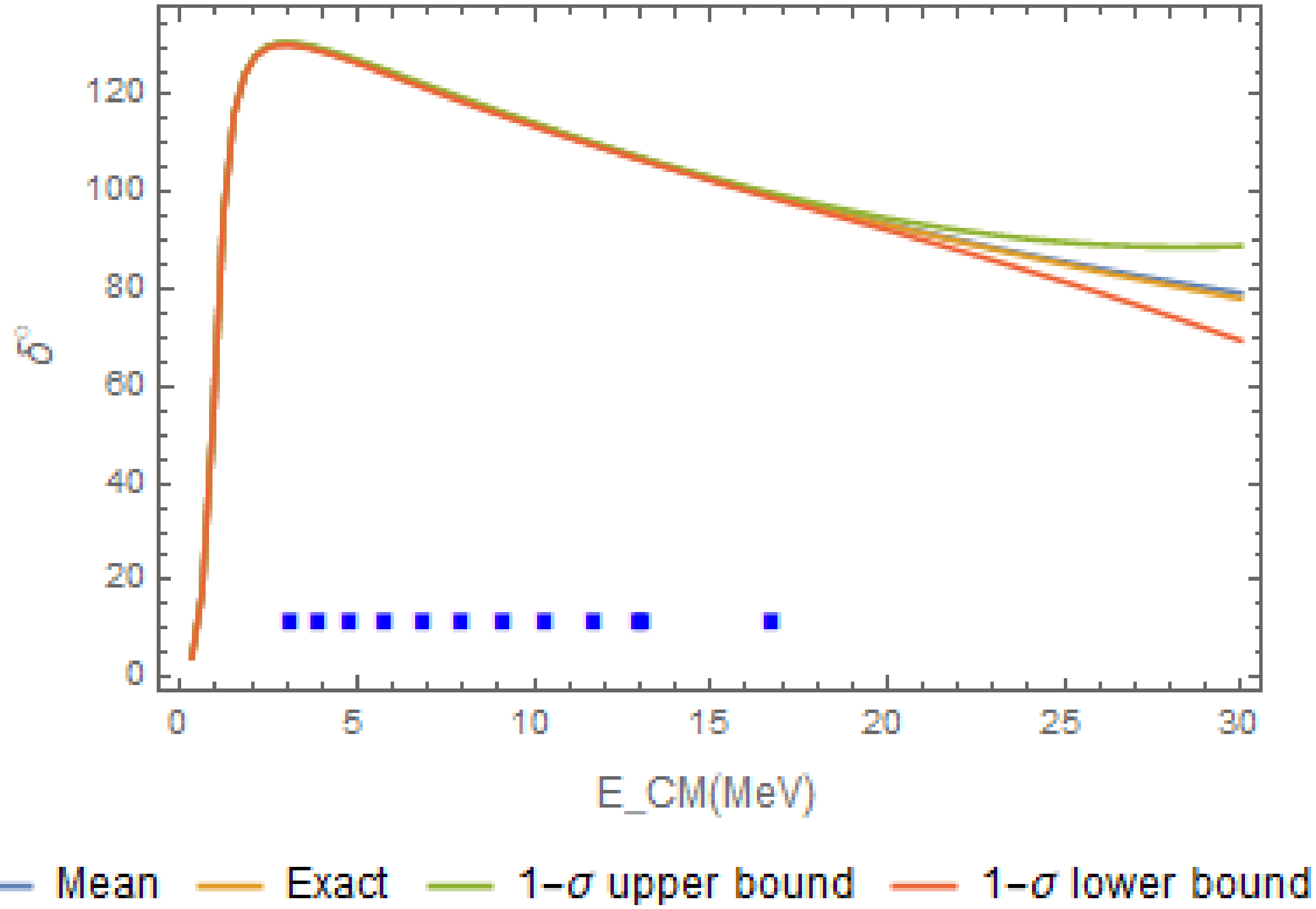
Prior distribution

T: $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j} \left(\frac{b^{-4}}{\Lambda^4}\right)^i \left(\frac{p^2}{\Lambda^2}\right)^j = (-1)^{l+1} \left(\frac{4\, M_R \omega}{\Lambda^2}\right)^{l+1/2} \frac{\Gamma(\frac{3}{4}+\frac{l}{2}-\frac{E}{2\omega})}{\Gamma(\frac{1}{4}-\frac{l}{2}-\frac{E}{2\omega})}$

$$\left(\frac{p^2}{\Lambda^2}\right)^{l+\frac{1}{2}} Cot\delta_l = \sum_{j=0}^{\infty} C_{i=0,j} \left(\frac{p^2}{\Lambda^2}\right)^j$$

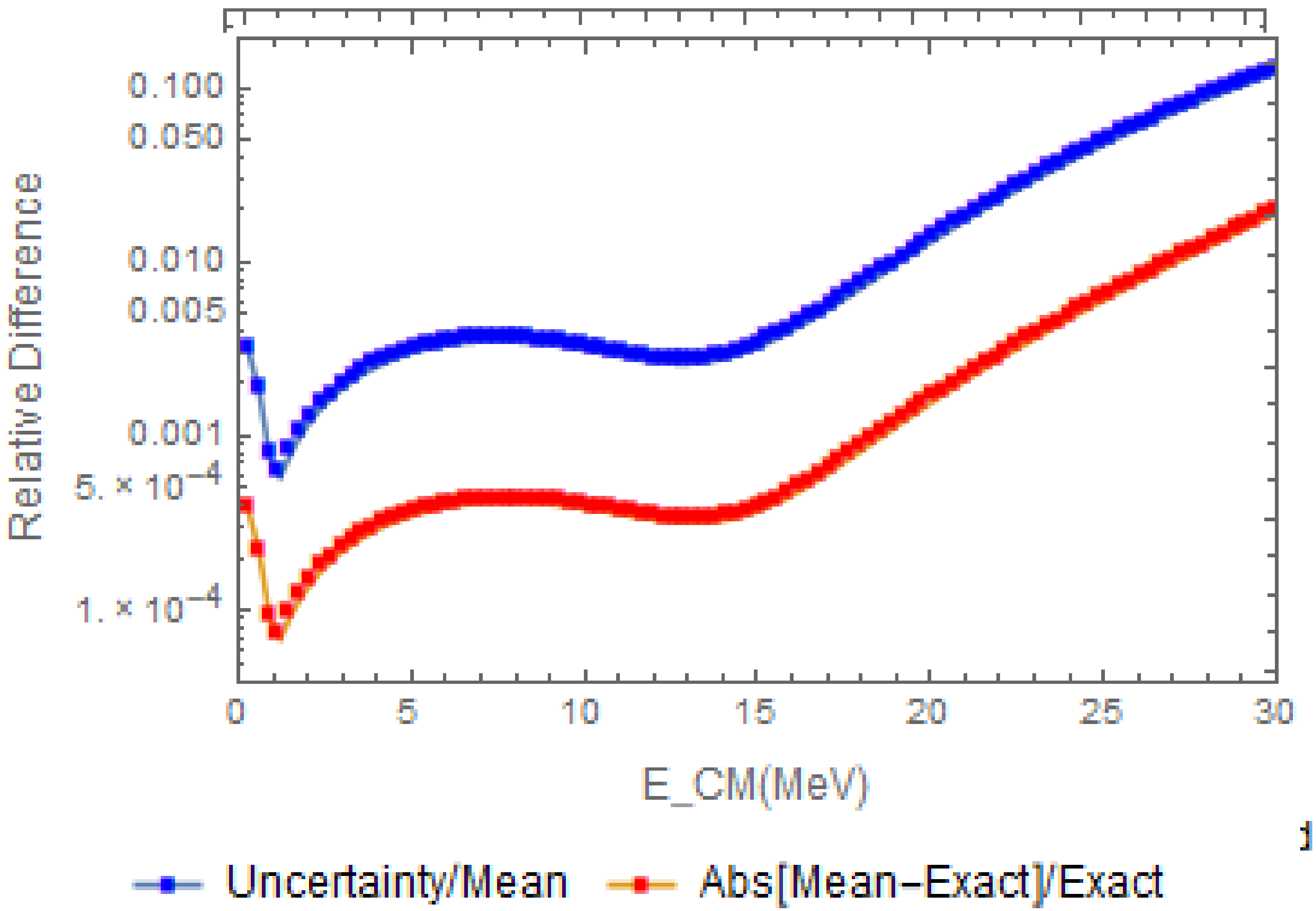
$3/2^-$ at N6LO

$3/2^-$ at N6LO



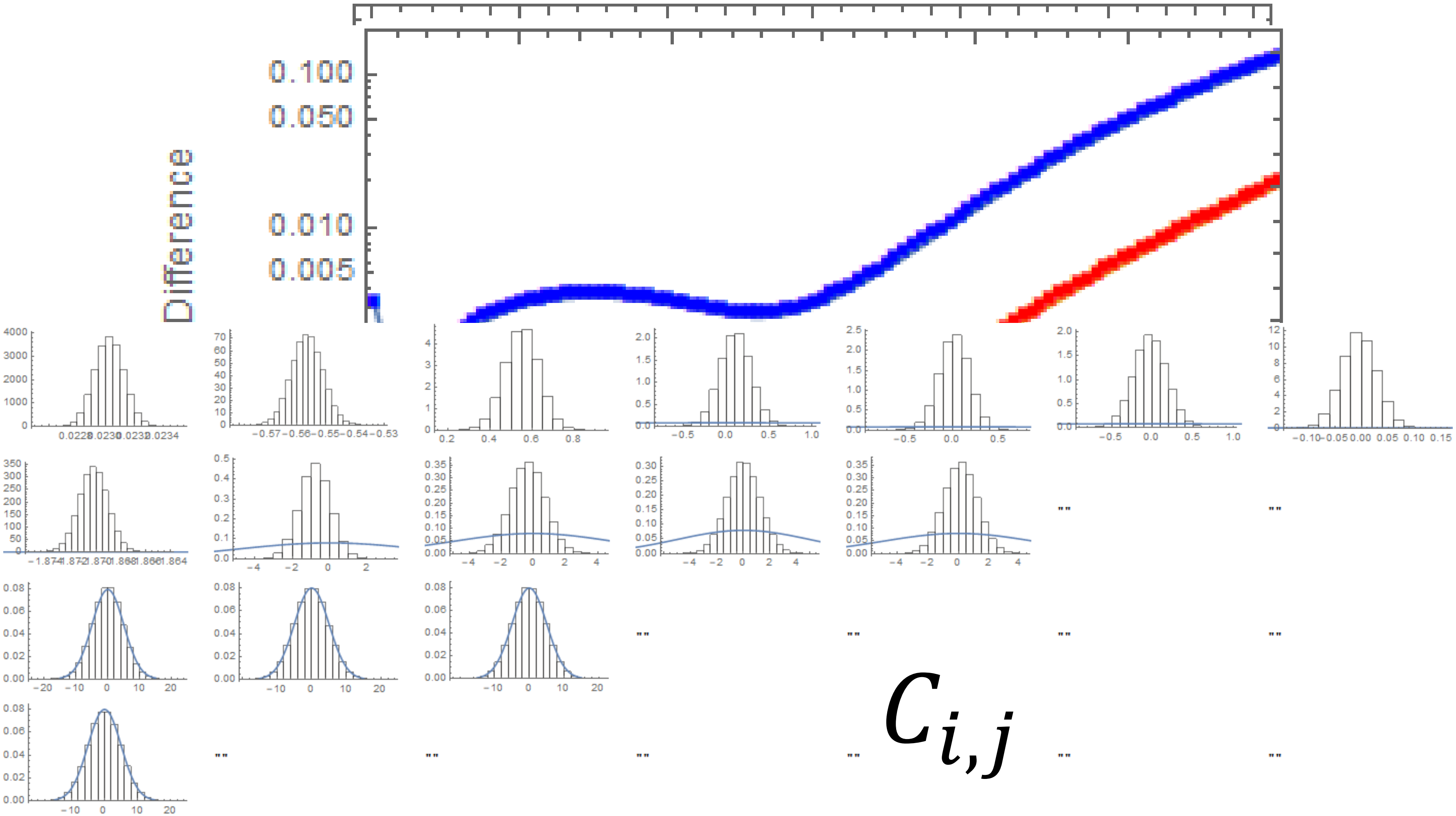
120
100
80
60
40
20
0
δ°
0
5
10
15
20
25
30
E_CM(MeV)
Mean
Exact
1-σ upper bound
1-σ lower bound

$3/2^-$ at N6LO



0.100
0.050
0.010
0.005
0.001
5. × 10^-4
1. × 10^-4
Relative Difference
0
5
10
15
20
25
30
E_CM(MeV)
Uncertainty/Mean
Abs[Mean-Exact]/Exact

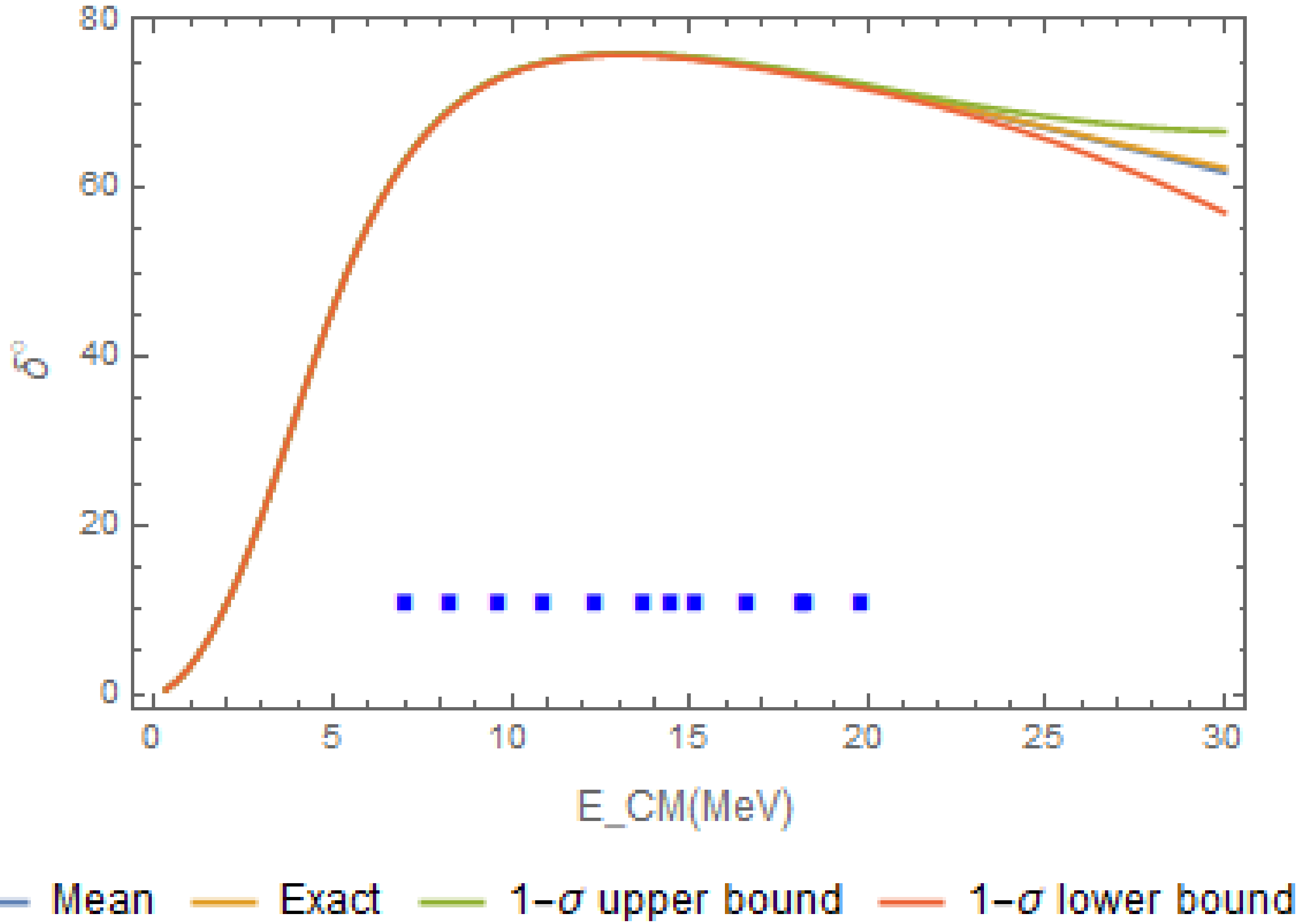
$3/2^{-}$ at N6LO



$c_{i,j}$

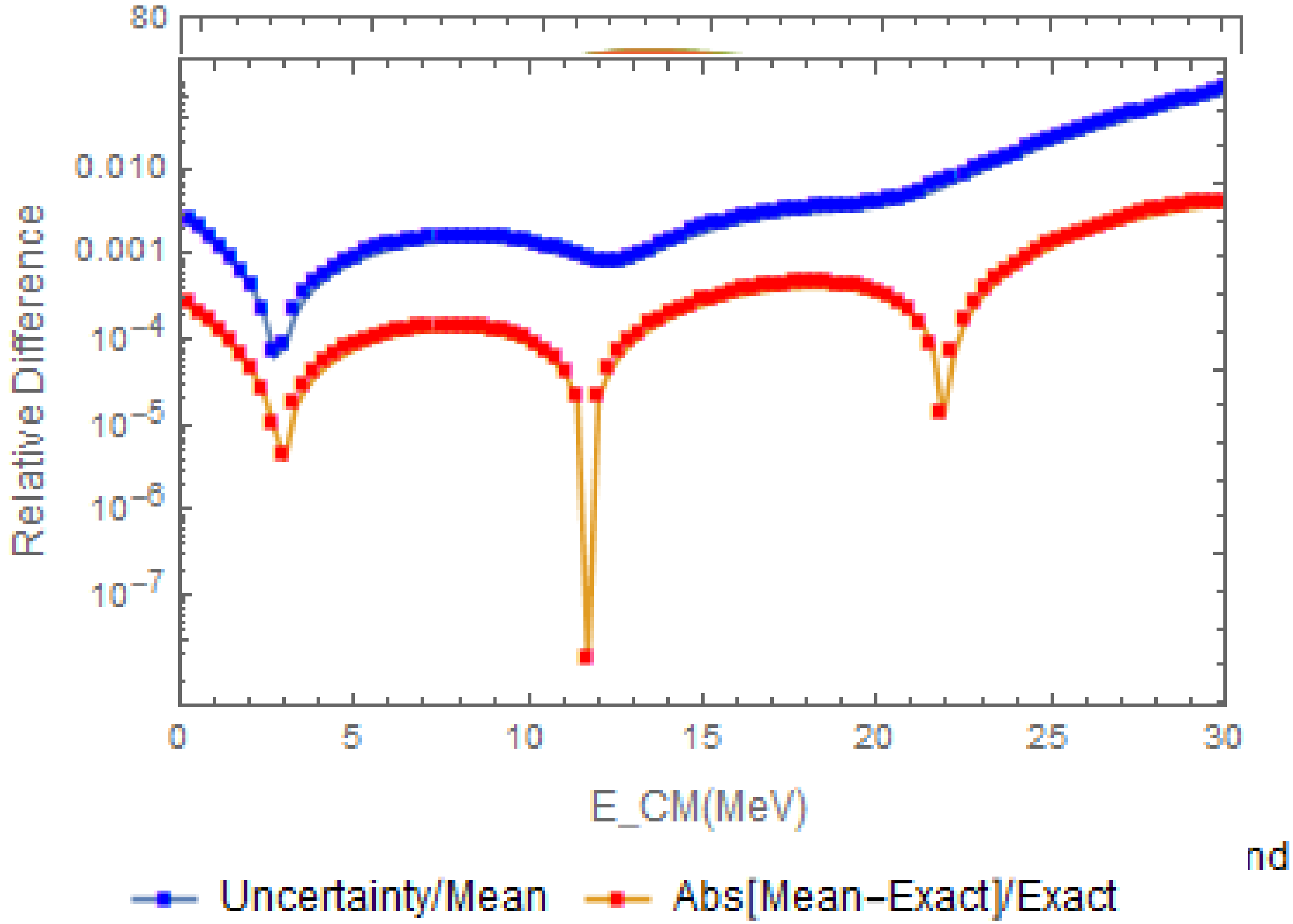
$1/2^-$ at N6LO

$1/2^-$ at N6LO

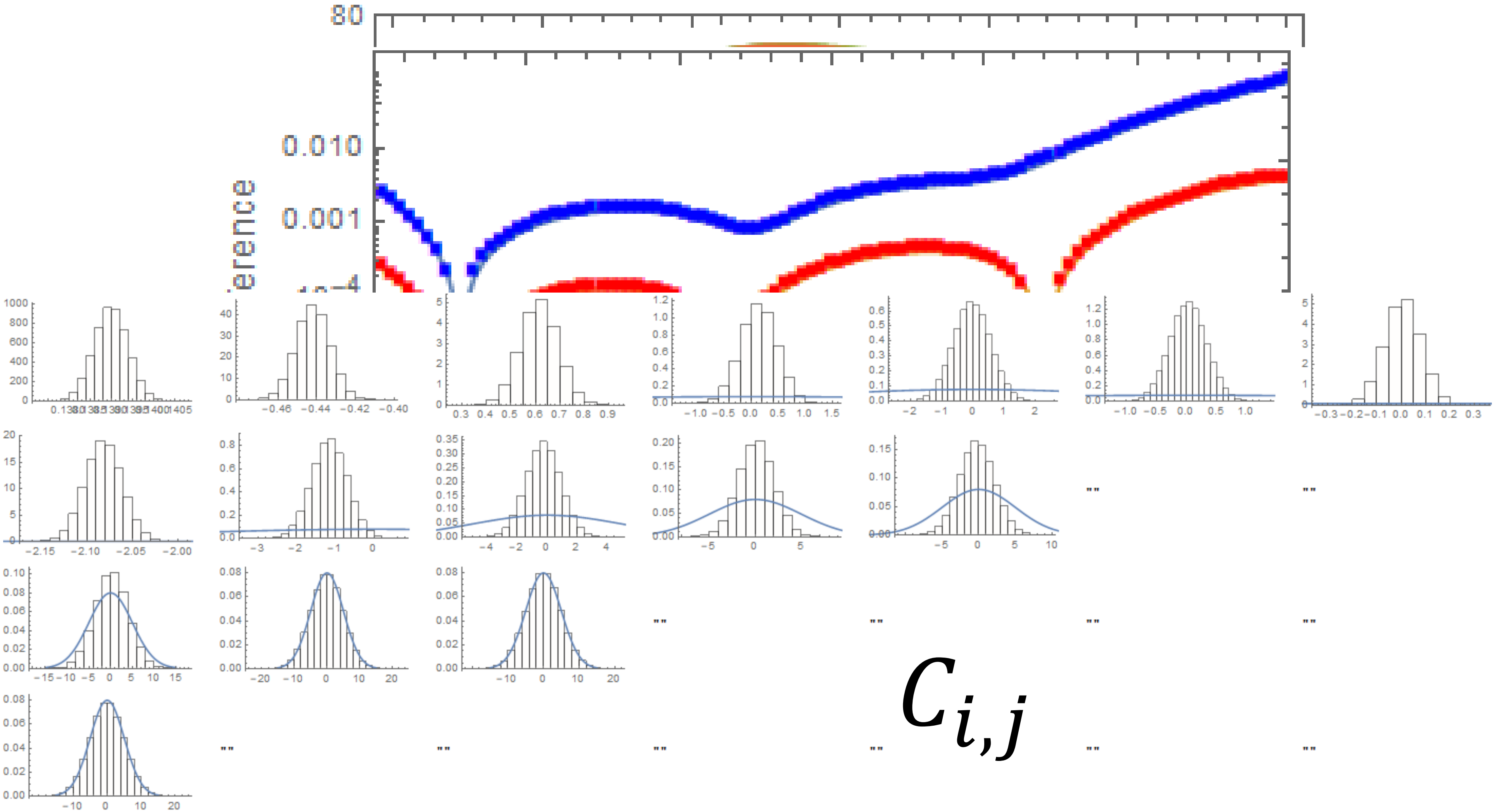


80
60
40
20
0
δ°
0
5
10
15
20
25
30
E_CM(MeV)
Mean
Exact
1−σ upper bound
1−σ lower bound

$1/2^-$ at N6LO


80
0.010
0.001
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
Relative Difference
0
5
10
15
20
25
30
E_CM(MeV)
nd
Uncertainty/Mean
Abs[Mean−Exact]/Exact

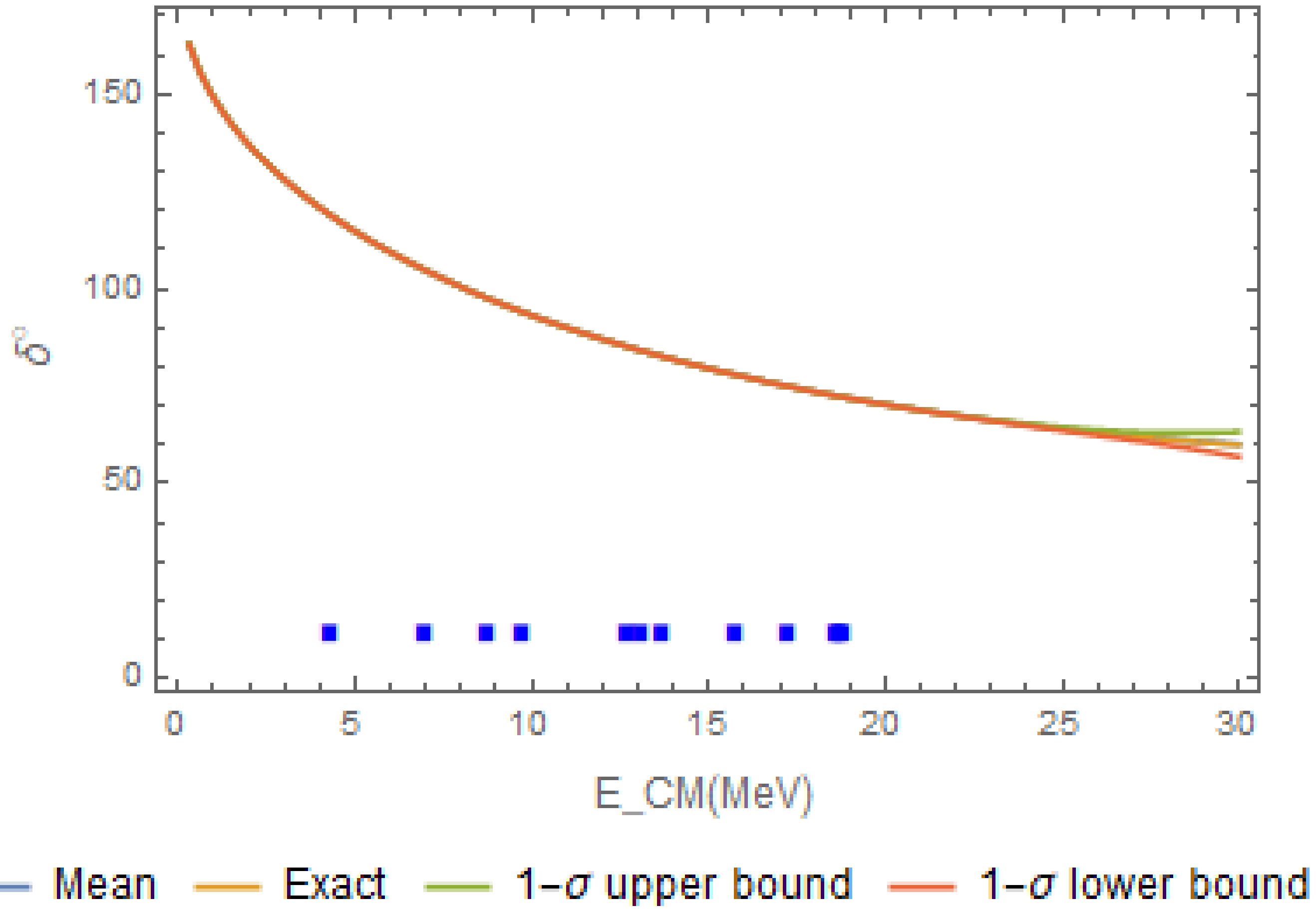
$1/2^-$ at N6LO



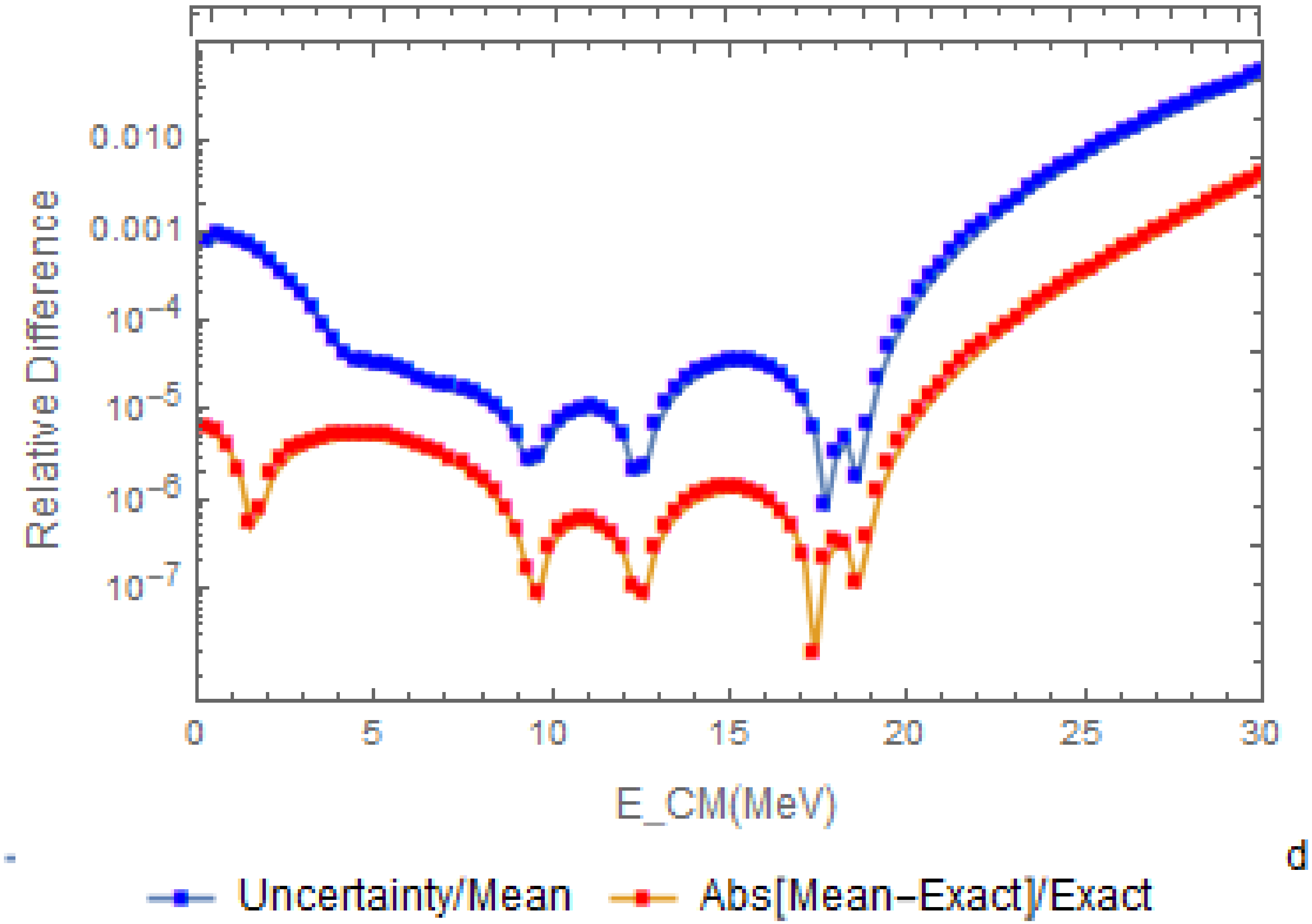
$C_{i,j}$

$1/2^+$ at N6LO

$1/2^+$ at N6LO

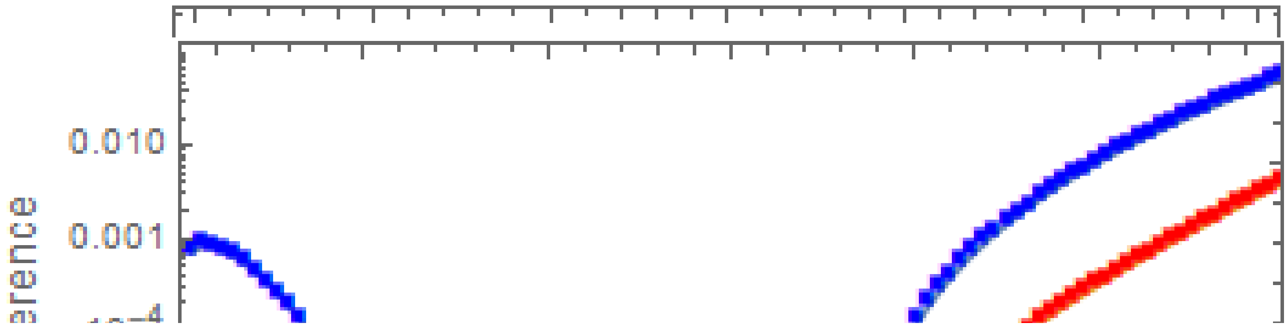

150
100
50
0
δ°
0
5
10
15
20
25
30
E_CM(MeV)
Mean
Exact
1−σ upper bound
1−σ lower bound

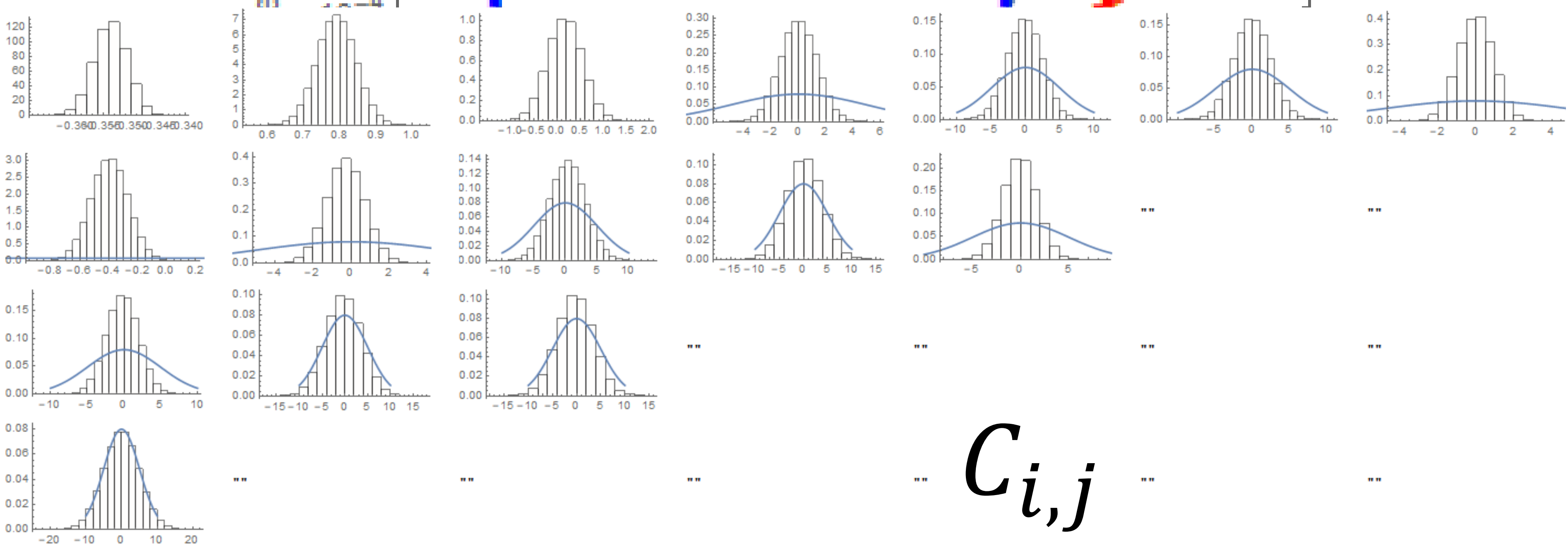
$1/2^+$ at N6LO



Relative Difference
0.010
0.001
10^-4
10^-5
10^-6
10^-7
0
5
10
15
20
25
30
E_CM(MeV)
d
Uncertainty/Mean
Abs[Mean−Exact]/Exact

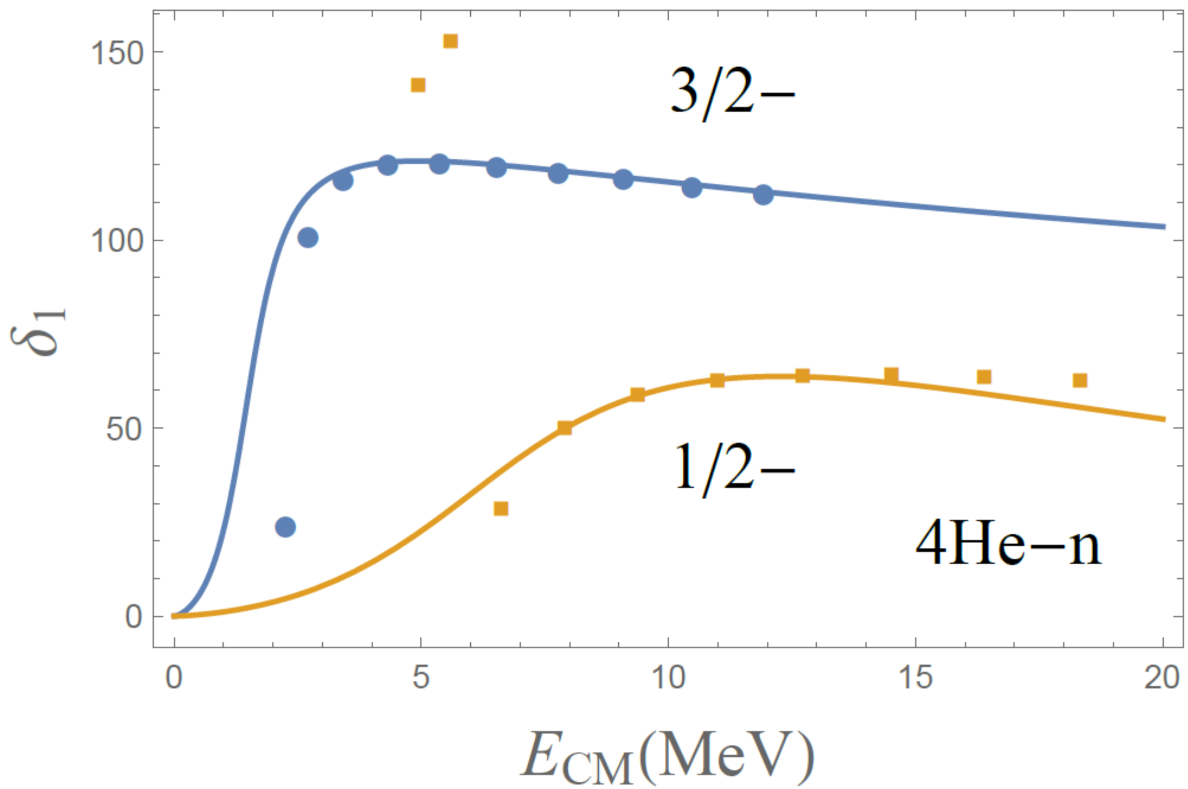
$1/2^+$ at N6LO





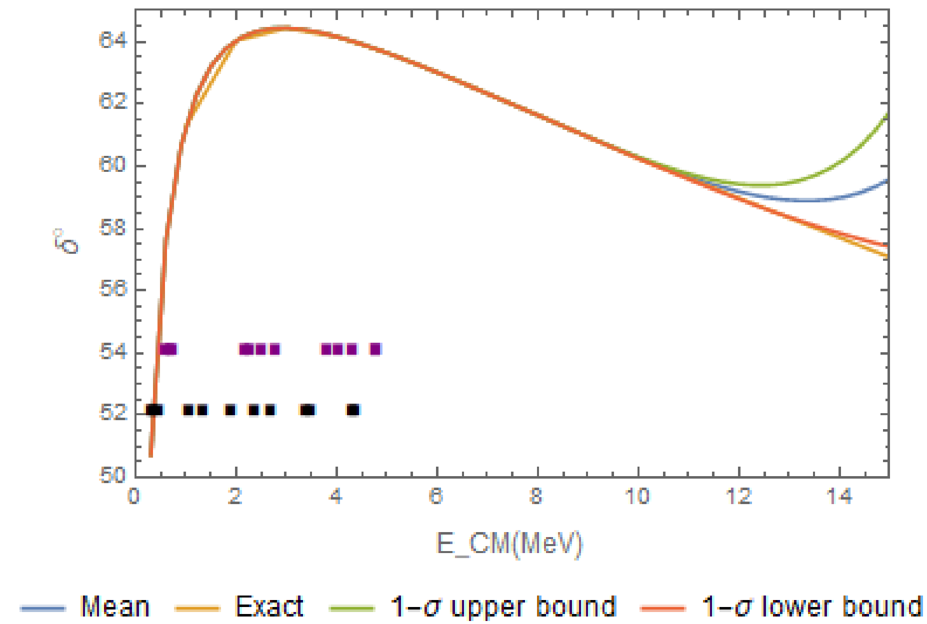
$C_{i,j}$

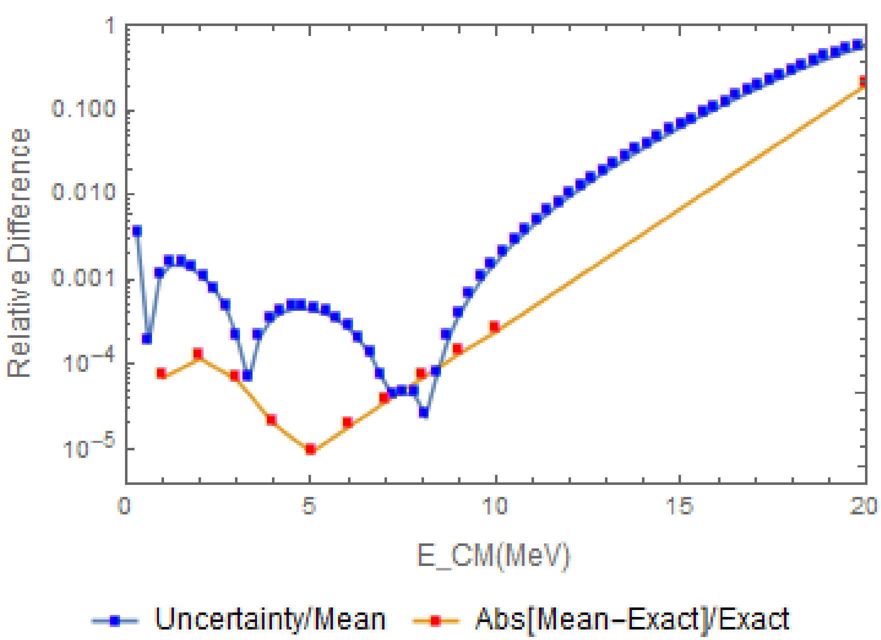
Trial results by analyzing IM-SRG "data" from G. Chan, R. Stroberg, and J. Holt

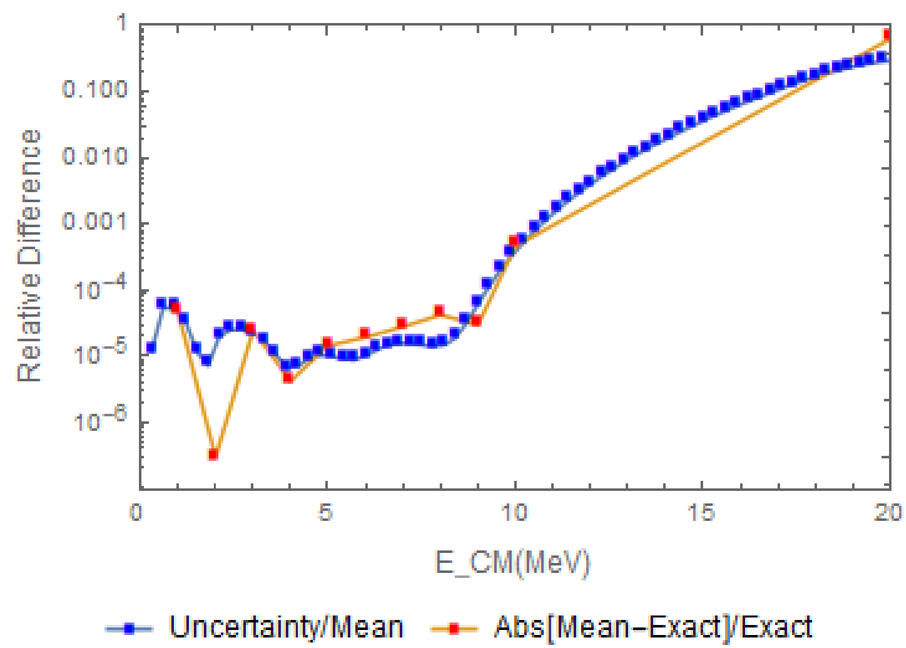


NN at N6LO

The energy spectrum are from the calculations by J. Vary et.al. [T. Luu, M. Savage, A. Schwenk, and J. Vary, PRC (2010)]







NN at N6LO

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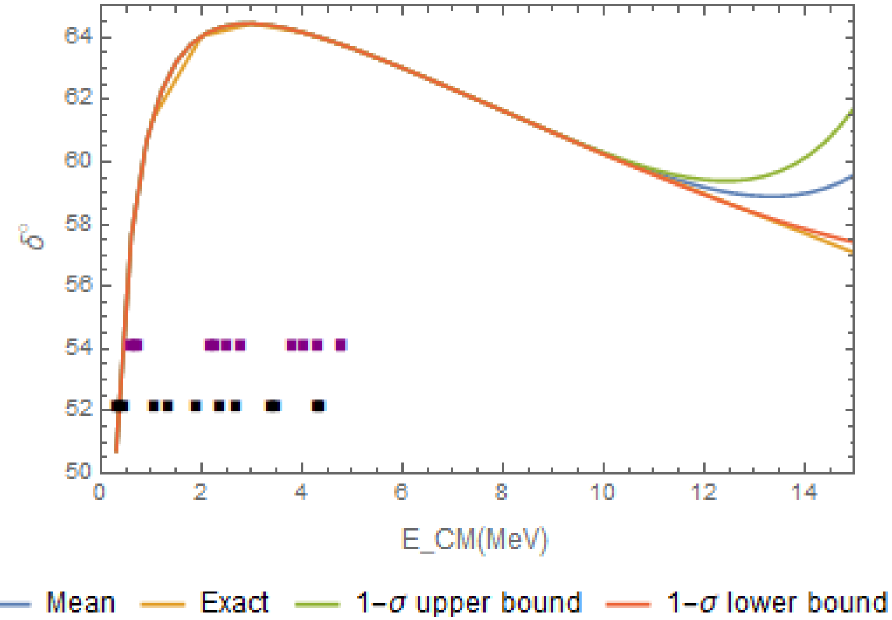

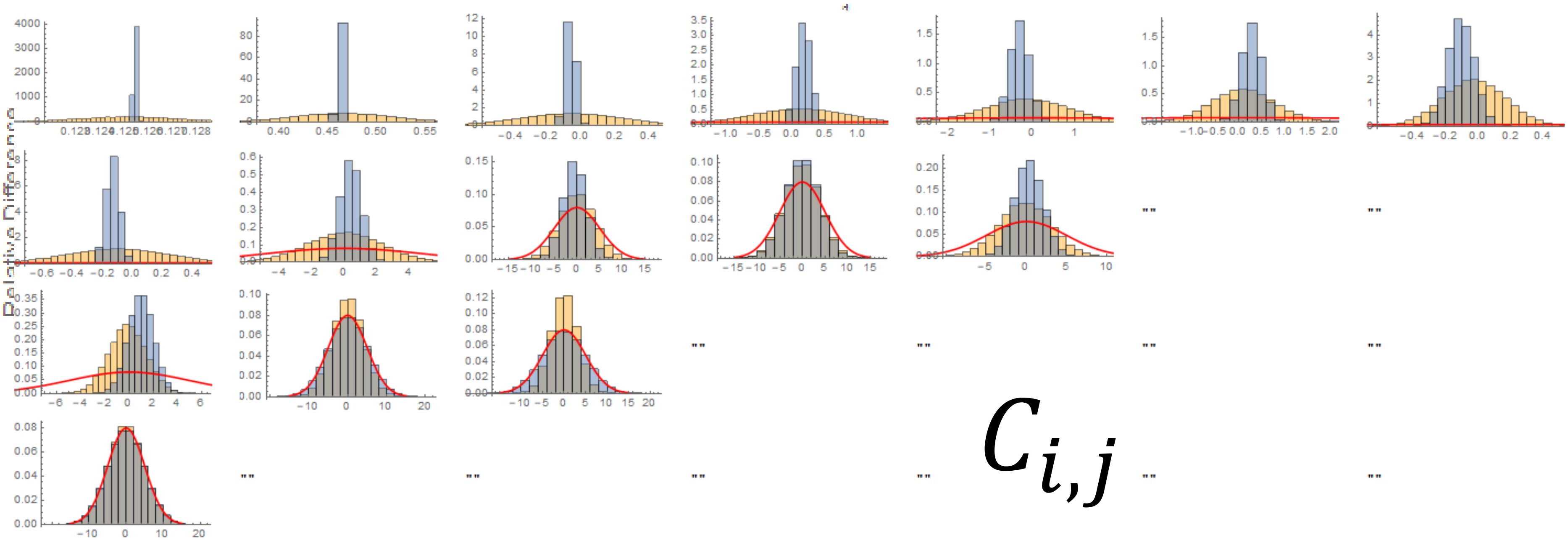

Summary and outlook

- The improved Busch formula can be used to infer scattering from structure calculation
- Test on $\mathrm{n} - \alpha$ is encouraging
- It works for NN system in the range of its validity
- Working with P. Narvatil on $\mathrm{n} - \alpha$
- Also applying it to study $\mathrm{n} -^{24} O$ with G. Chan, R. Stroberg, and J. Holt
- Consider generalizing it to study two-cluster reactions and three-cluster systems
- It would be interesting to consider the connection between this method and the infrared extrapolation used in structure calculation