An Operator Algebra Approach to Entropy Spread and Quantum Chaos

With Marius Junge
Some Parts with Li Gao
Information in Time & Subsystems
Channels as Open Systems

\[ \Phi(\rho) = \text{tr}_E \left( U_\Phi (\rho \otimes \omega_E) U^\dagger_\Phi \right) \]

Open system process evolution

Trace out environment for observer

Unitary evolution of full system

Initial state of environment

Density matrix

(Slengspring dilation)

Unitaries conjugate density matrix

We also consider what environment sees:

\[ \Phi^c(\rho) = \text{tr}_B \left( U_\Phi (\rho \otimes \omega_E) U^\dagger_\Phi \right) \]

Closed System

Open System

(Stinespring dilation)

Slight generalization of complementary channel...

\[ U = e^{-iHt} \]

(Quench)

E (Environment)
Channels & Shannon Theory

- Goal: maximize entropy transmission to Bob, minimize to Eve
- Capacity is hard to calculate
  - Inspires new perturbation techniques

\[ \Phi(\rho) = \operatorname{tr}_E(U(\rho \otimes |0\rangle_E)U^\dagger) \]
Channels as Views of a System

- Original: accessible subsystems
  - Inaccessible subsystems ~ environment

But we don’t always have the whole quantum system (e.g. classical registers)!

Source
\[ \rho \]

View
\[ \Phi \]

Perceived Ideal State
\[ \Phi(\rho) \]
Channels as Views of a System

- Original: accessible subsystems
  - Inaccessible subsystems ~ environment
  
  But we don’t always have the whole quantum system (e.g. classical registers)!

- Refinement: subalgebras of observables
  - Qubit ~ combinations of Pauli matrices
  - Classical ~ single Pauli matrix
  - Combination ~ conditional expectation
    - Combination of partial traces, measurements, etc.

  But we don’t always have an ideal probe!

Source \( \rho \)  
View \( \Phi \)  
Perceived Ideal State \( \Phi(\rho) \)
Channels as Views of a System

- Original: accessible subsystems
  - Inaccessible subsystems ~ environment
  But we don’t always have the whole quantum system (e.g. classical registers)!
- Refinement: subalgebras of observables
  - Qubit ~ combinations of Pauli matrices
  - Classical ~ single Pauli matrix
  - Combination ~ conditional expectation
    - Combination of partial traces, measurements, etc.
  But we don’t always have an ideal probe!
- Can represent system access via channel
Channels as Views of a System

Resource Theory

Operations: can’t pull info out of environment!

This is equivalent to Heisenberg-picture notion that Bob may only apply operations to his own channel (in this case $\Phi$).

2 Well-Known Conditional Expectations

$$\mathcal{E}_A(\rho) = \text{tr}_{A^c}(\rho) \otimes (1/|A^c|)_{A^c}$$

$$\mathcal{E}_{X_A} \sim \text{measure } X \text{ on } A$$

Recover some established models...

$$\mathcal{E}_\mathcal{N} \sim \text{conditional expectation to subalgebra } \mathcal{N}$$

Tensor-Decomposable Spaces

Traces are conditional expectations.

\[ \text{tr}_B \sim \mathcal{E}_{AC}, \quad \text{tr}_A \sim \mathcal{E}_{BC} \]

\[ H(AC) + H(BC) \geq H(ABC) + H(C) \]

(Strong Subadditivity, Local Operations)
Mutually Unbiased Bases

\[ H(X_A, C) + H(Z_A, C') \geq H(AC') + H(C) + \log |A| \]

(Uncertainty Principle w/ Memory – Berta et. al. 2010)
(relates to coherence resource theories)

...again, conditional expectations.
Common Aspect: Commuting Squares

Observe...

\[ \text{tr}_A \text{tr}_B = \text{tr}_B \text{tr}_A \]

\[ \mathcal{E}_X \mathcal{E}_Z = \mathcal{E}_Z \mathcal{E}_X \]

\[ H(A^c) + H(B^c) \geq H(A^c \cup B^c) + H(A^c \cap B^c) \]
Common Aspect: Commuting Squares

\[ \Phi^c \cdot \Psi^c = \tilde{\Psi}^c \cdot \Phi^c \]

\[ A^c \leftarrow \Phi^c \]

\[ \tilde{\Psi}^c \]

\[ "A^c \cup B^c" \leftarrow \Phi^c \]

\[ "A^c \cap B^c" \]

\[ \Psi^c \]

\[ B^c \]

Observe...

\[ \text{tr}_A \text{tr}_B = \text{tr}_B \text{tr}_A \]

\[ \mathcal{E}_X \mathcal{E}_Z = \mathcal{E}_Z \mathcal{E}_X \]

\[ H(A^c) + H(B^c) \geq H(A^c \cup B^c) + H(A^c \cap B^c) \]

Now interpret both channels as views of the system?
Common Aspect: Commuting Squares

\[ \Phi^c \cdot \Psi^c = \tilde{\Psi}^c \cdot \Phi^c \]

\[ A^c \leftarrow \Phi^c \]

\[ \tilde{\Psi}^c \]

\[ \Psi^c \rightarrow \Psi^c \]

Now interpret both channels as views of the system?

\[ \text{Requires: } \Psi^c(U^\dagger_{\Phi}(1^A \otimes \eta^E)U_{\Phi}) = 1^A \otimes \tilde{\Psi}^c(\eta^E) \forall \eta^E \]

General Theorem

\[ H(\Phi^c(\rho)) + H(\Psi^c(\rho)) \geq H((\Phi^c \cdot \Psi^c)(\rho)) + H(\rho) \]

Known for conditional expectations by Petz (1991). Now for quantum channels!*

Compatible with recovery map enhancement of arXiv:1509.07127

*(with some caveats around factorizability & unitality)
Proof Sketch (Data Processing)

Let...

\[ \tilde{\rho} = U_\Phi (\rho \otimes \omega^E) U_\Phi^\dagger \]

Then (in sandwiched p-Rényi relative entropy)...

\[ \tilde{D}_p \left( U_\Phi (\rho \otimes \omega^E) U_\Phi^\dagger \left| \begin{array}{c} \frac{1}{|A|} \\ \otimes \Phi^c (\sigma) \end{array} \right. \right) \]

\[ \geq \tilde{D}_p \left( \tilde{\Psi}^c (U_\Phi (\rho \otimes \omega^E) U_\Phi^\dagger) \left| \begin{array}{c} \frac{1}{|A|} \\ \otimes \Phi^c (\sigma) \end{array} \right. \right) \]

\[ = \tilde{D}_p \left( \tilde{\Psi}^c (\tilde{\rho}) \left| \begin{array}{c} \frac{1}{|A|} \\ \otimes \tilde{\Psi}^c (\Phi^c (\sigma)) \end{array} \right. \right) \]

\[ = \tilde{D}_p \left( U_\Phi (\Psi^c (\rho) \otimes \omega^E) U_\Phi^\dagger \left| \begin{array}{c} \frac{1}{|A|} \\ \otimes \Phi^c (\Psi^c (\sigma)) \end{array} \right. \right) \]
Applications

Field Theory

- No tensor product structure
  (Witten 2018)
  arXiv:1803.04993
- Observable algebras still valid, SSA useful
  e.g. (Casini & Huerta 2006)
  arXiv:cond-mat/0610375
Applications

Field Theory

- No tensor product structure (Witten 2018)
  arXiv:1803.04993
- Observable algebras still valid, SSA useful
  e.g. (Casini & Huerta 2006)
  arXiv:cond-mat/0610375

Q Computing

- Traditional model: uniform qubit errors, error correction
- Realistic errors may be more complicated (basis bias, multi-qubit errors, etc.)
- Channels can build in noise
Applications

Field Theory
● No tensor product structure (Witten 2018) arXiv:1803.04993
● Observable algebras still valid, SSA useful e.g. (Casini & Huerta 2006) arXiv:cond-mat/0610375

Q Computing
● Traditional model: uniform qubit errors, error correction
● Realistic errors may be more complicated (basis bias, multi-qubit errors, etc.)
● Channels can build in noise

A Unified Resource Theory of Entanglement & Coherence
● Combines & converts between entanglement & coherence
  See arXiv:1710.10038 w/ Li Gao & Marius Junge
An Abridged Example...

\[
\rho = \left( |0\rangle_A \right) \otimes \left( \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)_C \otimes \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)_D \right) \otimes \left( |0\rangle_B \right)
\]

Alice
\[
\Psi^c = \text{tr}_B \otimes \mathcal{E}_{XC} \otimes \mathcal{E}_{XD}
\]
Alice has A, X basis access to C & D

Bob
\[
\Phi^c = \text{tr}_A \otimes \mathcal{E}_{ZC} \otimes \mathcal{E}_{ZD}
\]
Bob has B, Z basis access to C & D

Converts to...

\[
\rho_f = \frac{1}{\sqrt{2}} \left( |00\rangle_{AB} + |11\rangle_{AB} \right) \otimes (\ldots)_{CD}
\]
This is an EPR pair!

Single qubit also has some entanglement analogue!
Many-body Physics
Continuous Time Quantum Channels

Quantum Hamiltonian + Stochastic Markov Process = Lindbladian

\[ \frac{d}{dt} \rho = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \left\{ L_i^\dagger L_i, \rho \right\} \right) \]

Hamiltonian Part

Quantum Markov Semigroup Part

\[ \mathcal{L} \equiv \text{Lindbladian - open system time evolution generator} \]

(not to be confused with the Lagrangian)

\[ \frac{d}{dt} \rho = \mathcal{L}(\rho) \implies \rho(t) = \Phi_t(\rho_0) = e^{\mathcal{L}t}(\rho_0) \]

Semigroup: \( \Phi_t(\Phi_s(\rho)) = \Phi_{t+s}(\rho) \)

\[ U = e^{-iHt} \]
Coherent Regime

\[
\frac{d}{dt}\rho = -\frac{i}{\hbar}[\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right)
\]

- Favorable regime for quantum computing
  - Ideal: noise below error correction threshold
  - NISQ: Hamiltonian speed > stochastic

- Complicated to predict
  - Capable of quantum advantage => hard to simulate classically or predict analytically
These are easier to understand – admit *decay estimates*. But bad for quantum advantage.
Stochastic Regime

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right)
\]

These are easier to understand – admit decay estimates. But bad for quantum advantage.

Let \( R \) generate a Markov process.

Let \( \mathcal{N} \) be \( R \)'s subalgebra of invariant observables.

Recall the Fisher information \( \mathcal{I}_R(\rho) \equiv \text{tr}(R(\rho) \ln(\rho)) \)
Stochastic Regime

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \left\{ L_i^\dagger L_i, \rho \right\} \right)
\]

These are easier to understand – admit decay estimates. But bad for quantum advantage.

Let \( R \) generate a Markov process.

Let \( \mathcal{N} \) be \( R \)'s subalgebra of invariant observables.

Recall the Fisher information \( \mathcal{I}_R(\rho) \equiv \text{tr}(R(\rho) \ln(\rho)) \)

If \( \nu \mathcal{I}_{Id-\mathcal{E}_\mathcal{N}}(\rho) \leq \mathcal{I}_R(\rho) \) for some constant \( \nu > 0 \), then...

\[
\mathcal{I}_{Id-\mathcal{E}_\mathcal{N}}(e^{-\mathcal{L}_t}(\rho)) \leq e^{-\nu t} \mathcal{I}_{Id-\mathcal{E}_\mathcal{N}}(\rho)
\]

See arXiv:1807.08838 w/ Marius Junge & Li Gao
How Does This Work?

Heat equation...

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

...becomes equation for entropy!

Optimal Mass Transport
(minimal cost to move a pile of dirt)

Concentration Inequalities
(e.g. Talagrand)
Entropy Spread in ‘Local’ Hamiltonians

$q$-site interactions: \( \vec{i} = (i_1, \ldots, i_q) \), \( H_{\vec{i}} = j_{\vec{i}} c_{i_1} \ldots c_{i_q} \), \( n \) sites

\( q \)-local Hamiltonian: \( H = \sum_{\vec{i}} H_{\vec{i}} \)

Any \( q \) sites can interact!

- Computer Science → local
- Physics → highly non-local

\( j_{\vec{i}} \) is often a random variable

Chaotic: \( \langle W(t), V, W(t), V \rangle = f_0 - \frac{f_1}{n} \exp(\lambda_L t) + \ldots \)

Decay of out-of-time-order correlator

Input-Output Mutual Info \( \sim 1 - \# \exp(\lambda_L (t - t_*) ) + \ldots \)


Complete graph by By David Benbennick - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=511711
Unitary $\rightarrow$ Stochastic

As $n \rightarrow \infty$, coupling $\langle j^2_i \rangle \sim 1/n^{q-1}$...

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{ L_i^\dagger L_i, \rho \} \right)
\]

...subsystem Lindbladian $\rightarrow$ dominant stochastic.
Comparison & Decay Estimates

Mostly proven for Majorana Clifford Models (e.g. SYK)

\[ D(e^{-itH} \rho e^{itH} \big| \mathcal{E}_A(e^{-itH} \rho e^{itH})) \leq c_q t \left( \frac{|A|^2}{\sqrt{n^{q-1}}} + t|A|^4 \right) \]

for \( \rho \in \text{Cl}_A \)

...for Radomacher-weighted coefficients.

Gaussian \( \Rightarrow \) \( c_q \rightarrow 8c_q \sqrt{q \log n} \)

(with high probability)

Conditional expectation onto A (subsystem or subalgebra)

q-dependent, constant in n
Analog ↔ Digital Approximation

**Usual Tactic:** perturb around ground state, weak interactions (scattering), etc.

- This breaks down for *strong interactions far from a ground or invariant state*

**Instead:** Approximate by or perturb around random circuit for highly interconnected systems.

For \( t \left\langle |j_\vec{i}|^2 \right\rangle q^q[H_\vec{i}, H] \ll 1 \ldots \)

\[
\exp(itH) \approx \text{Avg}_{\sigma \in S_N} \prod_{\vec{i}} \exp(itH_{\sigma(\vec{i})})
\]

...but this is potentially too simple.
Comparison & Decay Estimates

Mostly proven for Majorana Clifford Models (e.g. SYK)

\[ D(e^{-itH} \rho e^{itH} | \mathcal{E}_A(e^{-itH} \rho e^{itH})) \leq c_q t \left( \frac{|A|^2}{\sqrt{n^q-1}} + t|A|^4 \right) \]

for \( \rho \in \mathbb{C}l_A \)

Conditional expectation onto \( A \) (subsystem or subalgebra)

\( \text{q-dependent, constant in n} \)

Effective density matrix for environment

\[ \mathcal{E}_A(e^{-itH} \rho e^{itH}) \approx \sum_{k,l} f_{k,l} c_k \rho c_l^\dagger \rightarrow \mathcal{E}_{X_A}(\rho) \text{ as } \langle |j_i|^2 \rangle \rightarrow \infty \]

Subalgebra of X-basis of A

\[ \Rightarrow H(\mathcal{E}_{X_A}(\rho)) \geq H(\mathcal{E}_A(e^{-itH} \rho e^{itH})) \geq H(\mathcal{E}_{X_A}(\rho)) - \tau(f \log f) \]

Via complex interpolation methods of arXiv:1609.08594 w/ Gao & Junge

"Capacity Estimates via Comparison with TRO Channels"
Concluding Remarks
Analogy: Chaos, Entropy, Computing

Turing Table by Wvbailey - Raw material created in Excel then cut, pasted and massaged in Autodesk SketchBook Pro., CC BY-SA 3.0, https://en.wikipedia.org/w/index.php?curid=48901372


Relatively Doable

- Long-range photonics
- Small ion arrays
- Long-term storage

Isolated Qudits

Interacting Qudits

- Superconducting processors
- Large ion arrays
- Adiabatic QC
Much Harder: Do Both, Dynamically

\[ \frac{d}{dt} \rho = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \sum_{i=1}^{d^2-2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \left\{ L_i^\dagger L_i, \rho \right\} \right) \]

- Want *malleable*, *selective*, *strong* interactions
- Want *dynamically redefinable* ‘locality’
- *When is the noise not a problem?*
Acknowledgements

- Li Gao & Marius Junge
- Fermilab & the conference organizers
  - Kiel Howe
  - The Boulder Summer School

- This research is supported by the NSF Graduate Research Fellowship Program under Grant Number DGE-1144245. Marius Junge is partially supported by NSF-DMS 1501103. The authors thank the Institut Henri Poincaré at Pierre and Marie Curie University for hosting them during September–October 2017.

\[
\frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho] + \sum_{i=1}^{d^2 - 2} \gamma_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right)
\]

\[
H(\Phi^c(\rho)) + H(\Psi^c(\rho)) \geq H((\Phi^c \cdot \Psi^c)(\rho)) + H(\rho)
\]