Integrable and chaotic mappings of the plane with polygon invariants.

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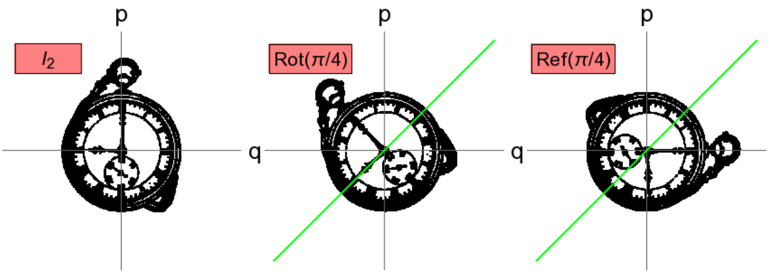
1. DEFINITIONS, HISTORICAL REMARKS AND TOOLS WE NEED

We will consider area-preserving mappings of the plane

$$\begin{aligned} q' &= q'(q,p), \\ p' &= p'(q,p), \end{aligned} \qquad \det \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix} = 1.$$



Identity, Id | Rotation, Rot | Reflection*,**, Ref

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

*The reflection is anti area-preserving transformation, $\det J = -1$.
**In addition, $\mathrm{Ref}^2 = \mathrm{Id}$ (or $\mathrm{Ref} = \mathrm{Ref}^{-1}$). Transformations which satisfy this property are called *involutions*.

More on reflections and rotations

$$\begin{aligned}
\mathrm{Rot}(\theta) \circ \mathrm{Rot}(\phi) &= \mathrm{Rot}(\theta + \phi) \\
\mathrm{Ref}(\theta) \circ \mathrm{Ref}(\phi) &= \mathrm{Rot}(2\,[\theta - \phi]) \\
\mathrm{Rot}(\theta) \circ \mathrm{Ref}(\phi) &= \mathrm{Ref}(\phi + \tfrac{1}{2}\theta) \\
\mathrm{Ref}(\phi) \circ \mathrm{Rot}(\theta) &= \mathrm{Ref}(\phi - \tfrac{1}{2}\theta)
\end{aligned}$$

A map T in the plane is called *integrable*, if there exists a non-constant real valued continuous functions $\mathcal{K}(q,p)$, called *integral*, which is invariant under T:

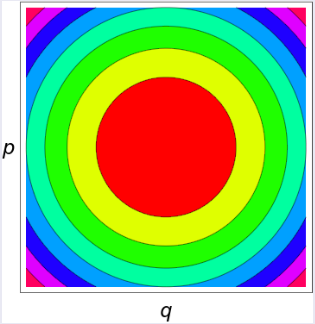
$$\forall\,(q,p): \qquad \mathcal{K}(q,p)=\mathcal{K}(q',p')$$

where primes denote the application of the map, $(q',p')=\mathrm{T}(q,p)$.

Example. Rotation transformation

$$\mathrm{Rot}(\theta): \quad \begin{aligned} q' &= q\,\cos\theta - p\,\sin\theta \\ p' &= q\,\sin\theta + p\,\cos\theta \end{aligned}$$

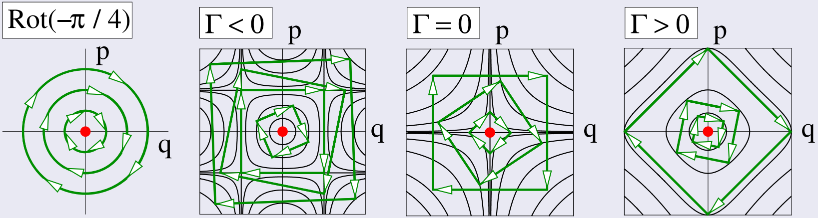
has the integral $\mathcal{K}(q,p)=q^2+p^2$.



If θ and π are commensurable, then transformation $\mathrm{Rot}(\theta)$ has infinitely many invariants of motion.

Example. Rotations through angles $\pm\pi/4$ has another invariant

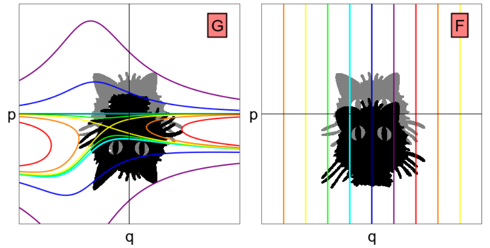
$$\mathcal{K}(q,p) = q^2 p^2 + \Gamma(q^2 + p^2), \qquad \forall\, \Gamma.$$



Thin lens transformation, F, and nonlinear vertical shear, G,

$$\mathrm{F}: \quad \begin{aligned} q' &= q, \\ p' &= p + f(q), \end{aligned} \qquad\qquad \mathrm{G}: \quad \begin{aligned} q' &= q, \\ p' &= -p + f(q), \end{aligned}$$

$$\mathrm{F} = \mathrm{G} \circ \mathrm{Ref}(0), \qquad\qquad \mathrm{G} = \mathrm{F} \circ \mathrm{Ref}(0).$$



Transformation G is anti area-preserving involution, $\mathrm{G}^2 = \mathrm{Id}$.

A map T is said to be *reversible* if there is a transformation R_0, called the *reversor*, such that

$$\mathrm{T}^{-1} = \mathrm{R}_0 \circ \mathrm{T} \circ \mathrm{R}_0^{-1}.$$

In the important special case, where R_0 is involutory

$$\mathrm{T}^{-1} = \mathrm{R}_0 \circ \mathrm{T} \circ \mathrm{R}_0 \qquad \text{or} \qquad \mathrm{R}_0 \circ \mathrm{T} \circ \mathrm{R}_0 \circ \mathrm{T} = \mathrm{Id}.$$

Hence, if we set $\mathrm{R}_1 = \mathrm{R}_0 \circ \mathrm{T}$, we see that R_1 is also involutory. Moreover we have

$$\mathrm{T} = \mathrm{R}_0 \circ \mathrm{R}_1 \qquad \text{or} \qquad \mathrm{T}^{-1} = \mathrm{R}_1 \circ \mathrm{R}_0$$

so that T is the product of two involutory transformations.

Arnold-Liouville theorem

Integrable map can be written in the form of a Twist map

$$\begin{aligned} J_{n+1} &= J_n, \\ \theta_{n+1} &= \theta_n + 2\,\pi\,\nu(J) \mod 2\,\pi, \end{aligned}$$

where $|\nu(J)| \leq 0.5$ is the rotation number, θ is the angle variable and J is the action variable, defined by the mapping T as

$$J = \frac{1}{2\,\pi} \oint p\,\mathrm{d}q.$$

Poincaré rotation number

Rotation number represents the average increase in the angle per unit time (average frequency)

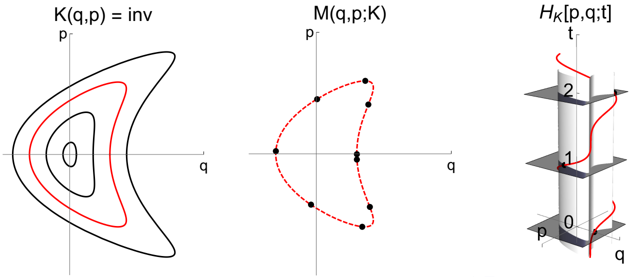
$$\nu = \lim_{n\to\infty} \frac{\mathrm{T}^n(\theta) - \theta}{n}.$$

Theorem (Danilov)

Let $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be the area-preserving integrable map with invariant of motion $\mathcal{K}(q,p) = \mathcal{K}(q',p')$. If constant level of invariant is compact, then a Poincaré rotation number is

$$\nu = \int_q^{q'} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q \Bigg/ \oint \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q$$

where integrals are assumed to be along invariant curve.



I. Contribution of Edwin McMillan

From "*A problem in the stability of periodic systems*" (1970)

In the Spring of 1967 I attended a theoretical seminar at which Professor René de Vogelaere spoke concerning the stability of non-linear periodic systems. The motivation was storage rings, with beams focused by azimuthally varying fields ("strong focusing"); the question, the effect of non-linear terms on an otherwise stable system; the presentation I found utterly fascinating. It recalled another seminar I attended at Princeton* over a third of a century earlier, at which G. D. Birkhoff discussed the stability of the solar system. I remember none of the detail of that earlier seminar, but I have a strong memory of how an apparently simple situation led rapidly and unavoidably into a maze of complexity, leaving the original question "Is the motion of the system stable for infinite time?" unanswered.

I-1. McMillan form of the map

McMillan considered a special form of the map

$$\boxed{\mathrm{M}: \quad \begin{array}{l} q' = p, \\ p' = -q + f(p), \end{array}}$$

where $f(p)$ is called *force function* (or simply *force*).

a. Fixed point

$$p = q \ \cap \ p = \frac{1}{2} f(q).$$

b. 2-cycles

$$q = \frac{1}{2} f(p) \ \cap \ p = \frac{1}{2} f(q).$$

1D accelerator lattice with thin nonlinear lens, $\mathrm{T} = \mathrm{F} \circ \mathrm{M}$

$$\mathrm{M}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha\,\sin\Phi & \beta\,\sin\Phi \\ -\gamma\,\sin\Phi & \cos\Phi - \alpha\,\sin\Phi \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

$$\mathrm{F}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ F(y) \end{bmatrix},$$

where α, β and γ are Courant-Snyder parameters at the thin lens location, and, Φ is the betatron phase advance of one period.

Mapping in McMillan form after CT to (q,p), $\mathrm{T} = \widetilde{\mathrm{F}} \circ \mathrm{Rot}(-\pi/2)$

$$q = y,$$
$$p = y\,(\cos\Phi + \alpha\,\sin\Phi) + \dot{y}\,\beta\,\sin\Phi,$$

$$\boxed{\widetilde{\mathrm{F}}(q) = 2\,q\,\cos\Phi + \beta\,F(q)\,\sin\Phi}.$$

Turaev theorem

INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **16** (2003) 123–135 PII: S0951-7715(03)35323-X

Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps

Dmitry Turaev

Recommended by C Liverani

Abstract

It is shown that every symplectic diffeomorphism of R^{2n} can be approximated, in the C^∞-topology, on any compact set, by some iteration of some map of the form $(x, y) \mapsto (y + \eta, -x + \nabla V(y))$ where $x \in R^n$, $y \in R^n$, and V is a polynomial $R^n \to R$ and $\eta \in R^n$ is a constant vector. For the case of area-preserving maps (i.e. $n = 1$), it is shown how this result can be applied to prove that C^r-universal maps (a map is universal if its iterations approximate dynamics of all C^r-smooth area-preserving maps altogether) are dense in the C^r-topology in the Newhouse regions.

I-2. McMillan condition for invariant curve

a. Consider a decomposition of map in McMillan form

$$\mathrm{T} = \mathrm{F} \circ \mathrm{Rot}(-\pi/2) = \mathrm{G} \circ \mathrm{Ref}(0) \circ \mathrm{Rot}(-\pi/2) = \mathrm{G} \circ \mathrm{Ref}(\pi/4).$$

b. Lines $p = q$ and $p = f(q)/2$ are sets of fixed points for reversors.

c. If $\mathcal{K}(q,p)$ is invariant under transformation T, then it is invariant under both, $\mathrm{Ref}(\pi/4)$ and G:

$$\mathcal{K}(q,p) = \mathcal{K}(p,q), \qquad \mathcal{K}(q,p) = \mathcal{K}(q,-p+f(q)).$$

d. Solving for $p = \Phi(q)$ from the invariant $\mathcal{K}(q,p) = \mathrm{const}$

$$\boxed{f(q) = \Phi(q) + \Phi^{-1}(q)}.$$

Example. Heńon map, $f(p) = 2\,p^2$.

Heńon map

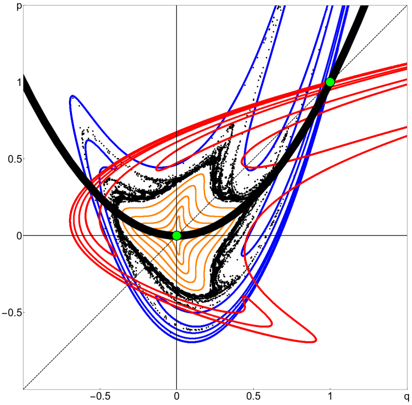
$$\mathrm{M}: \quad \begin{aligned} q' &= p, \\ p' &= -q + 2\,p^2. \end{aligned}$$

Symmetry lines:

$$p = q, \qquad\qquad p = q^2.$$

Fixed points:

$$(0,0), \qquad\qquad (1,1).$$



II. Suris theorem and recurrence $x_{n+1} + x_{n-1} = f(x_n)$.

INTEGRABLE MAPPINGS OF THE STANDARD TYPE

Yu. B. Suris

UDC 517.9

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon F(x_n, \varepsilon), \tag{1}$$

$$F(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k f_k(x), \quad |\varepsilon| < \varepsilon_0. \tag{2}$$

THEOREM. Equation (1) has a nontrivial symmetric integral of the form

$$\Phi(x, y, \varepsilon) = \Phi_0(x, y) + \varepsilon\Phi_1(x, y), \tag{4}$$

holomorphic in the domain $|x - y| < \delta_0$, in the following and only in the following three cases:

a) $F(x, \varepsilon) = (A + Bx + Cx^2 + Dx^3)/(1 - \varepsilon(E + Cx/3 + Dx^2/2))$,

$$\Phi_0(x, y) = (x - y)^2/2, \quad \Phi_1(x, y) = -A(x + y)/2 - Bxy/2 - Cxy(x + y)/6 - Dx^2y^2/4 - E(x - y)^2/2.$$

б) $$F(x, \varepsilon) = \frac{2}{\omega\varepsilon} \operatorname{arctg}\left\{ \frac{\frac{\omega\varepsilon}{2}(A \sin \omega x + B \cos \omega x + C \sin 2\omega x + D \cos 2\omega x)}{1 - \frac{\omega\varepsilon}{2}(A \cos \omega x - B \sin \omega x + C \cos 2\omega x - D \sin 2\omega x + E)} \right\},$$

$$\Phi_0(x, y) = (1 - \cos \omega(x - y))/\omega^2, \quad \Phi_1(x, y) = (A(\cos \omega x + \cos \omega y) - B(\sin \omega x + \sin \omega y) + C \cos \omega(x + y) - D \sin \omega(x + y) + E \cos \omega(x - y))/2\omega$$

в) $$F(x, \varepsilon) = \frac{1}{\alpha\varepsilon} \ln \frac{1 + \alpha\varepsilon(B \exp(-\alpha x) + D \exp(-2\alpha x) - E)}{1 - \alpha\varepsilon(A \exp(\alpha x) + C \exp(2\alpha x) + E)},$$

$$\Phi_0(x, y) = (\operatorname{ch} \alpha(x - y) - 1)/\alpha^2, \quad \Phi_1(x, y) = (-A(e^{\alpha x} + e^{\alpha y}) + B(e^{-\alpha x} + e^{-\alpha y}) - Ce^{\alpha(x+y)} + De^{-\alpha(x+y)} - 2E \operatorname{ch} \alpha(x - y))/2\alpha.$$

III. Recurrence $x_{n+1} + x_{n-1} = |x_n|$

THE AMERICAN
MATHEMATICAL MONTHLY

Volume 90, Number 8 October 1983

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by February 29, 1984. The solver's full post-office address should be on each sheet.

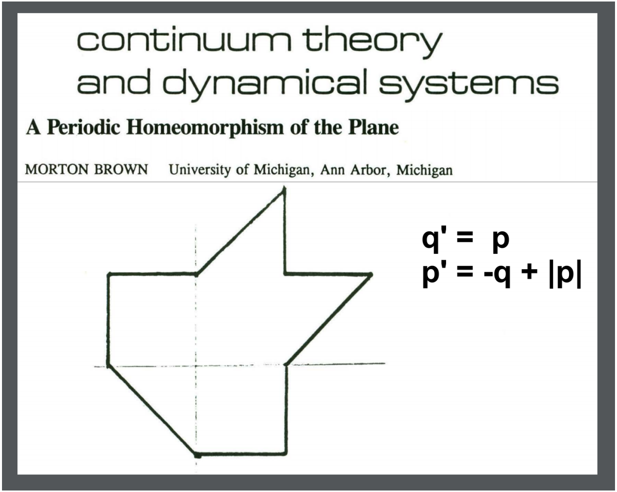
6439. *Proposed by Morton Brown, University of Michigan.*

Let $\{a_n\}$ be a sequence of real numbers satisfying the relation $a_{n+1} = |a_n| - a_{n-1}$. Prove that $\{a_n\}$ is periodic with period 9.

THE AMERICAN
MATHEMATICAL MONTHLY

Volume 92, Number 3 March 1985

IV. Periodic homeomorphism of the plane (1993)



V. Letter from Professor D. Knuth

"When I saw advanced problem 6439, I couldn't believe that it was 'advanced': a result like that has to be either false or elementary!"

"But I soon found that it wasn't trivial. There is a simple proof, yet I can't figure out how on earth anybody would discover such a remarkable result. Nor have I discovered any similar recurrence relations having the same property."

"So in a sense I have no idea how to solve the problem properly. Is there an 'insightful' proof, or is the result simply true by chance?"

VI. R. Devaney's Gingerbreadman map, $f(p) = |p| + 1$

Physica 10D (1984) 387–393
North-Holland, Amsterdam

A PIECEWISE LINEAR MODEL FOR THE ZONES OF INSTABILITY OF AN AREA-PRESERVING MAP

Robert L. DEVANEY*
Department of Mathematics, Boston University, Boston, Mass. 02215, USA

Received 14 March 1983

In this note we study the global behavior of the piecewise linear area-preserving transformation $x_1 = 1 - y_0 + |x_0|$, $y_1 = x_0$, of the plane. We show that there are infinitely many invariant polygons surrounding an elliptic fixed point. The regions between these invariant polygons serve as models for the "zones of instability" in the corresponding smooth case. For our model we show that some of these annular zones contain only finitely many elliptic islands. The map is hyperbolic on the complement of these islands and hence exhibits stochastic behavior in this region. Unstable periodic points are dense in this region.

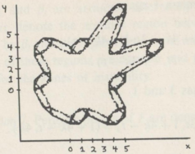


Fig. 2. 10,000 iterates of a single point in the region A_1. The inner boundary of A_1 is the outer boundary of B_0.

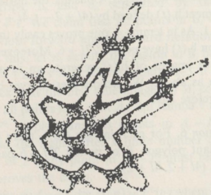
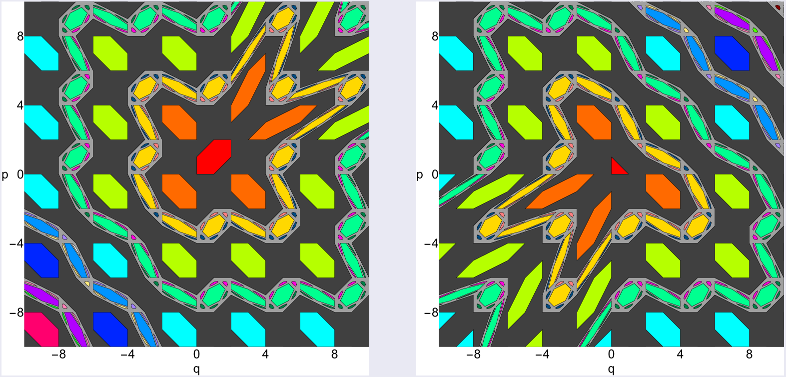


Fig. 3. The outer region is the ergodic region B_1; the inner region is B_0 as shown in fig. 1.

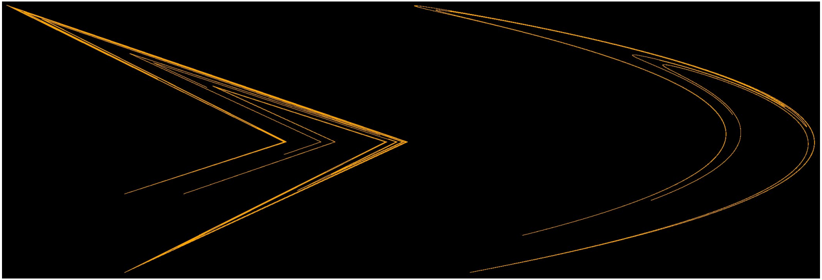
Gingerbreadman and Rabbit maps

$$q' = p$$
$$p' = -q \pm |p| + 1$$



V. Lozi and Hénon maps

$$\mathrm{M}_{\mathcal{L}}: \quad \begin{aligned} q' &= p \\ p' &= b\,q + 1 - a\,|p| \end{aligned} \qquad \mathrm{M}_{\mathcal{H}}: \quad \begin{aligned} q' &= p \\ p' &= b\,q + 1 - a\,p^2 \end{aligned}$$



2. PERIODIC INTEGER MAPS WITH POLYGON INVARIANTS

2.1 Linear maps with integer coefficients

Crystallographic restriction theorem

If A is an integer 2×2 matrix and $\mathrm{A}^n = \mathrm{Id}$ for some natural $n \in \mathbb{N}$, then

$$\boxed{n = 1, 2, 3, 4, 6}$$

corresponding to 2-, 3-, 4- and 6-fold rotational symmetries.

$n = 1, 2$

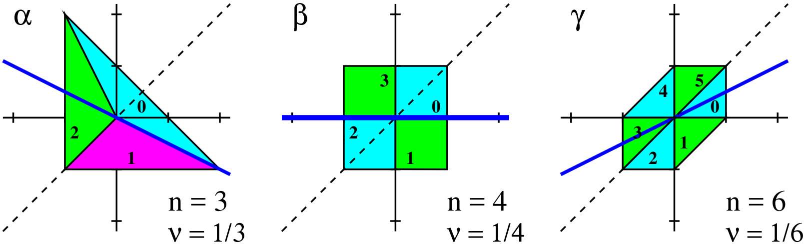
Transformations with a period $n = 1, 2$ are simply $\pm\mathrm{Id}$, which can be considered as a special cases of rotation through the angles θ equal to 0 or π,

$$\mathrm{Rot}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathrm{Rot}(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$n = 3, 4, 6$

Three other cases are given by mappings in McMillan form

$$\mathrm{M}_\alpha : \quad \begin{aligned} q' &= p, \\ p' &= -q - p, \end{aligned} \qquad \mathrm{M}_\beta : \quad \begin{aligned} q' &= p, \\ p' &= -q, \end{aligned} \qquad \mathrm{M}_\gamma : \quad \begin{aligned} q' &= p, \\ p' &= -q + p, \end{aligned}$$

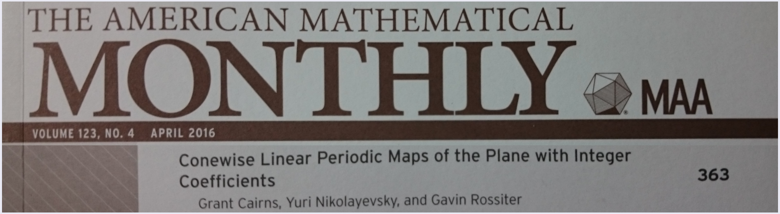


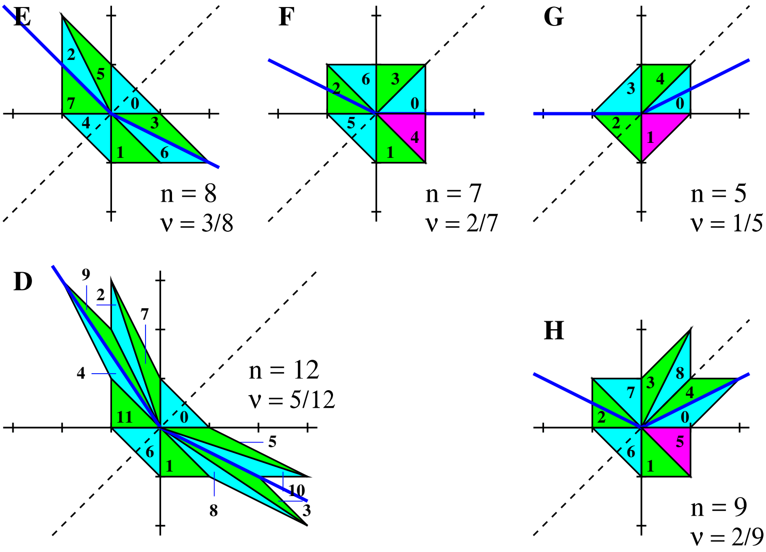
2.2 Maps linear on two half planes

CNR (Cairns, Nikolayevsky and Rossiter) theorem

Suppose that M is periodic continuous map of the plane that is linear with integer coefficients in each half plane $q \geq 0$ and $q < 0$. Then M has period

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \text{ or } 12.$$




E
n = 8
ν = 3/8
F
n = 7
ν = 2/7
G
n = 5
ν = 1/5
D
n = 12
ν = 5/12
H
n = 9
ν = 2/9

3. MAPS WITH POLYGON INVARIANTS

3.1 First good idea

Mappings E, F, G and D, H are in McMillan form with force

$$f(p) = \alpha\, p + \beta|p|, \qquad \alpha \pm \beta \in \mathbb{Z}.$$

What about affine generalization?

$$f(p) = \alpha\, p + \beta|p| + d.$$

Proposition 1. The change of coordinates $(q, p) \to (d\, q, d\, p)$ allows to reduce problem to cases $d = \pm 1$:

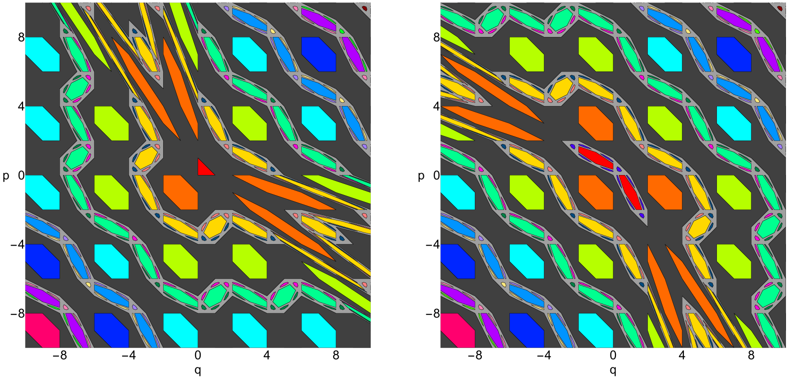
$$f(p) = \alpha\, p + \beta|p| \pm 1 \quad \text{for} \quad d \gtrless 0.$$

Proposition 2. The change of coordinates $(q, p) \to (-q, -p)$ allows to rewrite the force function as

$$f(p) = \alpha\, p \pm \beta|p| + 1 \quad \text{for} \quad d = \pm 1.$$

3.1.1 Zoo maps: Octopus and Crab

$$q' = p$$
$$p' = -q - 2\,p \pm |p| + 1$$

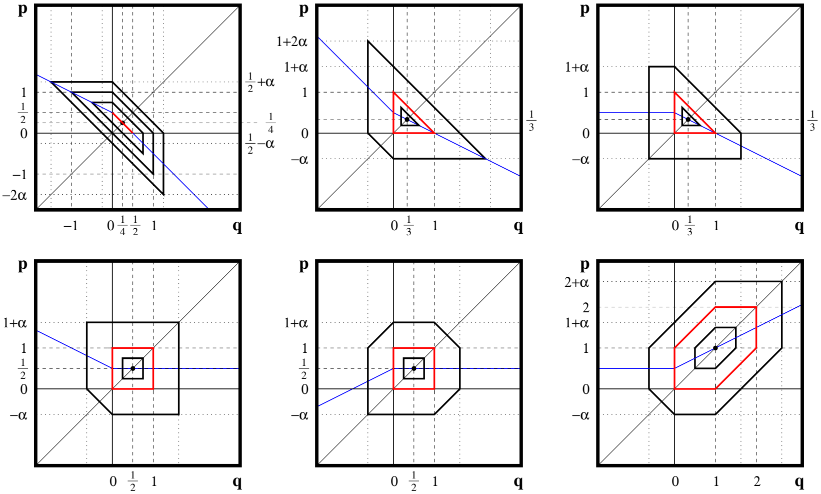


3.1.2 Nonlinear integrable maps with polygon invariants

$$\mathrm{M}_{\frac{1}{2}\,\frac{3}{8}} \text{ and } \mathrm{M}_{\frac{1}{3}\,\frac{3}{8}}: \qquad \begin{aligned} q' &= p, \\ p' &= -q + 1 \mp \frac{|p| \pm 3\,p}{2}, \end{aligned}$$

$$\mathrm{M}_{\frac{1}{3}\,\frac{2}{7}} \text{ and } \mathrm{M}_{\frac{1}{4}\,\frac{2}{7}}: \qquad \begin{aligned} q' &= p, \\ p' &= -q + 1 \mp \frac{|p| \pm p}{2}, \end{aligned}$$

$$\mathrm{M}_{\frac{1}{4}\,\frac{1}{5}} \text{ and } \mathrm{M}_{\frac{1}{6}\,\frac{1}{5}}: \qquad \begin{aligned} q' &= p, \\ p' &= -q + 1 \mp \frac{|p| \mp p}{2}, \end{aligned}$$


p
q
1
$\frac{1}{2}$
0
−1
−2α
−1
0
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{1}{2}$+α
$\frac{1}{4}$
$\frac{1}{2}$−α
1+2α
1+α
−α
$\frac{1}{3}$
2+α
2

Integrability and symmetries, $f(q) = \Phi(q) + \Phi^{-1}(q)$

$\mathrm{M}_{\frac{1}{4}\,\frac{2}{7}}$	$\mathcal{K}(q,p)=\alpha$	$(\partial\alpha/\partial p)^{-1}$	$-(\partial\alpha/\partial q)^{-1}$
$S_1:\ q=-\alpha$	$-q$	—	1
$S_2:\ p=-\alpha$	$-p$	-1	—
$S_3:\ q=1+\alpha$	$q-1$	—	-1
$S_4:\ p=1+\alpha$	$p-1$	1	—
$S_5:\ p=-q-\alpha$	$-q-p$	-1	1

$$f(q) = 1 + \frac{|q|-q}{2} = \begin{cases} S_2 + S_4 = 1, & q > 0, \\ S_4 + S_5 = 1 - q, & q < 0. \end{cases}$$

Poincaré rotation number

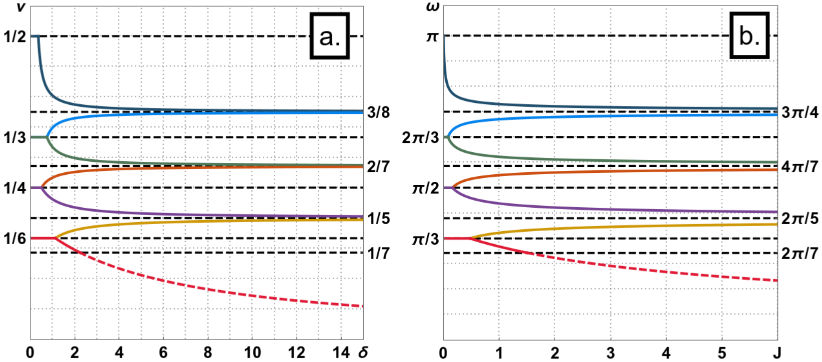
$$\int_q^{q'} \mathrm{d}t = \underbrace{\int_p^{1+\alpha} \mathrm{d}p}_{S_1} + \underbrace{\int_{-\alpha}^{q'} \mathrm{d}q}_{S_4} = 1 + 2\,\alpha,$$

$$\begin{aligned}\oint \mathrm{d}t &= \underbrace{\int_0^{1+\alpha} \mathrm{d}p}_{S_1} + \underbrace{\int_{-\alpha}^{1+\alpha} \mathrm{d}q}_{S_4} - \underbrace{\int_{1+\alpha}^{-\alpha} \mathrm{d}p}_{S_3} - \underbrace{\int_{1+\alpha}^{0} \mathrm{d}q}_{S_2} - \underbrace{\int_0^{-\alpha} \mathrm{d}q}_{S_5} \\ &= 4 + 7\,\alpha.\end{aligned}$$

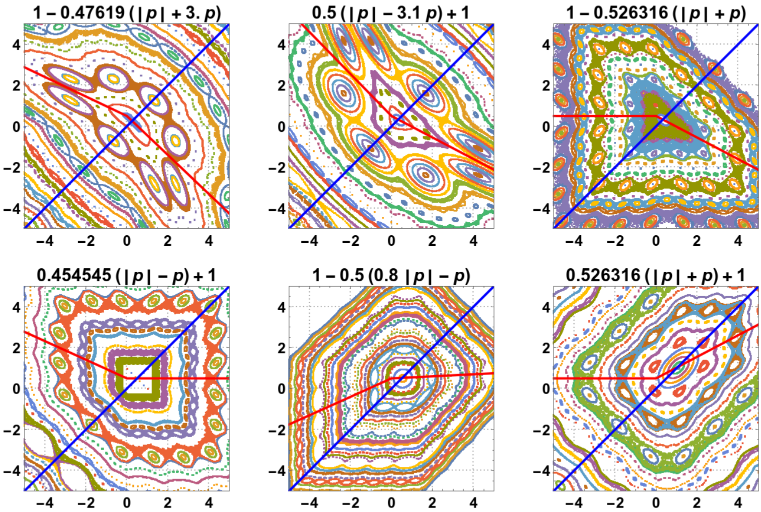
$$\boxed{\nu = \frac{1 + 2\,\alpha}{4 + 7\,\alpha}}$$

Action-angle variables

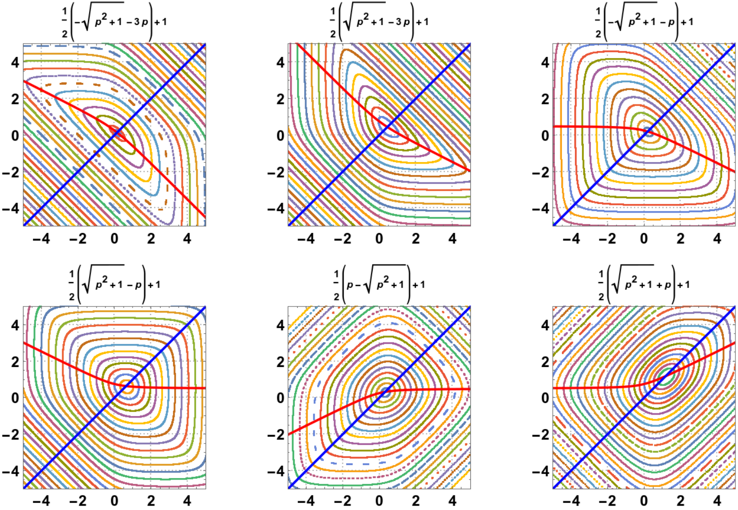
$$J_{n+1} = J_n = J(\alpha)$$
$$\theta_{n+1} = \theta_n + 2\,\pi\,\nu(\alpha) \pmod{2\pi}$$

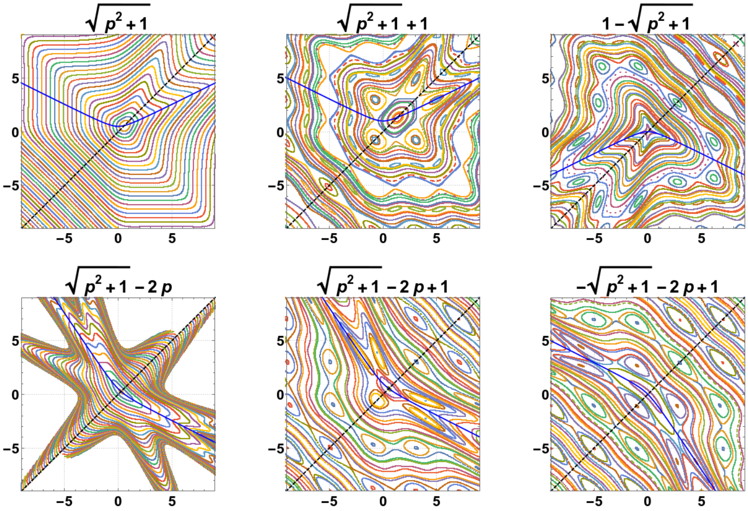


Perturbation of integrability



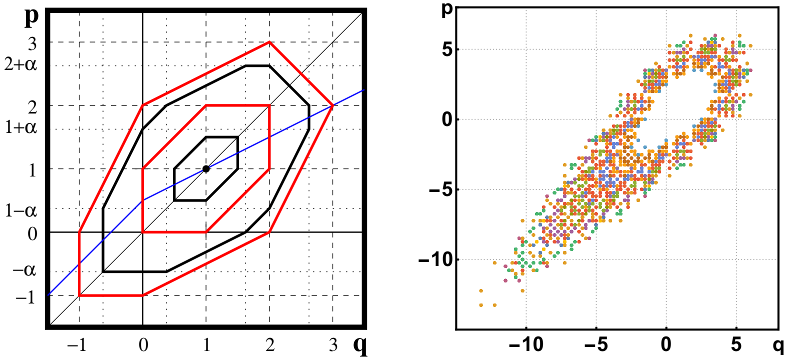
Application 1. Cohen-like mappings, $|p| \to \sqrt{p^2+1}$



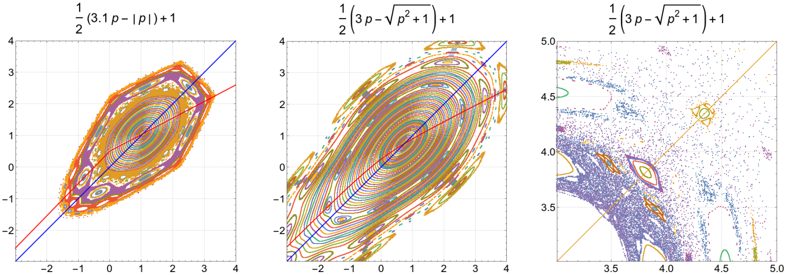

$\sqrt{p^2+1}$
$\sqrt{p^2+1}+1$
$1-\sqrt{p^2+1}$
$\sqrt{p^2+1}-2p$
$\sqrt{p^2+1}-2p+1$
$-\sqrt{p^2+1}-2p+1$
5
0
−5

3.1.3 Coexisting stochastic and integrable behavior

$$\mathrm{M}_{\frac{1}{6}\frac{1}{7}}: \quad q' = p$$
$$p' = -q + 1 - \frac{|p| - 3\,p}{2}$$

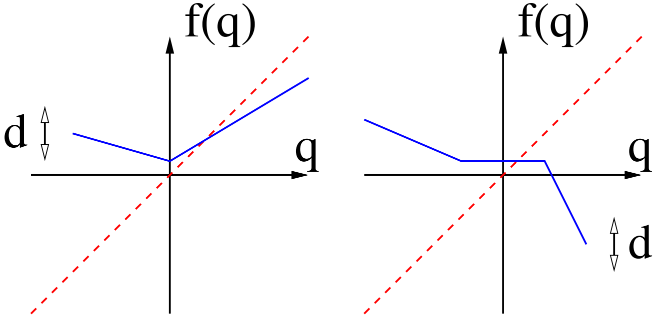


Perturbations of $\mathrm{M}_{\frac{1}{6}\,\frac{1}{7}}$



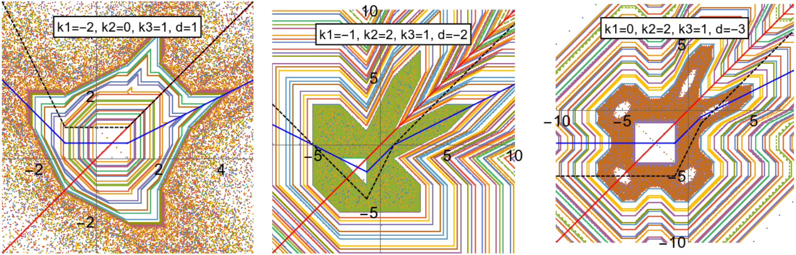
3.2 Second good idea: piecewise linear f with 3 segments

- Piecewise linear and continuous
- Integer coefficients
- 3 segments

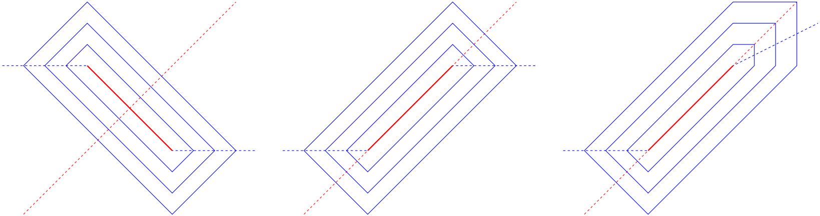


3.2.0 Pots and shards

3.2.0 Pots and shards

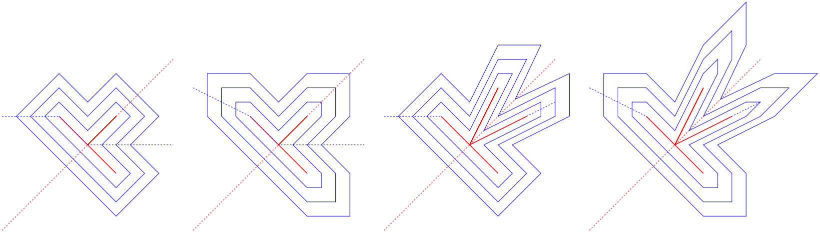


3.2.1 One layer maps 1



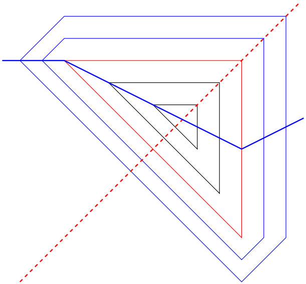
$$\nu = \frac{2+\alpha}{4+4\,\alpha} \qquad \nu = \frac{\alpha}{4+4\,\alpha} \qquad \nu = \frac{\alpha}{2+5\,\alpha}$$

3.2.1 One layer maps 2

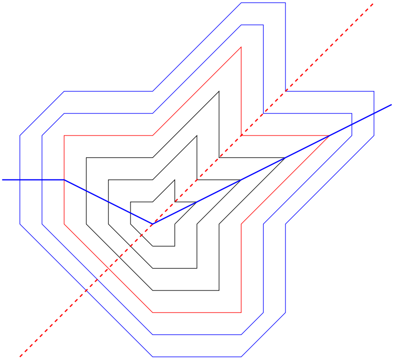


$$\nu = \frac{2+\alpha}{6+4\,\alpha} \qquad \nu = \frac{2+2\,\alpha}{6+7\,\alpha} \qquad \nu = \frac{2+\alpha}{14+5\,\alpha} \qquad \nu = \frac{2+2\,\alpha}{14+9\,\alpha}$$

3.2.2 Two layer maps

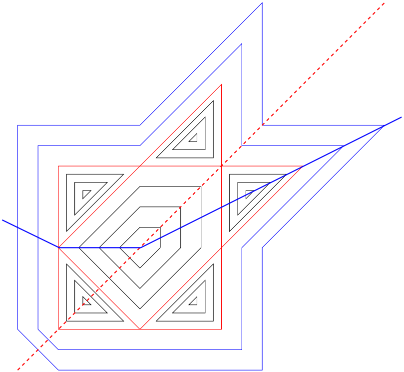


$$\nu = \frac{2+\alpha}{6+5\,\alpha}$$

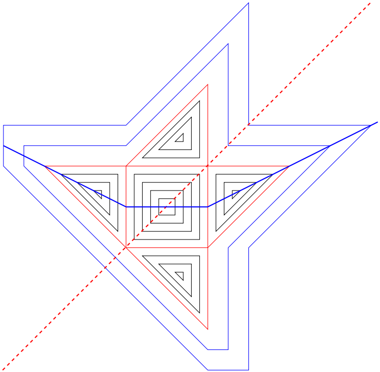


$$\nu = \frac{2+\alpha}{9+5\,\alpha}$$

3.2.3 Two layer maps with islands

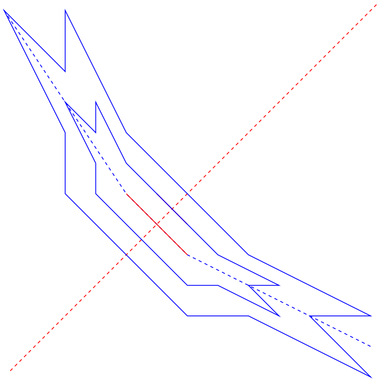


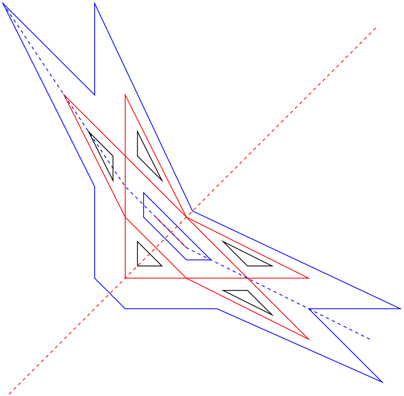
$$\nu = \frac{2 + 2\,\alpha}{10 + 9\,\alpha}$$



$$\nu = \frac{4 + 2\,\alpha}{16 + 9\,\alpha}$$

3.2.3 Two layer maps with islands 2



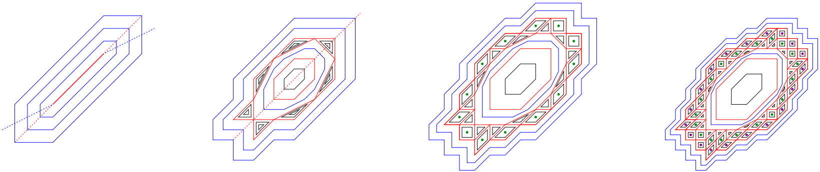


$$\nu = \frac{2 + 11\,\alpha}{4 + 24\,\alpha}$$

$$\nu =$$

3.2.4 System with discrete parameter 1

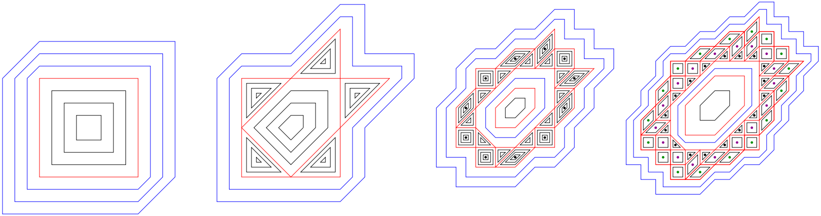
$$\boxed{\nu_1 = \frac{1}{6+\alpha}} \qquad \boxed{\nu_2 = \frac{2\,n+\alpha}{2+12\,n+6\,\alpha}}$$



$n=0$	$n=1$	$n=2$	$n=3$
$\nu_0 = 0$	$\nu_0 = \frac{1}{6}$	$\nu_0 = \frac{1}{6}$	$\nu_0 = \frac{1}{6}$
$\nu_1 = \frac{\alpha}{4+6\,\alpha}$	$\alpha^* = \frac{1}{1}$	$\alpha^* = \frac{1}{2}$	$\alpha^* = \frac{1}{3}$

3.2.4 System with discrete parameter 2

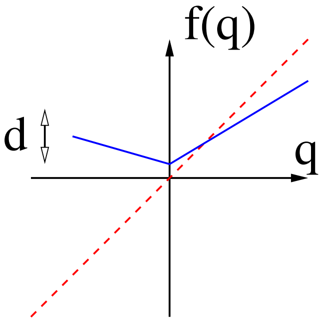
$$\boxed{\nu_1 = \frac{1+\alpha}{6+5\,\alpha}} \qquad \boxed{\nu_2 = \frac{2+2\,n+\alpha}{10+12\,n+6\,\alpha}}$$



$n = -$	$n = 0$	$n = 1$	$n = 2$
$\nu_0 = \frac{1}{4}$	$\nu_0 = \frac{1}{5}$	$\nu_0 = \frac{1}{6}$	$\nu_0 = \frac{1}{6}$
$\nu_1 = \frac{2+\alpha}{8+6\,\alpha}$	$\alpha^* = \frac{1}{0}$	$\alpha^* = \frac{1}{1}$	$\alpha^* = \frac{1}{2}$

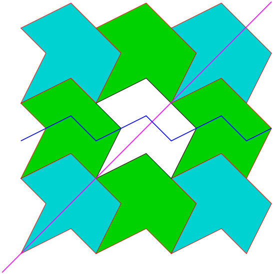
3.3 Third good idea: special periodic condition on f

$$\boxed{\forall\, q : \qquad f(q+T) = f(q) + f(T) - f(0)}$$

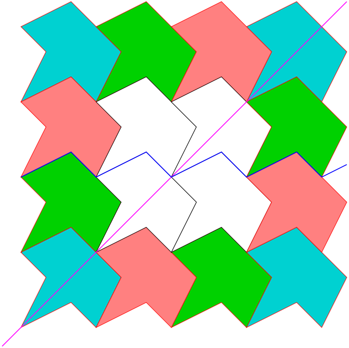


3.3.0 f with period made of 2 segments: Chaos in cell

d = 1

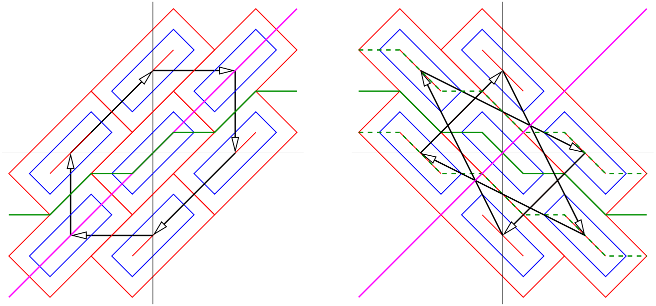


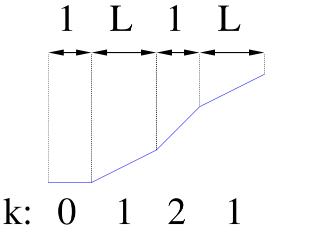
d = 4



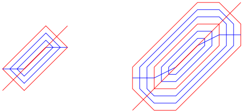
3.3.1 f with period made of 2 segments

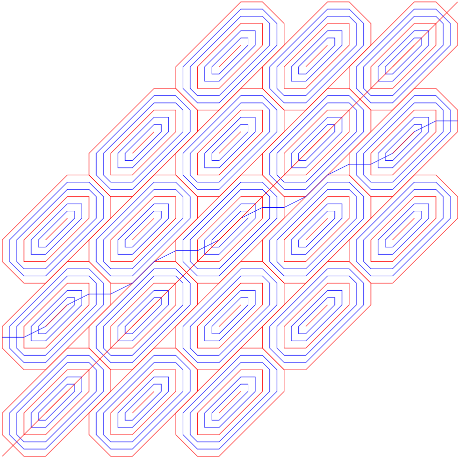
$$\boxed{\forall\, p: \qquad f(p+T) = f(p) + f(T) - f(0)}$$

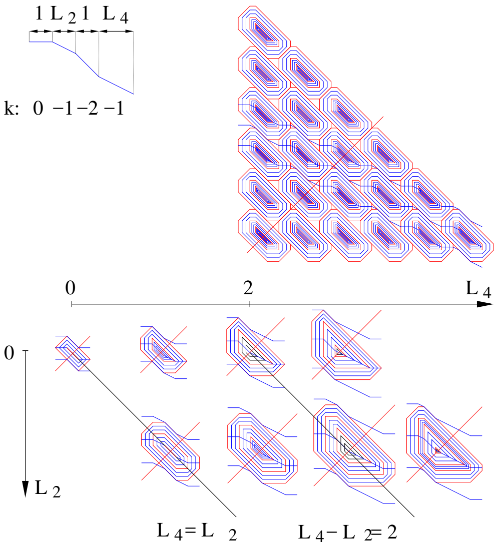



1 L 1 L
k: 0 1 2 1

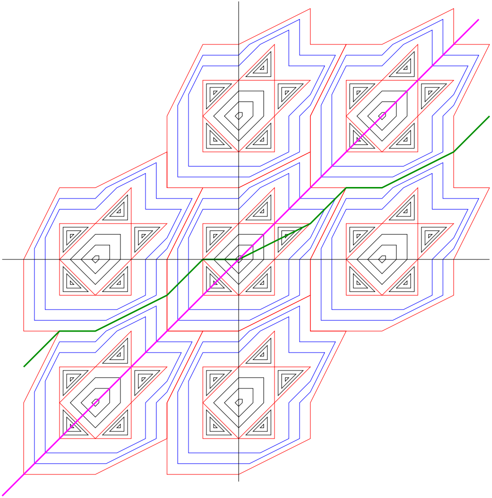

0
L

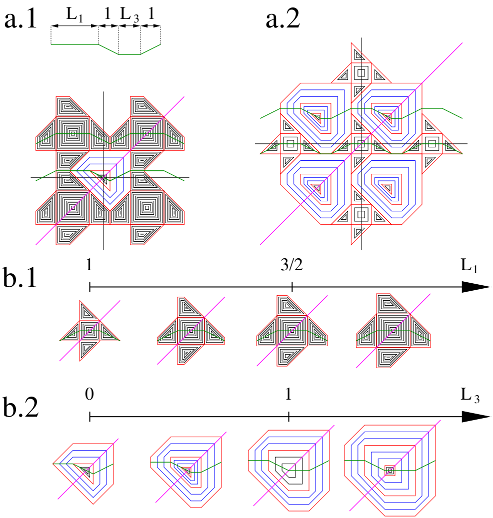




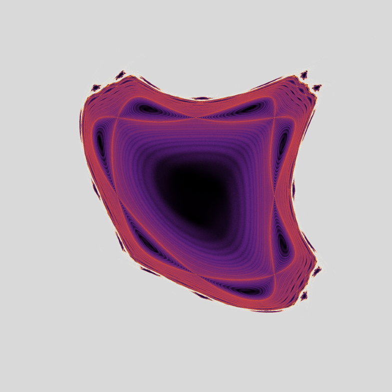

1 L 2 1 L 4
k: 0 −1 −2 −1
0 2 L 4
0
L 2
L 4 = L 2
L 4 − L 2 = 2

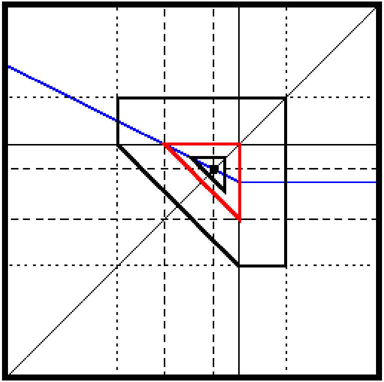
3.3.1 f with period made of 3 segments



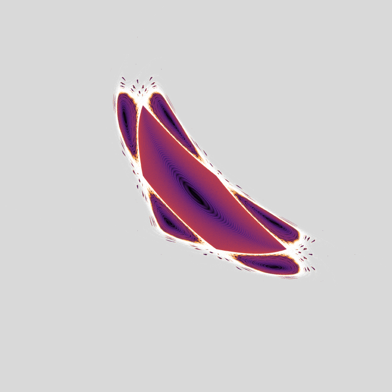

a.1
L1
1
L3
1
a.2
b.1
1
3/2
L1
b.2
0
1
L3

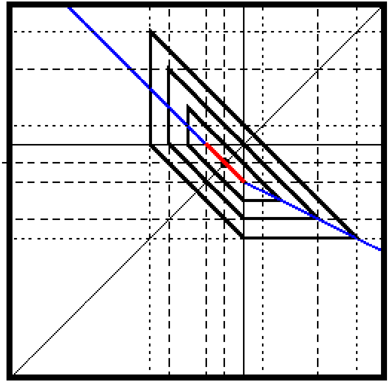
Application 2: Can we better understand the reality? Hénon map, $f(q) = a\,q + b\,q^2$.



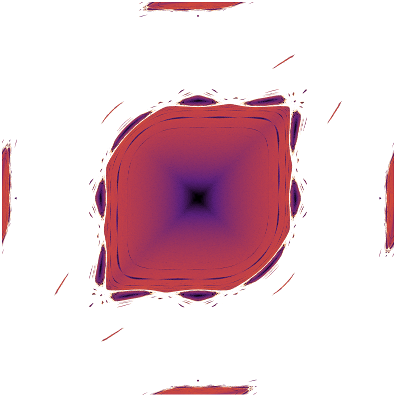


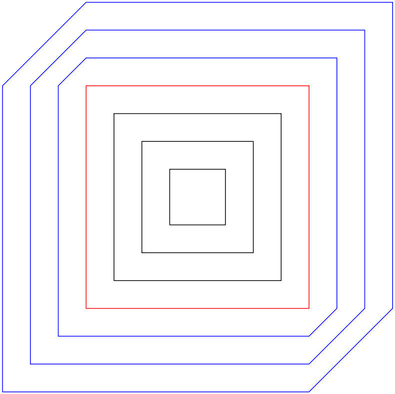
Application 2: Can we better understand the reality? Hénon map, $f(q) = a\,q + b\,q^2$.



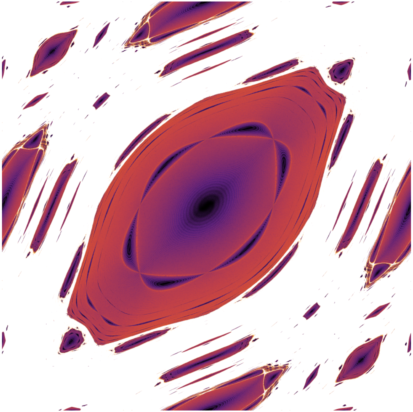


Application 2: Can we better understand the reality? Chirikov map, $f(q) = 2\,q + a\,\sin q$.

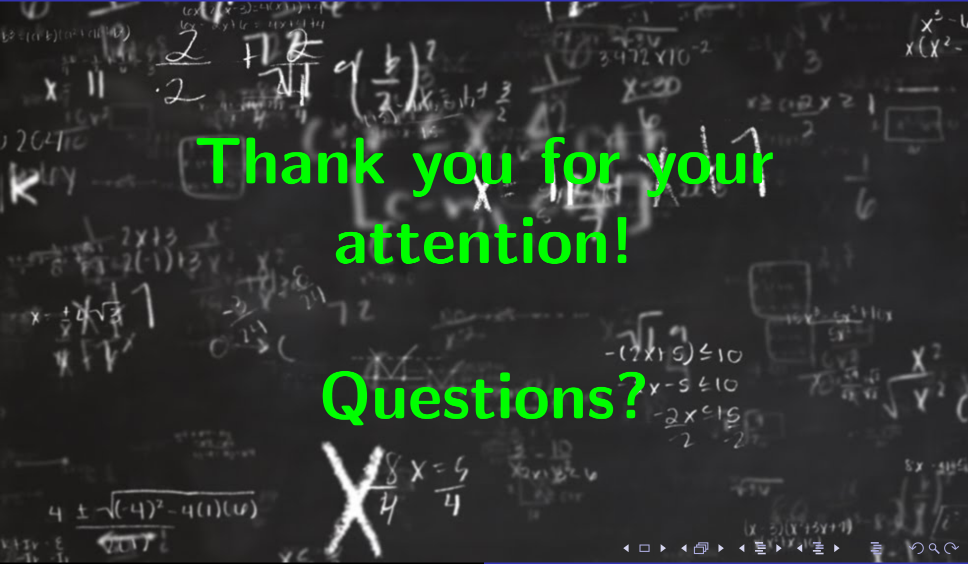




Application 2: Can we better understand the reality? Chirikov map, $f(q) = 2\,q + a\,\sin q$.



LAST SLIDE



Thank you for your attention!

Questions?