Rank of Visibilities Matrix

Summary

Definitions

The matrix of visibilities $\mathcal{V}_\nu^{i,j}$ (correlating feed i with feed j at frequency ν) for fixed ν but varying i and j is a Hermitian non-negative matrix we denote by $\boldsymbol{\mathcal{V}}_\nu$ and call the *visibilities matrix*. The contribution of the signal to $\boldsymbol{\mathcal{V}}_\nu$ we denote by $\boldsymbol{\mathcal{V}}_\nu^{\text{sky}}$. The *rank* of this matrix is the number of non-zero eigenvalues.

We denote the contribution of a single points source to $\boldsymbol{\mathcal{V}}_\nu^{\text{sky}}$ by $\boldsymbol{\mathcal{V}}_\nu^{\text{s}}$. A point source in direction $\hat{\boldsymbol{n}}_{\text{s}}$ has polarization tensor

$$\overleftrightarrow{\mathcal{P}}_\nu^{\text{s}}[\hat{\boldsymbol{n}}] = p_\nu^{\text{s}}\,\delta^{(2)}[\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}_{\text{s}}]$$

$$p_\nu^{\text{s}} = \frac{1}{2}\begin{pmatrix} f_\nu^{\text{I}} + f_\nu^{\text{Q}} & f_\nu^{\text{U}} + i\, f_\nu^{\text{V}} \\ f_\nu^{\text{U}} - i\, f_\nu^{\text{V}} & f_\nu^{\text{I}} - f_\nu^{\text{Q}} \end{pmatrix}$$

where $f_\nu^{\text{I}} \geq \sqrt{\left(f_\nu^{\text{Q}}\right)^2 + \left(f_\nu^{\text{U}}\right)^2 + \left(f_\nu^{\text{V}}\right)^2} > 0$. A point source is *completely polarized* when

$$f_\nu^{\text{I}} = \sqrt{\left(f_\nu^{\text{Q}}\right)^2 + \left(f_\nu^{\text{U}}\right)^2 + \left(f_\nu^{\text{V}}\right)^2}.$$

Then $\text{rank}[\boldsymbol{\mathcal{V}}_\nu^{\text{s}}] \leq \text{rank}[p_\nu^{\text{s}}] \leq 2$.

Two feeds, i and j, are said to be *aligned* at a point source when the *field patterns* obeys $\vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_{\text{s}}] = c[\nu]\,\vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_{\text{s}}]$ where $c[\nu]$ is any complex function of frequency.

Results

$$\text{rank}[p_\nu^{\text{s}}] = \begin{cases} 2 & f_\nu^{\text{I}} > \sqrt{\left(f_\nu^{\text{Q}}\right)^2 + \left(f_\nu^{\text{U}}\right)^2 + \left(f_\nu^{\text{V}}\right)^2} \quad \text{partially or non polarized} \\ 1 & f_\nu^{\text{I}} = \sqrt{\left(f_\nu^{\text{Q}}\right)^2 + \left(f_\nu^{\text{U}}\right)^2 + \left(f_\nu^{\text{V}}\right)^2} > 0 \quad \text{completely polarized} \\ 0 & f_\nu^{\text{I}} = f_\nu^{\text{Q}} = f_\nu^{\text{U}} = f_\nu^{\text{V}} = 0 \quad \text{no source} \end{cases}$$

Two conditions under which $\text{rank}[\boldsymbol{\mathcal{V}}_\nu^{\text{s}}] = 1$ are

1) a completely polarized point source

2) when all the feeds are aligned

If there is a *dominant point source* then the above will be approximately true of $\boldsymbol{\mathcal{V}}_\nu^{\text{sky}}$ in the sense that the number of "large" eigenvalues will be given by the previous rank formulae. The calibrations noise sources (ground or drone), the Sun approximate dominant point sources no matter the pointing of the dishes. The brightest radio sources, *e.g.* Cassiopeia A and Cygnus A, approximate dominant point sources when the dishes are pointing toward them.

For Tianlai the *calibration noise sources*, either on the ground or on the drone, should be nearly completely polarized since only one voltage stream is generated.

If the feeds on either the dishes and cylinders are identical and non-interacting then all the E-W feeds should be aligned as should all the N-S feeds.

Visibilities

The contribution of the sky signal to a *visibility*, $\mathcal{V}_\nu^{i,j}$, is given by

$$\mathcal{V}_\nu^{i,j} = g_i\, g_j^* \int d^2\hat{\boldsymbol{n}}\, \vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}] \cdot \overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] \cdot \vec{\mathcal{F}}_j[\nu, \hat{\boldsymbol{n}}]^* \, e^{i\, 2\pi \frac{\nu}{c} (\boldsymbol{x}_i - \boldsymbol{x}_j)\cdot \hat{\boldsymbol{n}}}$$

where $\hat{\boldsymbol{n}}$ is the direction on the sky, $\vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}]$ is the complex *field pattern* vector in the sky $\hat{\boldsymbol{n}} \cdot \vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}] = 0$ and $\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]$ is the *polarization tensor* in direction $\hat{\boldsymbol{n}}$

$$\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] = \frac{1}{2} \begin{pmatrix} I_\nu[\hat{\boldsymbol{n}}] + Q_\nu[\hat{\boldsymbol{n}}] & U_\nu[\hat{\boldsymbol{n}}] + i\, V_\nu[\hat{\boldsymbol{n}}] \\ U_\nu[\hat{\boldsymbol{n}}] - i\, V_\nu[\hat{\boldsymbol{n}}] & I_\nu[\hat{\boldsymbol{n}}] - Q_\nu[\hat{\boldsymbol{n}}] \end{pmatrix}.$$

$\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]$ is the non-negative Hermitian polarization tensor $\hat{\boldsymbol{n}} \cdot \overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] = 0$. Physically

$$\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] \propto \left\langle \tilde{\boldsymbol{E}}[\nu, \hat{\boldsymbol{n}}] \otimes \tilde{\boldsymbol{E}}[\nu, \hat{\boldsymbol{n}}]^* \right\rangle$$

where $\tilde{\boldsymbol{E}}[\nu]$ is the temporal Fourier transform of the electric field of incoming radiation from direction $\hat{\boldsymbol{n}}$ which is of course transverse $\hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{E}}[\nu, \hat{\boldsymbol{n}}] = 0$. Here i and j refer to different feeds and $\boldsymbol{x}_i$ is the nominal position of feed i.

The phase factors $e^{i\, 2\pi \frac{\nu}{c} \boldsymbol{x}_i \cdot \hat{\boldsymbol{n}}}$ in $\mathcal{V}_\nu^{i,j}$ represent a first order correction to the varying times of flight to different feeds, here $\boldsymbol{x}_i$ is the nominal position of feed i. These factors *do not* represent an approximation since any additional corrections will be absorbed in $\vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}]$. Rather these factors are only a convention in the definition of $\vec{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}]$.

The matrix of visibilities for all i and j is called the *visibilities matrix* and denoted by $\boldsymbol{\mathcal{V}}_\nu^{\text{sky}}$.

Stokes Parameters

The *Stokes parameters* are $I_\nu[\hat{\boldsymbol{n}}]$, $Q_\nu[\hat{\boldsymbol{n}}]$, $U_\nu[\hat{\boldsymbol{n}}]$ and $V_\nu[\hat{\boldsymbol{n}}]$ which are real. From its physical definition the Hermitian matrix $\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]$ is non-negative matrix and satisfies

$$I_\nu[\hat{\boldsymbol{n}}] \geq \sqrt{Q_\nu[\hat{\boldsymbol{n}}]^2 + U_\nu[\hat{\boldsymbol{n}}]^2 + V_\nu[\hat{\boldsymbol{n}}]^2} \geq 0.$$

Different types of light are

unpolarized light	$Q_\nu[\hat{\boldsymbol{n}}] = U_\nu[\hat{\boldsymbol{n}}] = V_\nu[\hat{\boldsymbol{n}}] = 0$
purely polarized light	$I_\nu[\hat{\boldsymbol{n}}] = \sqrt{Q_\nu[\hat{\boldsymbol{n}}]^2 + U_\nu[\hat{\boldsymbol{n}}]^2 + V_\nu[\hat{\boldsymbol{n}}]^2}$
linearly polarized light	$V_\nu[\hat{\boldsymbol{n}}] = 0$
circularly polarized light	$Q_\nu[\hat{\boldsymbol{n}}] = U_\nu[\hat{\boldsymbol{n}}] = 0$
elliptically polarized light	$Q_\nu[\hat{\boldsymbol{n}}]^2 + U_\nu[\hat{\boldsymbol{n}}]^2 \neq 0$ && $V_\nu[\hat{\boldsymbol{n}}] \neq 0$
no light	$I_\nu[\hat{\boldsymbol{n}}] = Q_\nu[\hat{\boldsymbol{n}}] = U_\nu[\hat{\boldsymbol{n}}] = V_\nu[\hat{\boldsymbol{n}}] = 0$

.

Generically $\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]$ has rank 2 but for pure polarization $\mathrm{Det}\left[\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]\right] = 0$ in which case it has rank 1 unless $\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] = \boldsymbol{0}$ except in the case of no light.

Eigen-Decomposition into 2 Pure Polarization States

One can decompose $\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}]$ into two polarization eigenstates:

$$\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] = I_\nu^+[\hat{\boldsymbol{n}}]\, \hat{\boldsymbol{e}}_+[\hat{\boldsymbol{n}}] \otimes \hat{\boldsymbol{e}}_+[\hat{\boldsymbol{n}}] + I_\nu^-[\hat{\boldsymbol{n}}]\, \hat{\boldsymbol{e}}_-[\hat{\boldsymbol{n}}] \otimes \hat{\boldsymbol{e}}_-[\hat{\boldsymbol{n}}]$$

where

$$I_\nu^\pm[\hat{\boldsymbol{n}}] = I_\nu[\hat{\boldsymbol{n}}] \pm \sqrt{Q_\nu[\hat{\boldsymbol{n}}]^2 + U_\nu[\hat{\boldsymbol{n}}]^2 + V_\nu[\hat{\boldsymbol{n}}]^2}$$

are real. Thus $I_\nu^+[\hat{\boldsymbol{n}}] \geq I_\nu^-[\hat{\boldsymbol{n}}] \geq 0$. The unit normalized eigenvectors satisfy

$$\hat{\mathbf{e}}_+[\hat{\boldsymbol{n}}]\cdot\hat{\mathbf{e}}_+[\hat{\boldsymbol{n}}]^* = \hat{\mathbf{e}}_-[\hat{\boldsymbol{n}}]\cdot\hat{\mathbf{e}}_-[\hat{\boldsymbol{n}}]^* = 1 \quad .$$

$$\hat{\mathbf{e}}_+[\hat{\boldsymbol{n}}]\cdot\hat{\mathbf{e}}_-[\hat{\boldsymbol{n}}]^* = 0$$

$$\hat{\mathbf{e}}_+[\hat{\boldsymbol{n}}]\cdot\hat{\mathbf{e}}_+[\hat{\boldsymbol{n}}]^* + \hat{\mathbf{e}}_-[\hat{\boldsymbol{n}}]\cdot\hat{\mathbf{e}}_-[\hat{\boldsymbol{n}}]^* = \overleftrightarrow{\boldsymbol{I}}$$

but are only determined up to a phase factor. In general $\hat{\mathbf{e}}_\pm[\hat{\boldsymbol{n}}]$ are complex vectors but can be chosen to be real for linearly polarized light. Purely polarized light has $I_\nu^+[\hat{\boldsymbol{n}}] > 0$ and $I_\nu^-[\hat{\boldsymbol{n}}] = 0$ while unpolarized light has $I_\nu^+[\hat{\boldsymbol{n}}] = I_\nu^-[\hat{\boldsymbol{n}}] > 0$ and of course no light has $I_\nu^+[\hat{\boldsymbol{n}}] = I_\nu^-[\hat{\boldsymbol{n}}] = 0$. For unpolarized light any orthonormal basis $\hat{\mathbf{e}}_\pm[\hat{\boldsymbol{n}}]$ can be chosen for the unit normalized eigenvectors.

Dominant Point Source

For an unresolved (point) source in direction $\hat{\boldsymbol{n}}_s$

$$\overleftrightarrow{\mathcal{P}}_\nu[\hat{\boldsymbol{n}}] = \overleftrightarrow{p}_\nu^{\,s}\,\delta^{(2)}[\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}_s]$$

where

$$\overleftrightarrow{p}_\nu^{\,s} = \frac{1}{2}\begin{pmatrix} f_\nu^I + f_\nu^Q & f_\nu^U + i\, f_\nu^V \\ f_\nu^U - i\, f_\nu^V & f_\nu^I - f_\nu^Q \end{pmatrix}$$

and $f_\nu^I \geq \sqrt{(f_\nu^Q)^2 + (f_\nu^U)^2 + (f_\nu^V)^2} \geq 0$. The quantity f_ν^I is more conventionally written f_ν and is called the *flux density*.

One can decompose into two pure polarization states

$$\overleftrightarrow{p}_\nu^{\,s} = f_\nu^+\,\hat{\mathbf{e}}_+^s \otimes \hat{\mathbf{e}}_+^s + f_\nu^-\,\hat{\mathbf{e}}_-^s \otimes \hat{\mathbf{e}}_-^s$$

where

$$\hat{\mathbf{e}}_+^s\cdot\hat{\mathbf{e}}_+^{s\,*} = \hat{\mathbf{e}}_-^s\cdot\hat{\mathbf{e}}_-^{s\,*} = 1$$

$$\hat{\mathbf{e}}_+^s\cdot\hat{\mathbf{e}}_-^{s\,*} = 0$$

and

$$f_\nu^\pm = f_\nu^I \pm \sqrt{(f_\nu^Q)^2 + (f_\nu^U)^2 + (f_\nu^V)^2}$$

are real and $f_\nu^+ > f_\nu^- > 0$. Purely polarized light has $f_\nu^+ > 0$ and $f_\nu^- = 0$ while unpolarized light has $f_\nu^+ = f_\nu^-$.

For a dominant point source the visibility is

$$\mathcal{V}_\nu^{i,j} = \left(g_i\, e^{i\,2\pi\frac{\nu}{c}\boldsymbol{x}_i\cdot\hat{\boldsymbol{n}}_s}\,\overrightarrow{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_s]\right)\cdot\overleftrightarrow{p}_\nu^{\,s}\cdot\left(g_j\, e^{i\,2\pi\frac{\nu}{c}\boldsymbol{x}_j\cdot\hat{\boldsymbol{n}}_s}\,\overrightarrow{\mathcal{F}}_j[\nu, \hat{\boldsymbol{n}}_s]\right)^*$$

$$= \sum_\pm f_\nu^\pm \left(g_i\, e^{i\,2\pi\frac{\nu}{c}\boldsymbol{x}_i\cdot\hat{\boldsymbol{n}}_s}\,\overrightarrow{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_s]\cdot\hat{\mathbf{e}}_\pm^s\right)\otimes\left(g_j\, e^{i\,2\pi\frac{\nu}{c}\boldsymbol{x}_j\cdot\hat{\boldsymbol{n}}_s}\,\overrightarrow{\mathcal{F}}_j[\nu, \hat{\boldsymbol{n}}_s]\cdot\hat{\mathbf{e}}_\pm^s\right)^*$$

so the visibilities matrix may be written.

$$\boldsymbol{\mathcal{V}}_\nu^s = f_\nu^+\,\boldsymbol{v}_+^s \otimes \boldsymbol{v}_+^{s\,*} + f_\nu^-\,\boldsymbol{v}_-^s \otimes \boldsymbol{v}_-^{s\,*}$$

where

$$\boldsymbol{v}_\pm^s = \left\{g_i\, e^{i\,2\pi\frac{\nu}{c}\boldsymbol{x}_i\cdot\hat{\boldsymbol{n}}_s}\,\overrightarrow{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_s]\cdot\hat{\mathbf{e}}_\pm^s\right\}.$$

Since $\boldsymbol{\mathcal{V}}_\nu^s$ is the sum of two rank 1 tensors it's rank must be 2 or less.

If the source is *purely polarized* then $f_\nu^- = 0$ and $\boldsymbol{\mathcal{V}}_\nu^s = f_\nu^+\,\boldsymbol{v}_+^s \otimes \boldsymbol{v}_+^s$ which has rank 1.

Another case where the visibilities matrix has rank 1 is when all the field patterns are *aligned at the source* ($\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}_s$) *i.e.*: $\overrightarrow{\mathcal{F}}_i[\nu, \hat{\boldsymbol{n}}_s] = z_i[\nu, \hat{\boldsymbol{n}}_s]\,\overrightarrow{\mathcal{F}}_0[\nu, \hat{\boldsymbol{n}}_s]$ where the z_i are non-zero complex numbers. In this case

$$\mathcal{V}_\nu^{i,j} = \left(g_i\, e^{i\, 2\pi \frac{\nu}{c} \mathbf{x}_i \cdot \hat{\mathbf{n}}_s}\, z_i[\nu, \hat{\mathbf{n}}_s]\right) \left(g_j\, e^{i\, 2\pi \frac{\nu}{c} \mathbf{x}_j \cdot \hat{\mathbf{n}}_s}\, z_j[\nu, \hat{\mathbf{n}}_s]\right)^* \left(\vec{\mathcal{F}}_0[\nu, \hat{\mathbf{n}}_s] \cdot \overleftrightarrow{p}_\nu^s \cdot \vec{\mathcal{F}}_0[\nu, \hat{\mathbf{n}}_s]^*\right)$$

or

$$\boldsymbol{\mathcal{V}}_\nu^s = p_\nu^s\, \boldsymbol{v}_s[\nu] \otimes \boldsymbol{v}_s[\nu]^*$$

where

$$p_\nu^s = \vec{\mathcal{F}}_0[\nu, \hat{\mathbf{n}}_s] \cdot \overleftrightarrow{p}_\nu^s \cdot \vec{\mathcal{F}}_0[\nu, \hat{\mathbf{n}}_s]^* \geq 0.$$

$$\boldsymbol{v}_s[\nu] = \left\{g_i\, e^{i\, 2\pi \frac{\nu}{c} \mathbf{x}_i \cdot \hat{\mathbf{n}}_s}\, z_i[\nu, \hat{\mathbf{n}}_s]\right\}$$

Since $\overleftrightarrow{p}_\nu^s$ is a non-negative Hermitian matrix it follows that $p_\nu^s \geq 0$. This condition for a rank 1 visibilities matrix only has to do with the telescope design (which determines the field patterns) and has *nothing* at all to do with the illumination pattern from the sky.

Could one use the rank of $\boldsymbol{\mathcal{V}}_\nu^s$ to "align" the feeds on a telescope by pointing it at a bright unpolarized point source? or use the eigenvalues to test the alignment?

By definition an array of *identical* feeds are aligned. Ideally all the Tianlai E-W feeds are aligned as are all the Tianlai N-S feeds and one should find Rank[$\boldsymbol{\mathcal{V}}_\nu^s$] = 1 when one only correlates either E-W or N-S feeds. However the E-W and N-S feeds should not be aligned so if one correlates both E-W and N-S feeds then one should find Rank[$\boldsymbol{\mathcal{V}}_\nu^s$] = 2 so long as the dominant source is not purely polarized.

The calibration noise source (on the ground or on the drone) should be purely polarized since they generate only a single voltage stream. In this case one should find Rank[$\boldsymbol{\mathcal{V}}_\nu^s$] = 1 even when correlating between E-W and N-S feeds.

Mathematica Computation