Compact perturbative expressions for oscillations with sterile neutrinos in matter

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Work done with S. Parke
Neutrino oscillations in vacuum

In a scheme with $N$ sterile neutrinos, the oscillation probabilities in vacuum are

$$P(\nu_{\alpha} \to \nu_{\beta}) = \left| \sum_{j=1}^{3+N} U_{\alpha j}^* U_{\beta j} e^{i\Delta m^2_{j1}/2E} \right|^2$$

$U$ is the PMNS matrix which converts the energy eigenstates to the flavor eigenstates.

$$H_{\text{vacuum}} = \frac{1}{2E} U \begin{pmatrix} 0 & \Delta m^2_{21} & \Delta m^2_{31} \\ \Delta m^2_{21} & 0 & \Delta m^2_{32} \\ \Delta m^2_{31} & \Delta m^2_{32} & \ddots \end{pmatrix} U^\dagger$$
Matter effect

In matters, propagation of the neutrinos will be altered by the L. Wolfenstein matter effect.

\[ V_{NC} = \mp \sqrt{2} G_F N_n / 2 \quad V_{CC} = \pm \sqrt{2} G_F N_e \]

\( N_n \) and \( N_e \) are the number densities of the neutrons and electrons, respectively, when \( N_n \approx N_e \), we have \( V_{NC} \approx -V_{CC} / 2 \). The sterile neutrinos will not be engaged in the matter effects.
Hamiltonian in matter

The Hamiltonian in the flavor basis becomes (free to add a multiple of the identity)

\[ H = H_{\text{vacuum}} + \frac{1}{2E} \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a/2 \\ 0 & a/2 & \ddots \\ \vdots & \ddots & \ddots & a/2 \end{pmatrix}, \]

where \( a = 2\sqrt{2}G_F N_e E. \)

Now the PMNS matrix in vacuum \( U \) can no longer diagonalize the Hamiltonian, the energy eigenstates and eigenvalues are altered by the matter effect.
Solve the eigensystem in matter

\[ H = \frac{1}{2E} V^\dagger \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \\ \vdots & & & \ddots \\ & & & & \lambda_{3+N} \end{pmatrix} V \]

Solve for \( V \) and \( \lambda_i \).

Analytic solutions

- Only possible for 3+1 model \( 1808.03985 \)

Perturbation expansions

- degeneracy of the zeroth order eigenvalues

Rotations+Perturbation expansions
The rotations can do...

- Disentangle the crossings of the $0^{th}$ order eigenvalues
- Diminish off-diagonal elements of the Hamiltonian
- Give $0^{th}$ order eigenvalues and mixing parameters (angles and phases)
Define \( \Delta m_{ee}^2 \equiv \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2 \), \( \epsilon \equiv \Delta m_{21}^2 / \Delta m_{ee}^2 \approx 0.03 \).

Orders of some important parameters

- weak mixing with sterile neutrinos, \( \sin \theta_{i(3+n)} \sim O(\sqrt{\epsilon}) \)
- heavy sterile neutrinos, \( \Delta m_{ee}^2 / \Delta m_{(3+n)}^2 \sim O(\epsilon) \).
- not extremely strong matter effect, \( a \sim \Delta m_{ee}^2 \), so \( a / \Delta m_{(3+n)}^2 \sim O(\epsilon) \)
Step 0: Convention of the vacuum PMNS matrix

A usual convention to define the PMNS matrix in vacuum, rotations mixing with the sterile neutrinos come after the ones in the active neutrino space

\[ U = U_{\text{sterile}} U_{23} U_{13} U_{12} \]

A different convention to define the PMNS matrix

\[ U = U_{23} U_{\text{sterile}} U_{13} U_{12} \]

The matter potential term in the Hamiltonian is invariant under a transformation in the (2-3) sector. If \( U_{23} \) is the last rotation, the following rotations process will be simplified.
Step 1: Vacuum $U_{23}$ rotation

$$H \Rightarrow U_{23}^\dagger(\theta_{23}, \delta_{23}) H U_{23}(\theta_{23}, \delta_{23})$$

$$= U_{23}^\dagger(\theta_{23}, \delta_{23}) H_{\text{vacuum}} U_{23}(\theta_{23}, \delta_{23}) + \frac{1}{2E} \begin{pmatrix} a & 0 & 0 \\ 0 & a/2 & \cdots \\ 0 & \cdots & a/2 \end{pmatrix}$$

$\theta_{23}$ and $\delta_{23}$ are in vacuum.
Step 2: Vacuum $U_{\text{sterile}}$ rotations.

$U_{23}^\dagger(\theta_{23}, \delta_{23}) \ H \ U_{23}(\theta_{23}, \delta_{23})$

$\Rightarrow \tilde{H} \equiv U_{\text{sterile}}^\dagger \ U_{23}^\dagger(\theta_{23}, \delta_{23}) \ H \ U_{23}(\theta_{23}, \delta_{23}) \ U_{\text{sterile}}$

Rotations parameter (angles and phases) in $U_{\text{sterile}}$ are still in vacuum
Step 3: $U_{13}$ rotation, explicit derivation in the 3+1 scheme

$$\tilde{H} = \frac{1}{2E} \begin{pmatrix} \lambda_a & \cdots & (\tilde{H})_{13} & \cdots \\ \vdots & \lambda_b & \vdots \\ (\tilde{H})_{13}^* & \cdots & \lambda_c \\ \vdots & \cdots & \cdots \end{pmatrix}$$

- **Kill** $(\tilde{H})_{13}$
- **Resolve the crossing of** $\lambda_a$ and $\lambda_c$ at $a \simeq \frac{\cos^2\theta_{13}}{c_{14}^2} \Delta m_{ee}^2$.

Normal Order

![Graph showing $\lambda$ versus $Y_e \rho E$](image-url)

- $\lambda_a$
- $\lambda_b$
- $\lambda_c$
Step 3: Continued

\[ \lambda_a = (s_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2 + (c_{14}^2 + \frac{\epsilon}{2} k_{11} c_{24}^2 c_{34}^2) a \quad k_{ij} = \frac{S_{i4} S_{j4}}{\epsilon} \sim O(1) \]

\[ \lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2 + \frac{\epsilon}{2} k_{22} c_{34}^2 a \]

\[ \lambda_c = (c_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2 + \frac{\epsilon}{2} k_{33} a \]

\[ (\tilde{H})_{13} = s_{13} c_{13} \Delta m_{ee}^2 + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{13}} \]
Step 3: Continued

Diagonalize the (1-3) sector of $\tilde{H}$ by implementing a complex rotation $U_{13}(\tilde{\theta}_{13}, \alpha_{13})$

$$\tilde{H} \Rightarrow \hat{H} \equiv U_{13}^\dagger(\tilde{\theta}_{13}, \alpha_{13}) \tilde{H} U_{13}(\tilde{\theta}_{13}, \alpha_{13})$$

$$\tilde{\theta}_{13} = \frac{1}{2} \arccos \frac{\lambda_c - \lambda_a}{\sqrt{\left|\lambda_c - \lambda_a\right|^2 + 4|s_{13}c_{13}\Delta m_{ee}^2 + \frac{\epsilon}{2}a k_{13}c_{24}c_{34}e^{-i\delta_{34}}|^2}}$$

$$\alpha_{13} = \text{Arg}\left[ s_{13}c_{13}\Delta m_{ee}^2 + \frac{\epsilon}{2}a k_{13}c_{24}c_{34}e^{-i\delta_{34}} \right]$$
Step 4: $U_{12}$ rotation, explicit derivation in the 3+1 scheme

$$\hat{H} = \frac{1}{2E} \begin{pmatrix} \lambda_- & (\hat{H})_{12} & 0 & \cdots \\ (\hat{H})^*_{12} & \lambda_0 & (\hat{H})_{23} & \cdots \\ 0 & (\hat{H})^*_{23} & \lambda_+ & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Kill $(\hat{H})_{12}$
- Resolve the crossing of $\lambda_-$ and $\lambda_0$ at the solar resonance.

![Normal Order Graph](image_url)
Step 4: Continued

\[ \lambda_- = \frac{1}{2} \left[ (\lambda_a + \lambda_c) - \right. \]

\[ \text{sign}(\Delta m^2_{ee}) \sqrt{(\lambda_a - \lambda_c)^2 + 4 |s_{13} c_{13} \Delta m^2_{ee} + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{34}}|^2} \]

\[ \lambda_0 = \lambda_b = \epsilon c_{12}^2 \Delta m^2_{ee} + \frac{\epsilon}{2} k_{22} c_{34}^2 a \]

\[ \lambda_+ = \frac{1}{2} \left[ (\lambda_a + \lambda_c) + \right. \]

\[ \text{sign}(\Delta m^2_{ee}) \sqrt{(\lambda_a - \lambda_c)^2 + 4 |s_{13} c_{13} \Delta m^2_{ee} + \frac{\epsilon}{2} a k_{13} c_{24} c_{34} e^{-i\delta_{34}}|^2} \]

\( (\hat{H})_{12} = \epsilon \left\{ s_{12} c_{12} (c_{13} \tilde{c}_{13} + s_{13} \tilde{s}_{13} e^{-i\alpha_{13}}) \Delta m^2_{ee} \right. \]

\[ + \frac{a}{2} \left[ k_{12} c_{24} c_{34}^2 \tilde{c}_{13} - k_{23} c_{34} \tilde{s}_{13} e^{i(\delta_{34} + \alpha_{13})} \right] e^{-i\delta_{24}} \right\} \]

\( (\hat{H})_{23} = \epsilon \left\{ s_{12} c_{12} (-s_{13} \tilde{c}_{13} + c_{13} \tilde{s}_{13} e^{i\alpha_{13}}) \Delta m^2_{ee} \right. \]

\[ + \frac{a}{2} \left[ k_{12} c_{24} c_{34}^2 \tilde{s}_{13} e^{i\alpha_{13}} + k_{23} c_{34} \tilde{c}_{13} e^{i\delta_{34}} \right] e^{i\delta_{24}} \right\} \]
Step 4: Continued

Diagonalize the (1-2) sector of $\hat{H}$ by implementing a complex rotation $U_{12}(\tilde{\theta}_{12}, \alpha_{12})$

$$\hat{H} \Rightarrow \tilde{\hat{H}} \equiv U_{12}^\dagger(\tilde{\theta}_{12}, \alpha_{12}) \hat{H} U_{12}(\tilde{\theta}_{12}, \alpha_{12})$$

$$\tilde{\theta}_{12} = \frac{1}{2} \arccos \frac{\lambda_0 - \lambda_-}{\sqrt{|\lambda_0 - \lambda_-|^2 + 4|\hat{H}_{12}|^2}}$$

$$\alpha_{12} = \text{Arg}[(\hat{H})_{12}]$$
In the 3+1 scheme

\[ V^{(0)} = U_{23} U_{34} U_{24} U_{14} U_{13} U_{12} \]

\[ \tilde{H} = \frac{1}{2E} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \\ \lambda_3 & \lambda_4 \end{pmatrix} + \tilde{H}_1 \]

\[ \tilde{H}_0 \]

All diagonal elements of \( \tilde{H} \) have been absorbed into the 0\(^{th}\) order Hamiltonian \( \tilde{H}_0 \)
$0^{\text{th}}$ order eigenvalues in the active neutrino space

$$\lambda_{1,2} = \frac{1}{2} \left[ (\lambda_- + \lambda_0) \mp \sqrt{(\lambda_- - \lambda_0)^2 + 4|\hat{H}_{12}|^2} \right]$$

$$\lambda_3 = \lambda_+$$

Normal Order

![Graph showing the relationship between $\lambda(10^{-3} \text{eV}^2)$ and $Y_e \rho E$ (g cm$^{-3}$ GeV).](image)
Active sectors of the perturbative Hamiltonian

All the diagonal elements of \( \hat{\mathcal{H}}_1 \) vanish, the off-diagonal elements in the sector of the active neutrinos (first three rows and columns) are

\[
\begin{align*}
(\hat{\mathcal{H}}_1)_{12} &= 0 \\
(\hat{\mathcal{H}}_1)_{13} &= -\tilde{s}_{12}(\hat{\mathcal{H}})_{23} e^{-i\alpha_{12}} \\
(\hat{\mathcal{H}}_1)_{23} &= \tilde{c}_{12}(\hat{\mathcal{H}})_{23}
\end{align*}
\]

Since \((\hat{\mathcal{H}})_{23} \sim \mathcal{O}(\epsilon)\), sectors of the active neutrinos \( \sim \mathcal{O}(\epsilon) \)
Sectors of the sterile neutrino

Crossings of the eigenvalues to $\lambda_4$

$$\lambda_4 = \Delta m^2_{41} - \frac{a^2 c_{14}^2 c_{24}^2 c_{34}^2}{2} \gg \Delta m^2_{ee} \sim a$$

Crossings to $\lambda_4$ only happen with very high neutrino energy ($E \gg 10\text{GeV}$), we are not interested in this energy scale.

4th row and column of $\tilde{H}_1$

Elements in the 4th row and column of the perturbative Hamiltonian

$$(\tilde{H}_1)_{i4} \propto \frac{a s_{i4}}{2E} \sim \mathcal{O}(\sqrt{\epsilon}), \; i = 1, 2, 3$$

However, they are not going to give $\mathcal{O}(\sqrt{\epsilon})$ corrections, because in perturbative expressions they will be divided by $\lambda_4$. 
Important special cases

Back to exact values in vacuum

In vacuum, $a = 0$, the $0^{\text{th}}$ order approximations will give exact vacuum values, i.e. $\tilde{\theta}_{13,12} = \theta_{13,12}$, $\alpha_{13,12} = 0$, $\lambda_i = \Delta m^2_{i1}$ and $\tilde{\mathcal{H}}_1 = 0$.

Related to the Standard Model

When $U_{\text{sterile}} = 1$, i.e. $s_{i4} = 0$ in the $3+1$ scheme, the results go to the DMP for the SM.
Perturbative expansion: Corrections to the eigenvalues

\[ \lambda_i^{(\text{ex})} = \lambda_i + \lambda_i^{(1)} + \lambda_i^{(2)} + \cdots \]

\( \lambda_i^{(n)} \) are the \( n^{\text{th}} \) order corrections.

\[ \lambda_i^{(1)} = 2E(\hat{H}_1)_{ii} \]

Since \( \hat{H}_1 \) has zero diagonal elements, the first order corrections are trivial.

\[ \lambda_i^{(2)} = \sum_{k \neq i} \frac{|2E(\hat{H}_1)_{ik}|^2}{\lambda_i - \lambda_k} \]

If \( i, k \in \{1, 2, 3\} \), \( |(\hat{H}_1)_{ik}|^2 \) will be zero or in scale of \( \epsilon^2 \). Otherwise either \( \lambda_i \) or \( \lambda_k \) will be \( \lambda_4 \), then the denominator will be \( \gtrsim \epsilon^{-1} \), moreover, since \( (\hat{H}_1)_{i4} \sim \sqrt{\epsilon} \), the square in the numerator provides another necessary \( \epsilon \).
Perturbative expansion: Corrections to the eigenstates

\[ V^{(ex)} = V^{(0)}(1 + W_1 + W_2 + \cdots) \]

\( W_n \) are \( n^{th} \) order corrections.

\[
(W_1)_{ij} = \begin{cases} 
0, & i = j \\
-\frac{2E(\tilde{H}_1)_{ij}}{\lambda_i - \lambda_j}, & i \neq j
\end{cases}
\]

Again if \( i, k \in \{1, 2, 3\} \), \( (\tilde{H}_1)_{ik} \) will be zero or in scale of \( \epsilon \), otherwise either \( \lambda_i \) or \( \lambda_k \) will be \( \lambda_4 \), then the denominator will be \( \gtrsim \epsilon^{-1} \).
Perturbative expansion: eigenstates continued

\[
(W_2)_{ij} = \begin{cases} 
-\frac{1}{2} \sum_{k \neq i} \frac{|2E(\tilde{H}_1)_{ik}|^2}{(\lambda_i - \lambda_k)^2}, & i = j \\
\frac{1}{\lambda_i - \lambda_j} \sum_{k \neq i, j} \frac{2E(\tilde{H}_1)_{ik} 2E(\tilde{H}_1)_{kj}}{\lambda_k - \lambda_j}, & i \neq j
\end{cases}
\]

It is a little more complicated to confirm the scale of \( W_2 \).

- \( i = j \) if \( i = 4 \), the denominator will be \( \gtrsim \epsilon^{-2} \); if \( i = j \neq 4 \) and \( k \neq 4 \) the numerator will be \( \sim \epsilon^2 \); if \( i = j \neq 4 \) and \( k = 4 \), the denominator will be \( \gtrsim \epsilon^{-2} \);
- \( i \neq j \) if \( i, j, k \in \{1, 2, 3\} \), the numerator will be \( \sim \epsilon^2 \); if \( i = 4 \) or \( j = 4 \), the denominator will be \( \gtrsim \epsilon^{-1} \) and the numerator will be \( \sim \epsilon^{3/2} \); if \( k = 4 \), the denominator will be \( \gtrsim \epsilon^{-1} \) and the numerator will be \( \sim \epsilon \).
Review of the calculation process

Vacuum Rotations

\[ \theta_{23}, \delta_{23} \rightarrow \text{sterile} \]

Matter Rotations

\[ \tilde{\theta}_{13}, \alpha_{13} \rightarrow \tilde{\theta}_{12}, \alpha_{12} \]

Correction

\[ \sim \mathcal{O}(\epsilon) \]

Perturbation Expantions

1st order

\[ \sim \mathcal{O}(\epsilon^2) \]

2nd order

\[ \sim \mathcal{O}(\epsilon^3) \]

\ldots
Precision test: active eigenvalues

Zeroth Order, NO

$\frac{|\Delta \lambda|}{\Delta m_{ee}^2}$ vs $E$ (GeV)

- $\lambda_1$
- $\lambda_2$
- $\lambda_3$
Presion test: oscillation possibilities

$$\nu_\mu \rightarrow \nu_e, \text{ L}=1300(\text{km}), \text{ NO}$$

$$P_{SM, P_{3+1}, |\Delta P_{3+1}^{(0)}|, |\Delta P_{3+1}^{(1)}|, |\Delta P_{3+1}^{(2)}|}$$
Possibilities shift from the Standard Model

\[ \nu_\mu \rightarrow \nu_e \]

\[ P_{3+1}, \text{Normal Order} \]

\[ P_{\text{SM}} - P_{3+1}, \text{Normal Order} \]
Possibilities shift from the Standard Model (Continued)

$$\nu_\mu \rightarrow \nu_\mu$$

![Heatmaps showing $P_{3+1}$, Normal Order and $|P_{SM} - P_{3+1}|$, Normal Order](image)
Possibilities shift from the Standard Model (Continued)

\( \nu_\mu \rightarrow \nu_\tau \)

\[ P_{3+1}, \text{Normal Order} \]

\[ |P_{SM} - P_{3+1}|, \text{Normal Order} \]
Resolve crossings of the 0\textsuperscript{th} order eigenvalues in the active neutrino space (crossings to sterile eigenvalues require very high neutrino energy)

Exact in vacuum

Accurate enough for current/future experiment