International Neutrino Summer School 2019 Group Exercise Questions

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1 Introduction to the Physics of Massive Neutrinos

Concha González-García (YITP-Stony Brook U. & ICREA-U.Barcelona)
Maria.Gonzalez-Garcia@StonyBrook.edu

1 Show that only for a free massless fermion the chirality eigenstates are also helicity eigenstates.

2 Show that the mass matrix M of a Majorana mass

$$-\mathcal{L}_{Maj} = -\frac{1}{2}\overline{\nu_{L,i}} M_{ij} (\nu_{L,j})^c + h.c \tag{1}$$

must be symmetric $M_{ij} = M_{ji}$.

3 Show that if there are $N = 3 + s$ massive neutrinos, the leptonic mixing matrix is dimension $3 \times N$ and contains $3s + 3$ physical angles and $2s + 1$ phases for Dirac neutrinos and $3s + 3$ for Majorana neutrinos

4 The decay width for β decay $N \to Pe^- \bar{\nu}_e$ after integrating over the proton momentum is

$$d\Gamma = G_F^2 \cos^2\theta_C F(E,Z) 2\pi \sum_{spin} \sum_i |U_{ei}|^2 |\bar{u}_e(p)\gamma^0(1-\gamma^5)v_{\nu_i(k)}|^2 \frac{d^3p}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2w} \delta(E_0 - E - w) \tag{2}$$

G_F is Fermi constant, θ_C is Cabbibo's angle, $F(E,Z)$ a Coulomb factor, E_0 is the mass difference between the initial and final nuclei, E is the electron energy and w is the neutrino energy. Obtain the electron energy spectrum $d\Gamma/dE$ and show that the Kurie function

$$K(T) \equiv \left[\frac{\frac{d\Gamma}{dE}}{C\, F(E,Z)\, |\vec{p}|\, E} \right]^{1/2} = \sqrt{(Q-T)\sum_i |U_{ei}|^2 \sqrt{(Q-T)^2 - m_i^2}} \tag{3}$$

$$\simeq \sqrt{(Q-T)\sqrt{(Q-T)^2 - \sum_i |U_{ei}|^2 m_i^2}}$$

with $Q = E_0 - m_e$ y $T = E - m_e$ y $C = G_F^2 \cos^2\theta_C/\pi^3$. Show that the last equality holds only for $Q - T \gg m_i$ and $\sum_i |U_{ei}|^2 = 1$.

Plot $K(T)$ as a function of T for tritium ($Q = 18.6$ KeV) for $m_\beta = \sqrt{\sum_i |U_e i|^2 m_i^2} = 0$ and for $m_\beta = \sqrt{\sum_i |U_e i|^2 m_i^2} = 2.2$ eV.

5 The charged pion decays almost 100% of the time into a muon and a (muon-type) neutrino ($\pi^+ \to \mu^+\nu_\mu$) In the reference frame where the parent pion is at rest, compute the muon energy as a function of the muon-mass (m_μ), the charged pion mass (m_π), and the neutrino mass (m_ν). What is the absolute value of the muon momentum (tri)vector? Numerically, what is the relative change of the muon momentum between $m_\nu = 0$ and $m_\nu = 0.1$ MeV? It is remarkable that the muon momentum from pion decay at rest has been measured at the 3.4×10^{-6} level (Phys. Rev. D53, 6065 (1996)). This provides the most stringent current constraint on the muon-neutrino mass.

6 Derive the characteristic E/L in eV^2 for atmospheric neutrinos, and for T2K and for Daya-Bay.

7 Show that in a disappearance experiment the survival probability for neutrinos and antineutrinos is the same.

8 a) What is the matter potential for $\nu_\tau \to \nu_s$? (ν_s is an sterile neutrino) Compare with the potential for $\nu_e \to \nu_{\mu,\tau}$
b) In the core of a supernova the matter density is $\rho \sim 10^{14}$ g/cm^3. Obtain the characteristic value for the matter potential for for$\nu_\tau \to \nu_s$ and $\bar{\nu}_\tau \to \nu_s$ in the core of the supernova c) For what characteristic mass differences (assume $E_\nu \sim 10MeV$) can occur the MSW effect in such supernova? In which of the two channels occurs?

2 Neutrino Detection

Mark Messier (Indiana University)
messier@indiana.edu

1 **Neutrino detector Livingston plot**

The Livingston plot is a famous representation of progress in the construction of particle accelerators. This problem asks you to prepare a similar plot to show progress in neutrino detectors.

The first neutrino detector "El Monstro" was constructed by Reines and Cowan in the 1950s. For each decade since then and looking ahead into the future, find a few representative neutrino detectors and tabulate some essential data summarizing the detector technology. For example: experiment name, dates of operation, detection technology, mass (in tons), granularity (is cm). If you can find it, also detector cost. You are encouraged to divide the task among several groups and share your data to ensure the best possible coverage.

With the data in hand, what plot or plots can you make which best represent progress in neutrino detector technology?

2 **Magnetic DUNE**

Suppose you wanted to magnetize the future DUNE detector to allow for 3σ particle-by-particle electron vs. positron separation at 2 GeV. How large a field is required? Compare the energy stored in the field to the total energy used by the residents of Lead, SD in a single day. Suggest how you might construct such a field and estimate the cost of the materials required.

3 **Side-by-side**

Many of you no doubt work on current neutrino experiments and are expert with the simulation programs of those experiments. Using your collaboration's simulation tools, make event displays for the following neutrino topologies ranging from 10 MeV to 1 TeV. To aid the problem, a text file containing exact neutrino interaction topologies to simulate are posted here:

`https://docs.google.com/document/d/14dfU7gHvea_iLyb_TKxVGvR0B-bG-BJP96OgOHjk6H0/edit?usp=sharing`

You may choose to skip energies that are well below and/or well above the detector's capabilities but try to push the simulated energy as range to be as wide as possible.

4 **Future beam**

Propose a design for a near detector for a future beta beam (see, for example, Phys.Lett. B532 (2002) 166-172) which is targeted toward a distant mega-ton scale water Cherenkov detector.

3 Phenomenology of Accelerator and Atmospheric Neutrinos

Concha González-García (YITP-Stony Brook U. & ICREA-U.Barcelona)

Maria.Gonzalez-Garcia@StonyBrook.edu

1 Understanding the Atmospheric Neutrino Data
Read the article "Super-Kamiokande Atmospheric Neutrino Results" by T. Toshito, hep-ex/0105023. It contains an old summary of the atmospheric neutrino data. The talk by T. Kajita, presented at the Neutrino 1998 Conference, may also prove helpful in understanding some of the Super-Kamiokande terminology: hep-ex/981001.

1. Numerically calculate and draw histograms of the average muon neutrino survival probability in ten equal-size bins of $\cos\theta_x$ where θ_x is the angle between the neutrino direction and the vertical-axis at the detector's location ($\theta_x = 0$ for neutrinos coming straight from above, and $\theta_x = \pi$ for neutrinos coming from below). For simplicity, assume two flavor $\nu_\mu \to \nu_\tau$ transitions. Make one histogram for $E_\nu = 0.2$ GeV, 2 GeV, and 20 GeV and
$\Delta m^2 = 2.5 \times 10^{-4}$ eV^2, 2.5×10^{-3} eV^2, and 2.5×10^{-2} eV^2, for a grand total of nine plots. Assume throughout that the mixing is maximal, i.e., $\sin 22\theta = 1$, and that neutrinos are produced 20 km above the surface of the Earth.
2. Look at Figure 1 in the paper and compare with the results you got in part 1. Can you verify that $\Delta m^2 = 2.5 \times 10^{-3}$ eV^2 and $\sin^2 2\theta = 1$ is a good fit to the data (200 MeV is characteristic of sub-GeV events, 2 GeV is typical of multi-GeV events, and 20 GeV is typical of upward stopping muons. The fourth category, upwardthrough- going muons, has an average energy above 100 GeV.)?
3. Use the number of observed sub-GeV "e-like" events (as these seem to agree well with Monte Carlo predictions) to obtain an order of magnitude estimate of the electron neutrino flux (neutrinos per unit time and unit area). The cross section for detecting neutrinos at this energy range is roughly 5 fb.

2 You want to design an accelerator neutrino oscillation experiment which is sensitive to oscillations with $|\Delta m^2 = 3 \times 10^{-3|}$ eV^2 and to the ordering. For this you want the contribution of the matter potential to the oscillation phase to be 20% of the Δm^2_{31} contribution at oscillation maximum. At what distance from the source do you have to put your detector?

4 Long Baseline Oscillation Experiments

Patricia Vahle (William and Mary), Jeff Hartnell (Sussex)

plvahle@wm.edu

Long-baseline neutrino oscillation experiments seek to measure CP violation, the mass hierarchy and the octant of θ_{23} using electron (anti)neutrino appearance and muon (anti)neutrino disappearance in a muon (anti)neutrino beam.

The NOvA experiment has a baseline of 810 km and a peak neutrino energy of 1.9 GeV. For the purposes of these problems, except as noted, consider the NOvA electron (anti)neutrino appearance and muon (anti)neutrino disappearance measurements as a "counting" experiment where you consider the beam to be monochromatic. Similarly, consider backgrounds and systematic uncertainties to be negligible compared to statistical uncertainties, again except as noted in individual problems.

For this question you should consider three flavor neutrino oscillations. The standard two flavor approximations will be insufficient.

Consider the duration of the NOvA experiment as an exposure of 36×10^{20} protons on target which may be divided between neutrino and antineutrino beams.

For an exposure of 6×10^{20} protons on target the following electron (anti)neutrino signal (S) and background (B) counts are expected. In neutrino-mode: B=7.75, S=24.19(13.76), 17.93(11.07), 27.85(18.88) for $\delta_{CP} = 0, \pi/2, 3\pi/2$ NH(IH) and $\sin^2\theta_{23} = 0.5$, $\sin^2(2\theta_{13} = 0.085$, $\sin^2(2\theta_{12} = 0.87$, $\Delta m^2_{12} = 7.5 \times 10^{-5}\text{eV}^2$ and $\Delta m^2_{23} = 2.5 \times 10^{-3}\text{eV}^2$. In antineutrino-mode: B=2.87, S=7.58(8.91), 8.53(11.40), 5.58(7.68) for the same parameter values.

Assume the backgrounds are independent of the oscillation parameters.

For background information please have a look at https://arxiv.org/abs/1210.1778.

1 NOvA must decide how to operate their beam. The choice ranges from operating the beam in neutrino mode 100% of the time, through to 100% in antineutrino mode. What is the optimal run plan for NOvA to determine specifically the mass hierarchy?

Consider the following scenarios for true oscillation parameters:

- Normal mass hierarchy, $\sin^2(\theta_{23}) = 0.6$ and $\delta_{CP} = 3\pi/2$
- Normal mass hierarchy, $\sin^2(\theta_{23}) = 0.4$ and $\delta_{CP} = 3\pi/2$
- Inverted mass hierarchy, $\sin^2(\theta_{23}) = 0.6$ and $\delta_{CP} = 3\pi/2$.
- Inverted mass hierarchy, $\sin^2(\theta_{23}) = 0.4$ and $\delta_{CP} = \pi/2$.

How does your proposed run plan depend on the oscillation parameters that Nature has chosen? What would your run plan be in those specific scenarios? What should the run plan be when we don't know what Nature has chosen?! Do you need to run antineutrinos? Invent a physics scenario of your own choosing that might cause you to make the incorrect hierarchy selection. Do a quantitative analysis of this scenario to see if such a thing is really possible.

If you have time, consider the sensitivity to the octant (and CP violation) and consider the optimal run plan to measure those parameters.

2 If you were designing NOvA from scratch, what choices might you make to increase the sensitivity of the NOvA experiment to the mass hierarchy? Consider the beam power at NuMI as a fixed input, and analyze your optimized experiment or experiments quantitatively.

5 Statistical Methods in Neutrino Physics

Tom Junk (Fermilab)

trj@fnal.gov

1 Show that the maximum-likelihood combination of several independent measurements with identical but uncorrelated Gaussian uncertainties is the average of the measurements.

2 If the uncertainties are different for each measurement, show that the maximum-likelihood combination is

$$x_{\text{comb}} = \frac{\sum x_i/\sigma_i^2}{\sum 1/\sigma_i^2}$$

where the ith measurement is $x_i \pm \sigma_i$. What is the total uncertainty on the combined x_{comb}?

3 For what range of μ is the median n_{med} of the Poisson probability distribution

$$P(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$$

equal to zero? For what range of μ is $n_{\text{med}} = 2$? Using a Gaussian approximation, for what range of μ is $n_{\text{med}} = 10000$?

4 Two measurements $m_1 = 10.1$ and $m_2 = 10.2$ of the same quantity have systematic uncertainties that are correlated $c_1 = 1.5$ and $c_2 = 1.6$, respectively (they come from the same source), and uncorrelated, $u_1 = 1.1$ and $u_2 = 1.0$.

a What is the correlation coefficient ρ from the covariance matrix between the two measurements?

b Combine the two measurements using BLUE, obtaining a central value and the total uncertainty. Find the weights w_1 and w_2.

c Suppose now that $u_1 = 0$ and $u_2 = 0$. Combine again. What are the weights? What is the total uncertainty on the combined measurement?

5 An experiment measuring a rare process using event counting runs for one year, expects 4.3 background events and observes ten events. The signal acceptance is 80%. Ignore systematic uncertainties on the background and acceptance for parts a through g.

a Calculate the maximum-likelihood value of the signal rate in events, and the upper and lower error bars using $\Delta \log L = 1/2$ rule.

b Calculate the Bayesian 68% confidence interval for the same result.

c Calculate the Feldman-Cousins 68% confidence interval for the same result.

d What is the frequentist coverage of the three methods assuming a true signal rate of 6 events?

e Calculate the 95% CL upper limit on the signal strength in events.

f Calculate the p-value for observing the number of events or more given only the background hypothesis. Is this enough to claim a discovery? Evidence?

g How many years must the experiment run in order for the median expected p-value to be 2.7×10^{-7}?

h Now add a $\pm 20\%$ relative uncertainty on the background, given a truncated Gaussian prior distribution, and a $\pm 10\%$ uncorrelated uncertainty on the acceptance. Repeat the above calculations.

6 Describe some statistical interpretation problems with this article:

L. Alunni Solestizi et al., 2018 JINST 13 P07003.

`https://iopscience.iop.org/article/10.1088/1748-0221/13/07/P07003`

7 Under what circumstances are profiling and marginalization expected to give the same result for inclusion of the effects of nuisance parameters?

6 Short Baseline Neutrino Experiments and Phenomenology

7 Introduction to Leptogenesis

Jessica Turner (Fermilab)

jturner@fnal.gov

1 Assuming a CP T conserving quantum field theory, three conditions (Sakharov's conditions) are required for a theory to generate the observed matter anti-matter asymmetry. Can you list these conditions and state why they are necessary.

2 Can you write the mass eigenstates of the active neutrinos assuming they acquire their mass from a type-I seesaw mechanism.

3 What is the minimal number of right-handed neutrinos, in the type-I seesaw mechanism, needed for leptogenesis?

4 How does the type-I seesaw mechanism satisfy Sakharov's three conditions?

5 Can you draw the tree and one-loop level Feynman diagrams that contribute to the lepton CP asymmetry?

6 At which temperatures do the three flavors of charged leptons come into thermal equilibrium?

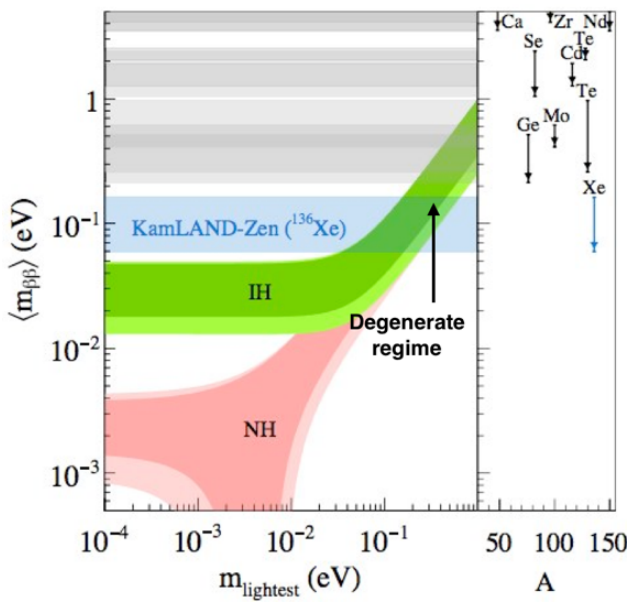
8 Neutrinoless Double Beta Decay Experiments

Cheryl Patrick (University College London)

c.patrick@ucl.ac.uk

1 **Consider the Lobster**

In the world of neutrinoless double-beta decay, there is a famous plot, which we call the "Lobster" plot (you can guess why). You can find it in PRL 117, 082503 (2016), but here is a copy:



The Lobster plots the Majorana neutrino mass of neutrinoless double-beta decay vs. the mass of the lightest of neutrino state. As we don't know the whether the hierarchy is inverted or normal, we don't know whether this is the mass of state ν_1 or of state ν_3.

The electron-neutrino mass we think about in neutrinoless double beta decay is given by the formula:

$$m\beta\beta = \left| \sum_{i=3}^{3} |U_{ei}|^2 \, m_i e^{i\alpha_i} \right|, \tag{4}$$

where U_{ei} correspond to the electron components of the PMNS mixing matrix, m_i are the three neutrino mass states (let's hope there are no sterile neutrinos!) and α_1 and α_2 are the two Majorana phases, which we don't know.

Your goal for this project is to break through the Lobster's shell and understand the shape of this odd plot.

1 The Lobster is wise, and knows some information about mass differences and neutrino mixing angles, from oscillation experiments. You will need to know this, so write it down now.

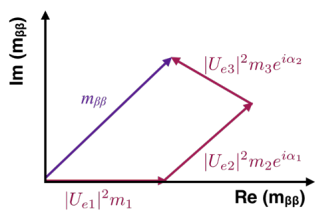
2 The double-beta mass is a kind of effective electron neutrino mass. Experiments like KATRIN, which look at the beta decay spectrum end-point, also measure an electron (anti)neutrino mass, which is also a combination of values from the PMNS matrix. Find out the expression for this mass $m\beta$.

   a Does it depend on CP violating phases or Majorana phases? If it does, continue this question assuming that they are all zero.

   b Is it the same as $m\beta\beta$? If not, how does it differ?

   c Is it possible for $m\beta$ to be zero, given the constraints we have from oscillation experiments? Using just the oscillation information, can you put any (upper or lower) limits on $m\beta$?

3 Now let's go back to $m\beta\beta$. Because is a coherent sum of the three mass states, it is affected by the Majorana phases, which introduce a complex element to the calculation. We can represent the three components of $m\beta\beta$ as vectors in the complex plane. $m\beta\beta$ will correspond to the length of the sum of the three vectors. As there are only two Majorana phases, the $m1$ component will be real. Remember, we don't know the Majorana phases. Here's what a double-beta sum could look like:



x

We are going to consider a few different scenarios, now. What would the diagram above look like in these situations? Is it possible for $m\beta\beta$ to be 0? If so, try to be as quantitative as you can about when that would be possible. Can you relate what you learn (qualitatively or quantitatively) to features of the Lobster plot?

   a In the inverted hierarchy, when $m2 \approx m1 >> m3$

   b In the inverted hierarchy, when $m2 \approx m1 >> m3 = 0$

   c In the inverted hierarchy, when $m2 \approx m1 > m3 >> 0$

   d In the normal hierarchy, when $m3 > m2 > m1 >> 0$

   e In the normal hierarchy, when $m3 >> m2 > m1 = 0$

4 If the lightest neutrino is massless, what are the maximum and minimum values for $m\beta\beta$, for the inverted and the normal hierarchy? How would the vectors in the plot above be arranged in these cases?

5 What's the case (or set of cases) that allows $m\beta\beta$ to be 0?

6 If the lightest neutrino is heavy, the inverted and normal hierarchies effectively merge into a "degenerate regime" where all the masses are effectively the same. Can you work out at what sort of masses we enter this regime, and get an equation for how the lightest mass relates to $m\beta\beta$ in this case?

7 (Bigger project, if you have time) Current double-beta decay experiments can measure masses of the order of 100 meV. Next generation measurements will be an order of magnitude better and cover the inverted hierarchy. But oscillation experiments at the moment favour the normal hierarchy. If that's what nature has given us, what is our chance of $m\beta\beta$ being in our range?

   a Use a Monte Carlo technique to calculate values of $m\beta\beta$, in the inverted and normal hierarchies, using the numbers we know from oscillation experiments, and using random throws to generate values for the lightest neutrino mass and the two Majorana phases. Assume that:

      i There is no CP violating phase

ii The probability for the log of the lightest neutrino mass (in eV) is evenly distributed between 0 and ~ 4.

iii The probabilities for the two Majorana phases are evenly distributed between 0 and 2π.

b Plot the values you get for the randomly-chosen lightest neutrino mass and your calculated $m\beta\beta$ on 2d histograms (one each for normal and inverted hierarachy). You made your own lobster plots! What did you learn?

c Now you're at it, why not try calculating $m\beta$ as well and plot $m\beta$ vs $m\beta\beta$?

2 **Your $0\nu\beta\beta$ future**

It's 2030, England has just won the World Cup, and you've just landed a top job at Fermilab! What a great year! Even more exciting, LEGEND `http://legend-exp.org` and NEXO `https://nexo.llnl.gov` experiments have just announced that they have observed neutrinoless double-beta decay. LEGEND measured a half-life of 5×10^27 years for their isotope, while NEXO measured a half-life of 7.5×10^27 years. In each case, they have a 20% uncertainty on their measurement.

- What neutrino mass limits do these correspond to? (Hint: arXiv:1610.06548 [nucl-th])
- Do you think the two measurements are compatible? (Think about the different isotopes and how they behave differently).
- What half-life would you expect SNO+ https://www.snolab.ca/science/experiments/snoplus to measure?
- Do they tell us anything else about the nature of the neutrino?

Now that neutrinoless double beta decay is in the news, Fermilab wants you, their hottest new scientist, to design a new experiment to learn more about it. There's lots of cash available, and you have access to the resources of the lab and the current far detector locations of the Fermilab oscillation experiments. Propose your new design! Some things to think about:

- What do you aim to learn with your new experiment? (Think about nuclear physics, mass hierarchy, double-beta decay mechanisms).
- What isotope(s) would you use? How much will you need, and what do you estimate it will cost? Why is this a good choice? Do you save on other detector components? Can your detector be used for anything else as well?
- What detector technology will you use? Feel free to be creative and steal the best ideas from current detectors like KamLAND Zen, GERDA, SNO+, SuperNEMO, and CUORE.

9 Lepton-Nucleus Cross Section Theory

Noemi Rocco (Argonne National Lab & Fermilab)

c.patrick@ucl.ac.uk

1. Consider the charge-current scattering process: $\nu_\ell^- + n \to \ell^- + p$ where $\ell = e, \mu, \tau$.

   - What is the minimum value of the energy transfer ω for which the reaction can take place?
   - Consider the case in which $\ell = \mu^-$ and the elastic scattering happens on a neutron at rest, *i.e.* the neutron quadri-momentum is given by $(m_n, 0)$ and the proton $(m_p + \omega, \mathbf{q})$. Show that the reconstructed neutrino energy reads

$$E_\nu = \frac{m_p^2 - m_n^2 - m_\mu^2 + 2m_n E_\mu}{2(m_n - E_\mu + p_\mu \cos\theta_\mu)} \ , \tag{5}$$

   where θ_μ is the muon angle relative to the neutrino beam.

- Write down the analogous expression for a moving neutron in the initial state, *i.e.* $(E_n, \mathbf{p}_n)$. To derive the neutrino energy expression, define the angle between the neutron and the neutrino beam momentum as $\cos\theta_n = (\mathbf{k}_\nu \cdot \mathbf{p}_n)/(|\mathbf{k}_\nu||\mathbf{p}_n|)$.

2. The most general expression for the hadronic tensor is constructed out of $g^{\mu\nu}$ and the independent moment of the initial nucleon p and the momentum transfer q, yielding

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_N^2} p^\mu p^\nu + i\epsilon^{\mu\nu\alpha\beta} \frac{W_3}{2m_N^2} p_\alpha q_\beta + \frac{W_4}{m_N^2} q^\mu q^\nu + \frac{W_5}{m_N^2}(p^\mu q^\nu + p^\nu q^\mu) \tag{6}$$

where W_i are called structure functions and m_N is the nucleon mass.

- In the electromagnetic case parity-violating effects are NOT present. The hadronic electromagnetic current matrix elements are polar-vectors and so the tensor must have specific properties under spatial inversion. In particular, in this case $W_3 = 0$. The current conservation condition at the hadronic vertex requires

$$q_\nu W^{\mu\nu} = q_\mu W^{\mu\nu} = 0 \tag{7}$$

As a result of this relation, verify that only two structure functions are independent and the hadronic electromagnetic tensor reads

$$W^{\mu\nu} = W_1 \left(g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_N^2} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right) \tag{8}$$

- For the scattering process $\nu_\mu^- + n \to \mu^- + p$, the double-differential cross section is proportional to

$$\frac{d^2\sigma}{dE_\mu d\Omega_\mu} \propto L_{\mu\nu} W^{\mu\nu} \,. \tag{9}$$

The contraction between the leptonic and hadronic tensor can be rewritten as

$$L_{\mu\nu} W^{\mu\nu} = \frac{16}{m_N^2} \sum_{i=1}^{5} L_i W^i \tag{10}$$

where the W^i are the structure functions reported above and L_i are leptonic factors, you can find the explicit expression in in Nucl. Phys. A 789, 379 (2007) or try to compute them. How would this generic expression change for the $\bar{\nu}_\mu$ scattering process?

10 Origin and Nature of Neutrino Mass

Goran Senjanović (ICTP)

goran@gssi.infn.it

These problems cover the essence of my course. The first problem is simply a summary of Dirac and Majorana spinors in the two-component form. This material is a pre-requisite, so I strongly advice you to find some time to go through it in case you are not familiar with the math and physics of Dirac and Majorana mass. The other problems cover the lectures' material and doing them would help tremendously understand the subtleties of the physics involved. I wish to warn you that there could be misprints or even an occasional error, so please contact me if you have any comments or questions regarding this material.

Problem 1

A four-component Dirac spinor transforms under the Lorentz group as

$$\Psi \to S\,\Psi, \qquad S \equiv e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}}, \tag{11}$$

with

$$\Sigma^{\mu\nu} \equiv \frac{1}{4i}[\gamma^\mu, \gamma^\nu], \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{12}$$

with the explicit form for Dirac matrices in my conventions

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

1. Show that the $\Sigma_{\mu\nu}$ satisfy the Lorentz algebra. You can do in a generic manner, or even better, by separating the rotations and boosts, and then check the usual relations among rotations and boosts.

2. Introduce Left and Right chiral spinors

$$\Psi_{L,R} \equiv \frac{1 \pm \gamma_5}{2} \Psi \equiv L\,(R)\,\Psi \tag{13}$$

$$\gamma_5 = -i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$$

or

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Using Eq. (1), show that

$$u_{L(R)} \to e^{i\vec{\sigma}/2(\vec{\theta} \pm i\vec{\phi})} u_{L(R)} \tag{14}$$

where

$$\Psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ u_R \end{pmatrix}$$

What does $\vec{\theta}$ represent? And $\vec{\phi}$?

3. Take a boost in the z-direction and find an expression for $\vec{\phi}$.

4. Define charge-conjugation transformation

$$\Psi^c \equiv C\overline{\Psi}^T \tag{15}$$

with

$$C^T \gamma^\mu C = -\gamma_\mu^T, \; C^T = C^\dagger = C^{-1} = -C$$

An explicit choice: $C = i\gamma_2\gamma_0$.

Show that

$$\Psi^c \to S\Psi^c \tag{16}$$

when $\Psi \to S\Psi$. In other words, Ψ^c transforms the same way as Ψ, i.e. it is also a proper spinor.

5. Take

$$\Psi = \Psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}$$

Compute Ψ^c. What is its chirality ?

6. What happens to u_L and u_R under parity? By definition, under parity transformation

$$\Psi \to \gamma_0 \Psi$$

in order to have the four vector $\bar{\Psi}\gamma_\mu\Psi$ behave correctly (three-vector and a scalar).

Now take a four-component Majorana spinor Ψ_M, defined by

$$\Psi_M = \Psi_M^c$$

1. Show that Ψ_M can be written as

$$\Psi_M = \begin{pmatrix} u_L \\ -i\sigma_2 u_L^* \end{pmatrix}$$

   or equivalently

$$\Psi_M = \Psi_L + C\,\overline{\Psi_L}^T. \tag{17}$$

2. Take the free Majorana Lagrangian

$$\mathcal{L} = i\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - m\bar{\Psi}_M \Psi_M \tag{18}$$

   Show that this is equivalent to

$$\mathcal{L} = i u_L^\dagger \bar{\sigma}^\mu \partial_\mu u_L - \frac{1}{2} m \left(u_L^T\, i\sigma_2 u_L + \text{h.c.} \right) \tag{19}$$

   where $\bar{\sigma}^\mu = (1; -\sigma_i)$ (and $\sigma^\mu = (1; \sigma_i)$, which enters for RH spinors).

3. Show the following vector current vanishes

$$\bar{\Psi}_M \gamma^\mu \Psi_M$$

   Can you explain why?

   `Hint: can there be invariance under` $\Psi_M \to e^{-i\alpha}\Psi_M$`?`

Problem 2

Multi-generation see-saw mechanism. This exercise serves to understand the properties of Majorana spinors and the seesaw mechanism of neutrino mass - an important aspect of the course.

We introduce a gauge singlet fermion, the so-called RH neutrino ν_R, per generation and write down the most general Yukawa interaction for n generations

$$\mathcal{L}_Y = \bar{\nu}_R \, \tilde{\Phi}^\dagger \, Y_D \, \ell_L \, + \, \nu_R^T \, C \frac{M_N^\dagger}{2} \, \nu_R + h.c. \tag{20}$$

where we use a compact notation for the following vectors in the generation space

$$\nu_R = \begin{pmatrix} \nu_1 \\ \cdot \\ \cdot \\ \nu_n \end{pmatrix}_R , \qquad \ell_L = \begin{pmatrix} \ell_1 \\ \cdot \\ \cdot \\ \ell_n \end{pmatrix}_L \tag{21}$$

with the usual notation for leptonic and Higgs doublets

$$\ell_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L , \quad \tilde{\Phi} = i\sigma_2 \Phi^* \tag{22}$$

Thus, in (20), Y_D and M_N are $n \times n$ matrices in the generations space.

- Show that M_N is a symmetric matrix. This is a general property of Majorana mass matrices of the same species of particles.
- Show that (20) is invariant under the SM $SU(2) \times U(1)$ gauge symmetry.
- Introducing
$$N_{iL} \equiv C \bar{\nu}_{iR}^T \tag{23}$$
show that (20) can be written as
$$\mathcal{L}_Y = N_L^T \, C \, \Phi^T \, (-i\sigma_2) \, Y_D \, \ell_L \, + \, N_L^T \, C \, \frac{M_N}{2} N_L + h.c. \tag{24}$$
- From the usual form for the Higgs doublet in the unitary gauge
$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h + v \end{pmatrix}$$
show
$$\mathcal{L}_{\text{mass}} = N_L^T \, C M_D \nu_L \, + \, N_L^T \, C \, \frac{M_N}{2} N_L + h.c. \tag{25}$$
where
$$M_D = \frac{Y_D \, v}{\sqrt{2}} \tag{26}$$
- Show next
$$\mathcal{L}_{\text{mass}} = \frac{1}{2} \left[\nu_L^T \, M_D^T \, C \, N_L \, + \, N_L^T \, M_D \, C \, \nu_L \, + \, N_L^T \, M_N \, C \, N_L \right] + h.c.. \tag{27}$$
In other words, one has a mass matrix (up to a factor 1/2) between ν_L and N_L
$$M_{\nu N} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_N \end{pmatrix} \tag{28}$$

- Assume $M_N \gg M_D$ and show that (28) can be put in block-diagonal form as

$$D_{\nu N} = \begin{pmatrix} M_\nu & 0 \\ 0 & M_N \end{pmatrix} \tag{29}$$

by a unitary transformation

$$U^T M_{\nu N} U = D_{\nu N}$$

where

$$U = \begin{pmatrix} 1 & \Theta^\dagger \\ -\Theta & 1 \end{pmatrix} \tag{30}$$

and

$$M_\nu = -M_D^T M_N^{-1} M_D \tag{31}$$

with $\Theta = M_N^{-1} M_D$, and where only terms up to order Θ^2 are considered.

- Next diagonalize symmetric matrices M_ν and M_N by unitary matrices V_L and V_R

$$M_\nu = V_L^* m_\nu V_L^\dagger \tag{32}$$

and similarly for M_N. The convention above corresponds to V_L being the leptonic mixing matrix (the PMNS matrix) in the basis of diagonal charged leptons

$$\frac{g}{\sqrt{2}} \bar{\nu}_L \gamma^\mu V_L e_L W_\mu^+ \tag{33}$$

where ν and e stand for all the neutrinos and charged leptons respectively.

Recall that one can always fix such a basis since mixings depend on the relative rotations between neutrinos and charged leptons - just as in the case of up and down quarks discussed in the class.

- Once obtained, the physical states (eigenstates of mass matrices) can be cast in the 4-dimensional Majorana form as in the problem 1. Do that and show that the factor 1/2 in (27) is not physical.

- Once neutrinos are massive, there will be flavor mixings in the leptonic sector. As usual, there will be in general CP phases.

If neutrinos are Majorana particles, is the number of phases the same as in the Dirac case (i.e. the quark case)? More precisely, we speak here of physical phases that cannot be rotated away by phase redefinitions. In particular, does one need three generations in order to have CP violation? Compute the number of physical phases for n generations.

Problem 3

a. Let us imagine that neutrino is a Majorana particle, i.e. $\nu \equiv \nu_L + C\,\overline{\nu_L}^T$. This happens naturally in the seesaw mechanism of the above problem, when there exist a heavy neutral Majorana lepton $N \equiv N_L + C\,\overline{N_L}^T$, with a mass m_N. Neutrino gets a mass through the mixing with N which then leads to N having weak interaction

$$\mathcal{L} = \frac{g}{\sqrt{2}}\theta\, W_\mu^+ \bar{N}\gamma^\mu L e + \text{h.c.}, \tag{34}$$

where L is the usual chiral projector $L = (1+\gamma_5)/2$. In the above formula $\theta = m_D/m_N = \sqrt{m_\nu/m_N}$ is the mixing between ν and N, and $m_D = y_D v$ is the Dirac mass term between ν and N. In the usual notation, v is the vacuum expectation value v of the Higgs field and y_D is what we call the neutrino Dirac Yukawa coupling defined through the interaction

$$y_D N_L^T C \nu_L (h+v) + h.c. \tag{35}$$

i. Compute the decay rate $N \to W^+ + e$. Neglect the electron mass.

ii. What is the result in the limit of the vanishing W mass? Does the result depend on how you take the limit, i.e. whether the gauge coupling g goes to zero or the vacuum expectation value v? Can you explain what is going on in both cases? This is the test of your knowledge of the Yang-Mills theory.

iii. Is there another decay channel? Recall that N is Majorana particle, or better half-particle and half-antiparticle.

iv. Compute the lifetime of N.

b. Let us imagine that there is a another heavy W' boson with the following interaction

$$\mathcal{L}' = \frac{g'}{\sqrt{2}}\, W_\mu'^+ [\bar{N}\gamma^\mu L(R) e + \bar{u}\gamma^\mu L(R) d] + \text{h.c.} \tag{36}$$

and assume $M_{W'} > m_N$. The notation says that the W' has either purely LH or RH couplings.

i. Compute the three body decay rate $N \to e + u + \bar{d}$. You can set all masses to zero, except for m_N. If you find it it too long or too hard, use the analogy with the muon 3-body decay, so that you can do the rest below.

ii. Is this the only channel?

iii. Does the decay rate depend whether the coupling is L or R?

iv. Compute the lifetime of N in this case.

v. Take $g' = g$, $m_\nu = 0.1$eV, $m_N = 240$GeV, $m_{W'} = 4.8$TeV and compare the values for the 2 and 3-body decay rates that you computed. Which is bigger? This situation is a realistic possibility for the so-called LR symmetric theory that predicted originally massive neutrinos and led to the seesaw mechanism.

Problem 4

Left-Right Symmetry and Neutrino mass. As discussed in the lectures, the LR symmetric extension of the SM, put forward in order to understand parity violation in weak interactions, led originally to the existence of the RH neutrino and neutrino mass. Moreover, it provides a natural framework for the seesaw mechanism.

Take a LR symmetric theory based on the gauge group

$$G_{LR} = SU(2)_L \times SU(2)_R \times U(1)_{B-L}$$

with fermions transforming as

$$\begin{aligned} \ell_L &= \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad ; \quad \begin{pmatrix} \nu \\ e \end{pmatrix}_R = \ell_R \\ q_L &= \begin{pmatrix} u \\ d \end{pmatrix}_L \quad ; \quad \begin{pmatrix} u \\ d \end{pmatrix}_R = q_R \end{aligned} \tag{37}$$

so that the covariant derivative is

$$D_\mu = \partial_\mu \, - \, ig\vec{T}_L \cdot \vec{A}_{\mu L} \, - \, ig\vec{T}_R \cdot \vec{A}_{\mu R} \, - \, ig' \frac{(B-L)}{2} B_\mu \tag{38}$$

where

$$(B-L)\ell = -\ell, \qquad (B-L)q = \frac{1}{3}q \tag{39}$$

Assume the existence of Higgs triplets Δ_L and Δ_R in the adjoint representations of $SU(2)_L$ and $SU(2)_R$ respectively, with

$$\Delta_{L,R} \to U_{L,R}\, \Delta_{L,R}\, U^\dagger_{L,R} \tag{40}$$

where

$$\ell_{L,R} \to U_{L,R}\, \ell_{L,R}, \qquad q_{L,R} \to U_{L,R}\, q_{L,R} \tag{41}$$

- Show that the Yukawa interaction

$$\mathcal{L}_\Delta = y_\Delta \left(\ell_L^T\, C\, i\sigma_2\, \Delta_L\, \ell_L \, + \, \ell_R^T\, C\, i\sigma_2\, \Delta_R\, \ell_R + h.c. \right) \tag{42}$$

  is invariant under G_{LR}. This of course implies the $B-L$ gauge numbers of the triplets to be

$$(B-L)\Delta_{L,R} = 2\Delta_{L,R} \tag{43}$$

- Take the following potential

$$\begin{aligned} V \;\; = \;\; & -\mu^2 \left(\mathrm{Tr}\Delta_L^\dagger \Delta_L + \mathrm{Tr}\Delta_R^\dagger \Delta_R \right) + \frac{\lambda}{4} \left[(\mathrm{Tr}\Delta_L^\dagger \Delta_L)^2 + (\mathrm{Tr}\Delta_R^\dagger \Delta_R)^2 \right] \\ & + \frac{\lambda'}{2} \mathrm{Tr}\Delta_L^\dagger \Delta_L \, \mathrm{Tr}\Delta_R^\dagger \Delta_R \end{aligned} \tag{44}$$

  - Find under which conditions one gets a minimum

$$\langle \Delta_L \rangle = 0, \qquad \langle \Delta_R \rangle \neq 0 \tag{45}$$

  - Show that

$$\langle \Delta_R \rangle = \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix} \tag{46}$$

    preserves the SM gauge symmetry $SU(2)_L \times U(1)_Y$, with

$$\frac{Y}{2} \equiv T_{3R} + \frac{B-L}{2} \tag{47}$$

    and

$$Q = T_{3L} + T_{3R} + \frac{B-L}{2} \tag{48}$$

- Find the masses of the heavy gauge fields $W_R^\pm$ and Z_R, and determine Z_R as a function of the original fields in (38)
- Show that (42) produces the Majorana mass of the RH neutrino.
- Defining as in the seesaw picture (Problem 3) $N_{iL} \equiv C\bar{\nu}_{iR}^T$, show that LR symmetry dictates

$$M_N = V_R m_N V_R^T \tag{49}$$

in analogy with the light neutrino mass matrix

$$M_\nu = V_L^* m_N V_L^\dagger. \tag{50}$$

In other words, this guarantees that the LH and RH PMNS mixing matrices are given symmetrically. Show that the above equations correspond to the gauge interactions with the LH and RH W bosons

$$\mathcal{L}_{gauge} = \frac{g}{\sqrt{2}}(\bar{\nu}_L \gamma_\mu V_L^\dagger e_L W_L^\mu + \bar{\nu}_R \gamma_\mu V_R^\dagger e_R W_R^\mu) \tag{51}$$

The LH mixing matrix V_L is measured by neutrino oscillation experiments, neutrinoless double beta decay and other low energy experiments, whereas its RH analog V_R would be measured principally at hadron colliders such as the LHC.

- Let us take the LR symmetry a step further, and use it to compute the neutrino Dirac mass matrix. Take LR to be (generalised) charge conjugation as discussed in the course

$$\psi_L \to C\bar{\psi}_R^T \tag{52}$$

in which case it is to see that Yukawa couplings must be symmetric, and similarly the fermion mass matrices, in particular the neutrino Dirac mass matrix

$$M_D^T = M_D. \tag{53}$$

Use this to disentangle the seesaw

$$M_D = iM_N\sqrt{M_N^{-1}M_\nu}. \tag{54}$$

Next, for the sake of illustration assume $V_R = V_L^*$ which corresponds to the unrealistic unbroken LR situation, and show that M_D takes a simple form

$$M_D = V_L^*\sqrt{m_N m_\nu}V_L^\dagger \tag{55}$$

where m_N and m_ν are diagonal heavy and light neutrino mass matrices.

- In the problem 2, one shows that the $\nu - N$ mixing matrix is given by $\Theta = M_N^{-1}M_\nu$, so that for the above case of LR symmetry being charge conjugation, one obtains

$$\Theta = i\sqrt{M_N^{-1}M_\nu}. \tag{56}$$

Knowing Θ determines decay rates of N to charged leptons and W, $N_i \to \ell_j W^+$ just as the knowledge of charged fermion masses determines Higgs decays $h \to f\bar{f}$ in the SM. It shows that the LR symmetric theory does for neutrino mass what the SM does for charged fermions - allows to probe their Higgs mass origin. In the case of neutrinos, their Majorana nature requires the knowledge of both M_ν and M_N in analogy of m_f in the case of quarks and charged leptons.

Show that for $V_R = V_L^*$, this can be written as

$$\Theta = iV_L\sqrt{m_N^{-1}m_\nu}V_L^* \tag{57}$$

Use this expression to show that the decay rate of N_i into a ℓ_j charged lepton is proportional to

$$\Gamma(N_i \to W\ell_j) \propto |V_{ji}|^2 m_{\nu_i}\frac{m_{N_i}^2}{M_W^2}. \tag{58}$$

Determine the constant of proportionality using the result of problem 2.

Problem 5

Effective d=5 interaction for neutrino mass. Imagine that we add no new particles to the SM, in which case neutrino is massless at the renormalizable d=4 level.

1. Now, add a dimension five operator
$$\frac{(l_L^T \sigma_2 \Phi)\mathcal{C}(\Phi^T \sigma_2 l_L)}{M} \tag{59}$$
where l_L is the SM lepton doublet and Φ is the usual Higgs scalar doublet.

2. Show that the above interaction is allowed by the SM gauge symmetry.

3. What happens when the Higgs gets a vev? Compare with the see-saw formula.

4. Show that there are only three ways of writing the above operator. Then show that they are all equivalent.

5. Go to the physical unitary gauge and determine the leading Higgs boson-neutrino Yukawa interaction Show that it is proportional to neutrino mass as expected from the Higgs mechanism.

11 Neutrino Astronomy

Francis Halzen (University of Wisconsin)

francis.halzen@icecube.wisc.edu

Cosmic neutrinos are produced in Galactic and extragalactic cosmic accelerators that accelerate protons (nuclei) to more than a hundred million TeV. Cosmic neutrinos are produced in cosmic beam dumps where the target is provided by the radiation fields (yes, photons are the target) and dust or molecular clouds surrounding the accelerator. Given the large luminosity (and not just the high energy) of the accelerators that we observe, it is believed that the power is provided by the large gravitational energy associated with the collapse of stars, or with the particle flows into supermassive black holes residing at the centers of active galaxies (black holes feeding on their own galaxy). Shock waves are believed to be the mechanism by which 1 10

Research the principle of first and second order Fermi acceleration. If you find a really great explanation on the web or in a text book, let me know. I recommend the early references by E. Fermi.

1. Summarize the essential concept of shock acceleration in less than 3 ppt slides.

2. Make a plot of the velocity of the particles moving across the shock in 4 frames, the lab frame, the frame where the shock is at rest, the frame where upstream particles are at rest and the frame where the downstream particles are at rest. (Note the gain in energy when the particle crosses the shock in either direction).

3. Do the problem set #4 in the problems sets attached below.

Problems — Neutrino Astronomy

The problem sets accompanying the neutrino astronomy presentation are somewhat different. They provide a set of exercises (with solutions) to refresh concepts that may be unfamiliar to particle physicists and are relevant to understanding the talk. You may also want to consult Wikipedia if you are not familiar with some of the concepts. The 4 sets cover:

- Relativistic kinematics applied to cosmic rays, particle decays and interactions.
- Particle and photon fluxes from astronomical objects.
- Interaction lengths, superluminal motion and luminosity.
- Shock waves and Fermi acceleration.

Francis Halzen

Series 1 – Relativistic kinematics, Particle Decays

July 29, 2019

Notation: a vector $\mathbf{v}$ has a module v and a versor $\widehat{v}$

Ex. 1 — Relativistic kinematics
Atmospheric muons are created at an altitude of about 10 km, with velocity $\beta = 0.999$.

1. — Calculate the average distance that they travel before to decay. Do they reach the ground?
2. — Imagine to be a muon: do you reach the ground?
3. — Do pions generated with the same velocity reach the ground?

Answer (Ex. 1) —

1. — In the reference system of the Earth, the average distance covered by the muons is given by $d = \gamma\beta c\tau_0$, where $\gamma = 1/\sqrt{(}1-\beta^2) = 22.37$, $c\tau_0 = 658.7\ m$:

$$d = 22.37 \cdot 0.999 \cdot 658.7\ m = 14.7\ km. \tag{1}$$

   So muons can reach the ground.
2. — In the reference system of the muon obiouvsly its mean lifetime is invariant, τ_0, and the distance that it can cover in this case is:

$$d = 0.999 \cdot 658.7\ m = 658.0\ m. \tag{2}$$

   The reason why it can reach the ground is that the distance to be covered is shortened, $h = 10000/\gamma \simeq 447.03\ m$.
3. — Charged pions have a much shorter mean lifetime, $c\tau_0 = 7.8\ m$, so they can cover a much shorter length (assuming for simplicity the same γ):

$$d = 22.37 \cdot 0.999 \cdot 7.8\ m = 174.3\ m \tag{3}$$

   so they can not reach the ground.

Ex. 2 — Relativistic kinematics
Consider a neutron produced by an astrophysical phenomenon. The energy of the neutron is equivalent to the energy of a proton accelerated in a fixed gas-target experiment at a center-of-mass energy of 14 TeV. Consider the target made of hydrogen.

1. — If the neutron would be produced by a cosmic source, what is the order of its energy?

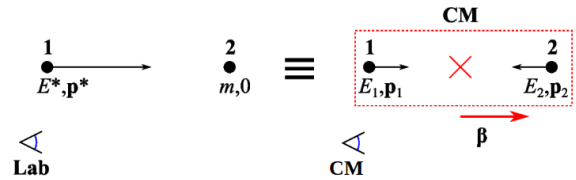


Figure 1: Boost from the frame of the center of mass to the laboratory frame. Looking at the Lab frame is equivalent to look the frame of the center of mass (dashed red box) that is moving with $\boldsymbol{\beta}$ in the opposite direction of the rest particle ("2" in the figure).

2. — Which is the maximum distance (in pc and ly units) of its source to be detected undecayed on Earth (use the most convenient set of units)? Can the neutron be detected before its decay if the source is in the milky way bulge? Ignore any interaction along its path.

Answer (Ex. 2) —

1. — Referring to Figure 1, in the center of mass, each proton has an energy $E_1 = E_2 = E = 7$ TeV and a momentum of

$$E = \sqrt{m^2 + p^2} \;\Rightarrow\; p_1 = p_2 = p = \sqrt{E^2 - m^2} \tag{4}$$
$$\simeq E. \tag{5}$$

To change from the frame of the center of mass (CM) to laboratory frame (Lab) in which a particle is at rest, the CM frame must be considered moving with a β able to stop one of the particle, i. e. the particle "2" as shown in Figure 1. This leads to the boost matrix for the particle at rest in the laboratory frame:

$$\begin{pmatrix} m \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \cdot \begin{pmatrix} E \\ p \end{pmatrix}. \tag{6}$$

Another way to see the boost matrix is with the vectorial notation[1]. The advantage is an overall equation, including in one expression the $\parallel$ and $\perp$ components of the momentum respect to the $\boldsymbol{\beta}$ vector; the disadvantage is that a proper consideration of the vectors must be taken into account. In this notation, the boost matrix to pass from CM to Lab is

$$\begin{pmatrix} m \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \boldsymbol{\beta} \\ \boldsymbol{\beta} & 1 \end{pmatrix} \cdot \begin{pmatrix} E \\ \mathbf{p}_2 \end{pmatrix}. \tag{7}$$

Still referring to Figure 1, the $\boldsymbol{\beta}$ vector is in opposite direction of $\mathbf{p}_2$, the particle at rest

[1] This notation will be used in all these solutions.

in the lab frame; that yields, using Equation (4),

$$m = \gamma\left(E - \beta\sqrt{E^2 - m^2}\right) = \gamma E\left(1 - \beta\sqrt{1 - \frac{m^2}{E^2}}\right), \tag{8}$$

$$0 = \cancel{\gamma}\left(\beta E - \sqrt{E^2 - m^2}\right) \;\Rightarrow\; \beta = \sqrt{1 - \frac{m^2}{E^2}} \simeq 1. \tag{9}$$

From Equation (9), the Equation (8) gives the γ factor:

$$m = \gamma E\left(1 - \beta^2\right) = \frac{E}{\gamma} \;\Rightarrow\; \gamma = \frac{E}{m} \approx 7000. \tag{10}$$

If the same boost is applied to the particle "1", which has $\mathbf{p}_1$ in the same direction of $\boldsymbol{\beta}$, and the approximations in Equations (5) and (9) are used, the estimation of the particle energy (and momentum) is

$$\begin{pmatrix} E^* \\ \mathbf{p}^* \end{pmatrix} = \gamma \begin{pmatrix} 1 & \boldsymbol{\beta} \\ \boldsymbol{\beta} & 1 \end{pmatrix} \cdot \begin{pmatrix} E \\ \mathbf{p}_1 \end{pmatrix} \simeq \gamma \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} E \\ E \end{pmatrix} \;\Rightarrow\; E^* = 2\gamma E \sim 10^{17}\ \text{eV}, \tag{11}$$

where the index $*$ denotes the laboratory frame.

2. — The maximum distance of its source is:

$$l^* = \tau^* c = \gamma\tau c\beta \simeq 2.9 \cdot 10^{16}\ \text{km} \tag{12}$$

$$\Rightarrow l^* \simeq 2.9 \cdot 10^3\ \text{ly} \tag{13}$$

$$\Rightarrow l^* \simeq 9.25 \cdot 10^2\ \text{pc}, \tag{14}$$

where $\gamma = E_n/m_n = 1.11 \cdot 10^8$ was used. From these results, it is quite obvious that any neutron produced in the milky way bulge cannot reach un-decayed the earth surface.

Ex. 3 — Relativistic kinematics

A proton travelling in the space interacts with the cosmic radiation background at 3K.

1. — Calculate the energy threshold of the proton to allow the reactions $p + \gamma \to p + \pi^0$ and $p + \gamma \to p + e^+ + e^-$ in case of frontal collisions.
2. — If the π photo-production cross-section is 100 μbarn and is constant, calculate the proton mean free path as a function of the energy. Remember that the mean free path is given by $\lambda = 1/\rho\sigma$ and $\rho = 0.24(kT/\hbar c)^3$ for a black body.

Answer (Ex. 3) —

1. — Let's do the calculations for the process $p + \gamma \to p + \pi^0$. The invariant mass of the system is conserved:

   $(\mathbf{p}_p + \mathbf{p}_\gamma)^2 = (\mathbf{p}'_p + \mathbf{p}'_{\pi^0})^2,$

   where $\mathbf{p}$ is the 4-momentum $\mathbf{p} = (E, \vec{p})$.

   For the left hand term of the equation:

$(\mathbf{p}_p + \mathbf{p}_\gamma)^2 = \mathbf{p}_p^2 + \mathbf{p}_\gamma^2 + 2\mathbf{p}_p\mathbf{p}_\gamma =$

$= E_p^2 - |\vec{p_p}|^2 + E_\gamma^2 - |\vec{p_\gamma}|^2 + 2(E_pE_\gamma - |\vec{p_p}||\vec{p_\gamma}|cos\theta) =$

$= m_p^2 + m_\gamma^2 + 2E_pE_\gamma - 2|\vec{p_p}||\vec{p_\gamma}|cos\theta,$

where θ is the angle between the proton and the pion.
Hence the full equation becomes:

$\cancel{m_p^2} + 2E_pE_\gamma - 2p_pp_\gamma cos\theta = \cancel{m_p^2} + m_{\pi^0}^2 + 2E'_pE'_{\pi^0} - 2p'_pp'_\pi cos\theta$

At the threshold, $p'_\pi = p'_p = 0$.
Under the hypothesis of high energies, $E_p \gg m_p$:

$2E_pE_\gamma(1 - cos\theta) = 2E'_pE'_{\pi^0} + m_{\pi^0}^2 = m_{\pi^0}^2 + 2m_pm_{\pi^0}$

Considering that:

$m_p = 938.27$ MeV $\approx 10^3$ MeV
$m_{\pi^0} = 134.9766$ MeV $\approx 10^2$ MeV
$2m_pm_{\pi^0} \approx 2 \cdot 10^5$ MeV
$m_{\pi^0}^2 \approx 10^4$ MeV $\rightarrow$ negligible wrt $2m_pm_{\pi^0}$

Considering only frontal collisions:

$4E_pE_\gamma = 2m_pm_{\pi^0}$
$E_{p,threshold} = \frac{m_pm_{\pi^0}}{2E_\gamma} \simeq 5 \cdot 10^{20}$ eV,
where $E_\gamma \sim kT = 25 \cdot 10^{-5}$ eV.

When the scattering is more complicated, as in the case $p + \gamma \rightarrow p + e^+ + e^-$:

$(\mathbf{p}_p + \mathbf{p}_\gamma)^2 = (\mathbf{p}'_p + \mathbf{p}'_{e^+} + \mathbf{p}'_{e^-})^2,$

$m_p^2 + 2E_pE_\gamma - 2|\vec{p_p}||\vec{p_\gamma}|cos\theta = m_p^2 + m_e^2 + m_e^2 + 2\mathbf{p'_p}\mathbf{p'_{e^+}} + 2\mathbf{p'_p}\mathbf{p'}_{\mathbf{e}}^- + 2\mathbf{p'_{e^-}}\mathbf{p'_{e^+}}$

$\cancel{m_p^2} + 2E_pE_\gamma - 2|\vec{p_p}||\vec{p_\gamma}|cos\theta = 4m_pm_e + \cancel{m_p^2} + 4m_e^2$

$E = \frac{(m_pm_e + m_e^2)}{E_\gamma} \simeq 2 \cdot 10^{18}$ eV.

2. — $\sigma(p + \gamma \rightarrow p + \pi) = 100\ \mu$barn $= 10^{-32}m^2$
$\lambda = 1/\rho\sigma$, with $\rho = 0.24(kT/\hbar c)^3 = 5 \cdot 10^{17}m^{-3} \rightarrow \lambda = 2 \cdot 10^5$ m.

Ex. 4 — Particles decay
In nature the stable particles are few. Most of them are unstable and they decay. In this exercise the decay $\pi^+ \rightarrow \mu^+ + \nu_\mu$ (a mesonic decay) is considered.

1. — Calculate the momentum of μ^+ and ν_μ in the rest frame, assuming the neutrino as massless.

2. — The neutrino is not massless but it has a small mass, so far measured only as upper limit. Measuring the muon momentum, the mass of the ν_μ can be constrained. How much must the precision on the measurement of the muon momentum be to determine the current upper limit of the ν_μ mass (190 keV).
3. — Assume that the π^+ has an initial momentum of 500 MeV (in natural units). If this momentum is perpendicular to the direction along which the decay in the rest frame of the pion occurs, how much is the (massless) neutrino energy and how much is the angle between μ^+ and ν_μ?

Answer (Ex. 4) —

1. — "Neutrino massless" means $E_\nu = p_\nu$. Therefore the energy and momentum conservation yield

$$m_\pi = \sqrt{p_\mu^2 + m_\mu^2} + E_\nu, \tag{15}$$

$$0 = \mathbf{p}_\mu + \mathbf{p}_\nu. \tag{16}$$

Through Equation (16), $p_\mu^2 = E_\nu^2$. Isolating the root square in Equation (15) and squaring gives

$$\begin{aligned} (m_\pi - E_\nu)^2 &= E_\nu^2 + m_\mu^2 \\ \Rightarrow m_\pi^2 + \cancel{E_\nu^2} - 2m_\pi E_\nu &= \cancel{E_\nu^2} + m_\mu^2, \end{aligned}$$

therefore, with $m_\pi = 139.6$ MeV and $m_\mu = 105.7$ MeV,

$$\Rightarrow E_\nu = p_\nu = p_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.7839183 \simeq 29.8 \text{ MeV (in natural units)}. \tag{17}$$

2. — With $m_\nu \neq 0$, the Equation (15) becomes

$$m_\pi = \sqrt{p_\mu^2 + m_\mu^2} + \sqrt{p_\nu^2 + m_\nu^2}, \tag{18}$$

that leads to p_μ:

$$\begin{aligned} \left(m_\pi - \sqrt{p_\mu^2 + m_\mu^2}\right)^2 &= \left(\sqrt{p_\mu^2 + m_\nu^2}\right)^2 \\ \Rightarrow m_\pi^2 + \cancel{p_\mu^2} + m_\mu^2 - 2m_\pi\sqrt{p_\mu^2 + m_\mu^2} &= \cancel{p_\mu^2} + m_\nu^2 \\ \Rightarrow \left(2m_\pi\sqrt{p_\mu^2 + m_\mu^2}\right)^2 &= \left(m_\pi^2 + m_\mu^2 - m_\nu^2\right)^2 \\ \Rightarrow 4m_\pi^2\left(p_\mu^2 + m_\mu^2\right) &= \left(m_\pi^2 + m_\mu^2 - m_\nu^2\right)^2 \\ \Rightarrow p_\mu^2 &= \frac{\left(m_\pi^2 + m_\mu^2 - m_\nu^2\right)^2}{4m_\pi^2} - m_\mu^2. \end{aligned} \tag{19}$$

In case of $m_\nu = 0$, the Equation (19) gives the Equation (17) ($p_{\mu|m_\nu=0} = 29.7839183$ MeV). In case of $m_\nu = 0.19$ MeV,

$$p_{\mu|m_\nu=0.19} = 29.7834416 \text{ MeV (in natural units)}. \tag{20}$$

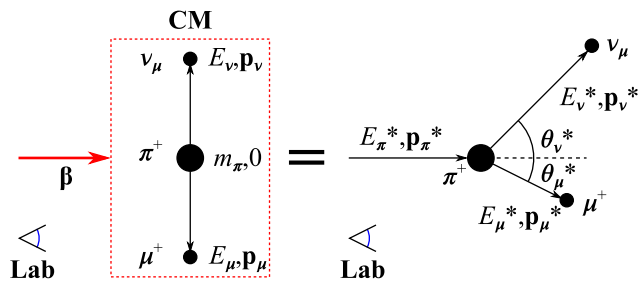


Figure 2: Boost for the π^+ decay, from the frame of the center of mass (dashed red box) to the laboratory frame. $\boldsymbol{\beta}$ is perpendicular to the momentum $\mathbf{p}_{\nu(\mu)}$ in the rest of frame.

Therefore the precision required is

$$\delta p_\mu = p_{\mu|m_\nu=0} - p_{\mu|m_\nu=0.19} \simeq 4.767 \cdot 10^{-4}\,\text{MeV} \sim 10^{-1}\ \text{keV (in natural units)}. \tag{21}$$

3. — Given $\boldsymbol{\beta}$ a boost vector in the direction perpendicular to the neutrino and the muon momentum in the center of mass frame as shown in Figure 2, the boost matrix equation to go from CM reference system to the Lab reference system is:

$$\begin{pmatrix} E^*_\pi \\ \mathbf{p}^*_{\pi,x} \end{pmatrix} = \gamma \begin{pmatrix} 1 & \boldsymbol{\beta} \\ \boldsymbol{\beta} & 1 \end{pmatrix} \cdot \begin{pmatrix} m_\pi \\ 0 \end{pmatrix}, \tag{22}$$

where the index $*$ denotes the laboratory frame. This yields

$$E^*_\pi = \gamma m_\pi, \tag{23}$$
$$\mathbf{p}^*_\pi = \gamma \boldsymbol{\beta} m_\pi, \tag{24}$$

as expected. The Equations (23) and (29), considering $m_\pi = 139.6$ MeV, give

$$\gamma = \frac{E^*_\pi}{m_\pi} = \frac{\sqrt{m_\pi^2 + p_\pi^{*\,2}}}{m_\pi} \simeq 3.72, \tag{25}$$
$$\beta = \frac{p^*_\pi}{\gamma m_\pi} \simeq 0.96. \tag{26}$$

To boost the neutrino the matrix is:

$$\begin{pmatrix} E^*_\pi \\ \mathbf{p}^*_{\nu,x} \\ \mathbf{p}^*_{\nu,y} \\ \mathbf{p}^*_{\nu,z} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_\nu \\ 0 \\ \mathbf{p}_{\nu,y} \\ 0 \end{pmatrix}, \tag{27}$$

where the index $*$ denotes the laboratory frame. This yields

$$E^*_\nu = \gamma E_\nu, \tag{28}$$
$$\mathbf{p}^*_{\nu,x} = \gamma\beta E_\nu, \tag{29}$$
$$\mathbf{p}^*_{\nu,y} = p_{\nu,y}. \tag{30}$$

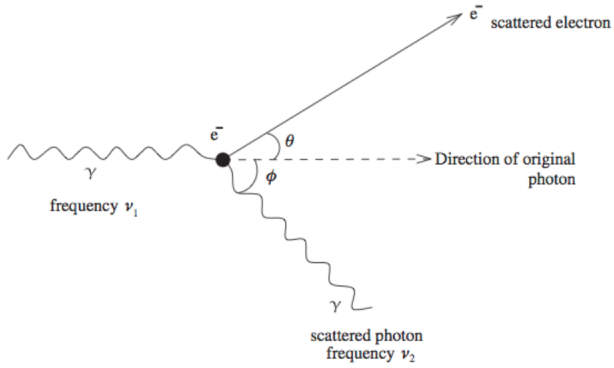


Figure 3: Illustration and reference of the Compton scattering process. **Suggestion:** consider the frame with the electron at rest...

So we obtain:

$$E^*_\nu = \gamma E_\nu, \simeq 110.4\text{MeV} \tag{31}$$

$$\tan\theta^*_\nu = \frac{\mathbf{p}^*_{\nu,y}}{\mathbf{p}^*_{\nu,x}} = \frac{1}{\beta\gamma} \simeq 0.28 \;\Rightarrow\; \theta^*_\nu \simeq 15.64^{\text{o}}. \tag{32}$$

For the μ, in analogous way:

$$\tan\theta^*_\mu = \frac{\mathbf{p}^*_{\mu,y}}{\mathbf{p}^*_{\mu,x}} = \frac{p_{\nu,y}}{E^2_\mu\beta\gamma} \simeq 0.07 \;\Rightarrow\; \theta^*_\mu \simeq 4^{\text{o}}, \tag{33}$$

therefore the aperture between ν and μ is $\simeq 15.64 + 4 = 19.64^{\text{o}}$.

Ex. 5 — Interaction photon-matter

When a γ interacts with the matter, several phenomena can occur. Two of them are the pair production and the Compton scattering.

1. — The pair production is the split of γs in matter/anti-matter pairs. This interaction occurs only in the presence of a nucleus or a particle. Considering $\gamma \to e^+e^-$, $\gamma \to \mu^+\mu^-$ and $\gamma \to \tau^+\tau^-$, how much must the minimum energy of the gamma particle be to produce these pairs? To which wavelength do they correspond and to which class of the electromagnetic spectrum (IR, Visible, UV, etc.) do they belong?
2. — The Compton scattering occurs when a photon hits a particle and they are both scattered away with an energy exchange between them. Considering the γe^- interaction as in Figure 3, derive the "Compton Scatter Formula", i. e. the functional form of the wavelength as a function of the scattered angle, and show it in a plot (**Suggestion:** show the plot of $\lambda_2 - \lambda_1$ as a function of ϕ from 0 to 180°).
3. — Name and explain a third interaction photon-matter process.

Answer (Ex. 5) —

1. — The energy is expressed as in **??**, therefore $E_\gamma = p$ and $E_\ell = E_{\bar{\ell}} = \sqrt{m^2_\ell + p^2}$, in natural units, where ℓ is e, μ or τ (NB: lepton and anti-lepton have the same mass). The

minimum case is when $\mathbf{p} = 0$; hence, for the energy conservation,

$$E_\gamma = E_\ell + E_{\bar{\ell}} = m_\ell + m_{\bar{\ell}} = 2m_\ell. \tag{34}$$

Therefore

$$\gamma \to e^+e^- : \; E_\gamma = 2 \cdot 0.511 \simeq 1.022 \text{ MeV}, \tag{35}$$

$$\gamma \to \mu^+\mu^- : \; E_\gamma = 2 \cdot 106 \simeq 212 \text{ MeV}, \tag{36}$$

$$\gamma \to \tau^+\tau^- : \; E_\gamma = 2 \cdot 1.777 \simeq 3.554 \text{ GeV} \tag{37}$$

and, remembering that we can divide a length x by $\hbar c \simeq 197$ MeV·fm, to convert a wavelength from natural units to SI,

$$\lambda_{e^+e^-} = \frac{2\pi}{1.022} = 6.14 \text{ MeV}^{-1} \;\Rightarrow\; 1.26 \text{ pm}, \tag{38}$$

$$\lambda_{\mu^+\mu^-} = \frac{2\pi}{212} = 0.03 \text{ MeV}^{-1} \;\Rightarrow\; 5.91 \text{ fm}, \tag{39}$$

$$\lambda_{\tau^+\tau^-} = \frac{2\pi}{3.554} = 1.77 \text{ GeV}^{-1} \;\Rightarrow\; 0.35 \text{ fm}. \tag{40}$$

All of them are γ-rays and only the electron-positron production is close to the X-rays.

2. — Labelling the initial and final states of the photon with subscripts 1 and 2 respectively as in Figure 3, the conservation laws in natural units are

$$2\pi\nu_1 + m_e = 2\pi\nu_2 + \sqrt{m_e^2 + p_e^2}, \tag{41}$$

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_e. \tag{42}$$

Squaring Equation (42) to find p_e^2 gives

$$p_e^2 = (\mathbf{p}_1 - \mathbf{p}_2)^2 = p_1^2 + p_2^2 - 2p_1p_2\cos\phi. \tag{43}$$

In natural units, $2\pi\nu = E = p$, therefore Equation (43) becomes

$$p_e^2 = (2\pi)^2\nu_1^2 + (2\pi)^2\nu_2^2 - 2(2\pi)^2\nu_1\nu_2\cos\phi. \tag{44}$$

Isolating the root square in Equation (41), squaring and substituting p_e^2 with Equation (44) yields

$$\begin{aligned}
& m_e^2 + p_e^2 = (2\pi\nu_1 - 2\pi\nu_2 + m_e)^2 \\
\Rightarrow\; & \cancel{m_e^2} + \cancel{(2\pi)^2(\nu_1^2 + \nu_2^2)} - 2(2\pi)^2\nu_1\nu_2\cos\phi = \\
& = \cancel{m_e^2} + \cancel{(2\pi)^2(\nu_1^2 + \nu_2^2)} - 2(2\pi)^2\nu_1\nu_2 + 2(2\pi)m_e(\nu_1 - \nu_2) \\
\Rightarrow\; & -(2\pi)\nu_1\nu_2\cos\phi = -(2\pi)\nu_1\nu_2\cos\phi + m_e(\nu_1 - \nu_2),
\end{aligned}$$

therefore, the Compton Scatter Formula is

$$\frac{\nu_1 - \nu_2}{\nu_1\nu_2} = \frac{2\pi}{m_e}(1 - \cos\phi)$$

$$\Rightarrow \lambda_2 - \lambda_1 = \frac{2\pi}{m_e}(1 - \cos\phi) \qquad \text{(in natural units)}, \tag{45}$$

$$\Rightarrow \lambda_2 - \lambda_1 = \frac{h}{m_e c}(1 - \cos\phi) \qquad \text{(in SI units)}. \tag{46}$$

shown in Figure 4. To make the plot, the following Python code was used:

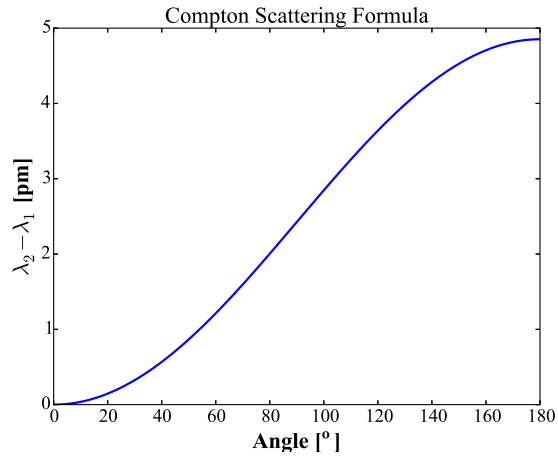


Figure 4: Plot of the Compton Scatter Formula

```
import numpy as np
import matplotlib.pyplot as pyp
from scipy.constants import c,h,m_e,pi # h, m_e, c are in SI.

comp = lambda x: 10**12 * h/(m_e*c) * (1-np.cos(x))
# 10**12 converts [m] in [pm]
x = np.linspace(0,pi,1000) # np.cos has radiant argument
pyp.plot(x*180./pi,comp(x)) # x-axis in degree
pyp.xlabel(r"Angle [$^\mathbf{o}$]",weight="bold")
pyp.ylabel(r"$\mathbf{\lambda_2 - \lambda_1}$ [pm]",weight="bold")
pyp.title("Compton Scattering Formula")
pyp.savefig("ComptonScatter.eps")
pyp.show()
```

3. — Absorption and photon emission. A γ is absorbed and let the electron of an atom to change in an excited state. When the electron return to its ground state, the atom emits a photon with a characteristic atomic wavelength. This is also called "Fluorescence radiation". If the photon energy is enough to unbound the electron from the orbits of the atom, the photoelectric effect occurs.

Series 2 – Messengers from the Universe

July 29, 2019

Ex. 1 — Particles from the universe
The flux of the cosmic rays is characterized by a power law due their acceleration mechanisms in their sources: $\mathrm{d}\phi/\mathrm{d}E \propto E^{-\gamma}$. The spectrum index γ is not a constant value for the full energy spectrum. Strong debates are around its precise values and the meaning of its evolution, especially for the energy range between $E_1 = 10^{18}$ eV and $E_2 = 10^{20}$ eV.

1. — Calculate the value of the A coefficient, so that the cumulative function of $\mathrm{d}N/\mathrm{d}E$ is normalized to 1 between E_1 and E_2.
2. — Once A is known, write a short python script that generates cosmic rays with a random energy E, with $E_1 \leq E \leq E_2$, according to the flux power law: $\mathrm{d}N/\mathrm{d}E = AE^{-\gamma}$. Plot the obtained energy distribution, using spectrum indices 2 and 3. (**Suggestion:** Use numpy.random to do the random sampling `https://docs.scipy.org/doc/numpy/reference/routines.random.html`. In the reference, you will find different methods to perform random extraction, find the one which is more suited for this case.)
3. — Studying the real distribution of the flux of the energy spectrum is still an ongoing research. In the scientific article "Measurement of the energy of cosmic rays above 10^{18} eV using the Pierre Auger Observatory" you can find a function that describes the spectrum between E_1 and E_2:

$$J(E; E < E_{\text{ankle}}) = BE^{-\gamma_1} \tag{1}$$

$$J(E; E > E_{\text{ankle}}) = \frac{E^{-\gamma_2}}{1 + \exp\left(\frac{\log_{10} E - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)}. \tag{2}$$

Find the values of the needed parameters in the paper. Then generate the random energy of cosmic rays with a distribution in agreement with this function, between E_1 and E_2. (**Suggestion:** Use the previous piece of code and re-weight the events with a weight w according to the function above. Be careful, you need to impose the continuity at E_{ankle} to find B). Finally, the flux is often multiplied by $E^{2.5}$ or E^3 to observe better its features; use one of these factors to plot your results. Choose the best y-axis to plot the simulated energy distribution.

Answer (Ex. 1) —

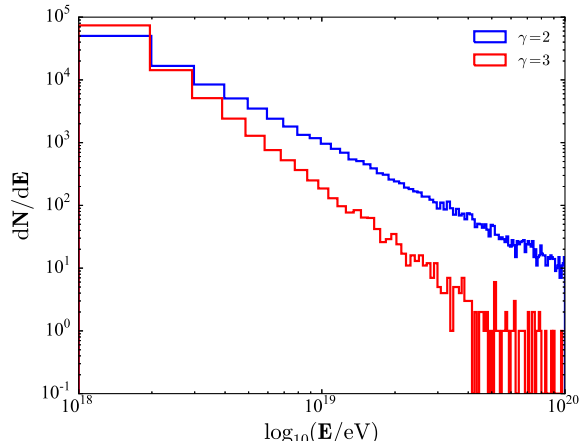


Figure 1: Event simulation with energy distribution with spectral indices $\gamma = 2$ (in blue) and $\gamma = 3$ (in red).

1. — To normalize,

$$1 = A \int_{E_1}^{E^2} E^{-\gamma} \mathrm{d}E = A \left. \frac{E^{-\gamma+1}}{1-\gamma} \right|_{E_1}^{E_2} = A \frac{E_2^{-\gamma+1} - E_1^{-\gamma+1}}{1-\gamma}$$
$$\Rightarrow A = \frac{1-\gamma}{E_2^{-\gamma+1} - E_1^{-\gamma+1}}, \tag{3}$$

where $\gamma \neq 1$.

2. — Given a generic cumulative number of events $N = \int_{E_1}^{E} \mathrm{d}E$,

$$N = A \int_{E_1}^{E} E^{-\gamma} \mathrm{d}E = A \frac{E^{-\gamma+1} - E_1^{-\gamma+1}}{1-\gamma} = \frac{E^{-\gamma+1} - E_1^{-\gamma+1}}{E_2^{-\gamma+1} - E_1^{-\gamma+1}}$$
$$\Rightarrow E = \left[E_1^{-\gamma+1} + N \cdot \left(E_2^{-\gamma+1} - E_1^{-\gamma+1} \right) \right]^{\frac{1}{1-\gamma}}. \tag{4}$$

In Figure 1 the plots of the following code are shown.

```
import numpy as np
import matplotlib.pyplot as pyp

# Set parameters
R = np.random.random((int)(1e5))
E1,E2 = 1e18, 1e20
# Convenient function to show in the plot
E = lambda g: (E1**(1.-g) + R*(E2**(1.-g)-E1**(1.-g)))**(1./(1.-g))
# Common kwd
kwd = dict(bins=100,log=True,facecolor='none',linewidth=2,histtype='step')
# Histograms
N1,bins,h1 = pyp.hist(E(2.),color='b',label=r"$\gamma=2$",**kwd)
```

```
N1,bins,h2 = pyp.hist(E(3.),color='r',label=r"$\gamma=3$",**kwd)
pyp.xscale('log')
pyp.xlabel(r"$\mathbf{\log_{10}(E/\mathrm{eV})}$")
pyp.ylabel(r"$\mathbf{\mathrm{d}N/\mathrm{d}E}$")
pyp.legend()
pyp.savefig("simSpectrumIndices.eps")
pyp.show()
```

3. — The smooth function provided by the article from the Pierre Auger collaboration is

$$J(E; E < E_{\text{ankle}}) \propto E^{-\gamma_1} \tag{5}$$

$$J(E; E > E_{\text{ankle}}) \propto \frac{E^{-\gamma_2}}{1 + \exp\left(\frac{\log_{10} E - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)}, \tag{6}$$

where the parameters are the following:

Parameter	Value
$\gamma_1 = \gamma(E < E_{\text{ankle}})$	3.26
$\gamma_2 = \gamma(E > E_{\text{ankle}})$	2.55
$\log_{10}(E_{\text{ankle}}/\text{eV})$	18.6
$\log_{10}(E_{1/2}/\text{eV})$	19.61
$\log_{10}(W_{\text{c}}/\text{eV})$	0.16

The idea is to use a simulation from the previous question and re-weight the events with a weight w according to this function. For the continuity,

$$A \cdot E_{\text{ankle}}^{-\gamma_1} = \frac{E_{\text{ankle}}^{-\gamma_2}}{1 + \exp\left(\frac{\log_{10} E_{\text{ankle}} - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)} \tag{7}$$

$$\Rightarrow A = \frac{E_{\text{ankle}}^{\gamma_1 - \gamma_2}}{1 + \exp\left(\frac{\log_{10} E_{\text{ankle}} - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)}, \tag{8}$$

therefore

$$w = \begin{cases} \dfrac{E_{\text{ankle}}^{\gamma_1 - \gamma_2}}{1 + \exp\left(\frac{\log_{10} E_{\text{ankle}} - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)} \cdot E^{-\gamma_1} & (E < E_{\text{ankle}}) \\ \dfrac{E^{-\gamma_2}}{1 + \exp\left(\frac{\log_{10} E - \log_{10} E_{1/2}}{\log_{10} W_{\text{c}}}\right)} & (E > E_{\text{ankle}}) \end{cases}. \tag{9}$$

Since the simulated events have already a distribution ($E^{-\gamma}$), to properly apply a weight the shape must be flattened by a factor E^{γ}. Rescaling the y-axis can be considered a weight too when the histogram is produced. Thus, following the suggestion in the article of rescaling by E^3, the final weight will be $w \cdot E^{3+\gamma}$.
Starting from the case with $\gamma = 2$, the check on the resulting distribution is performed and shown in Figure 2 with the following code. The code is a continuation of the one shown in the previous question

```
G, g1, g2, Eank, lgE12, lgWc = 2., 3.26, 2.55, 10**18.6, 19.61, 0.16
fsmooth = lambda e: (1+np.exp((np.log10(e)-lgE12)/lgWc))
```

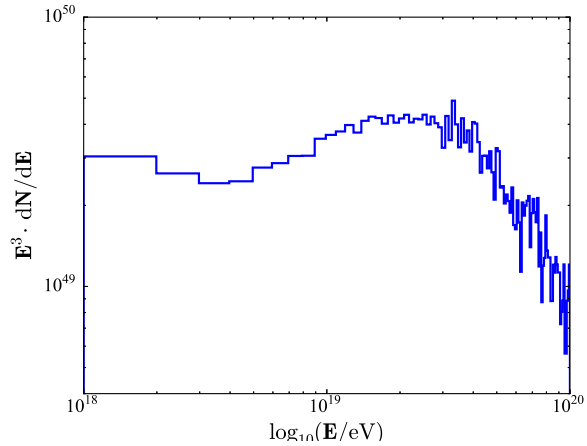


Figure 2: Energy distribution .

```
A = Eank**(g1-g2)/fsmooth(Eank)
Eg2 = E(2.)
# Weights for a generic scale
weights = lambda scale: np.where(Eg2>Eank, Eg2**(scale+G-g2)/fsmooth(Eg2),
                                 A*Eg2**(scale+G-g1))
# Scale chosen is 3., as said before
N3,bins,h3 = pyp.hist(Eg2,weights=weights(3.),**kwd)
pyp.xscale('log')
pyp.ylim(ymin=4e48)
pyp.xlabel(r"$\mathbf{\log_{10}(E/\mathrm{eV})}$")
pyp.ylabel(r"$\mathbf{E^3\!\cdot\,\mathrm{d}N/\mathrm{d}E}$")
pyp.savefig("simSpectrumRealistic.eps")
pyp.show()
```

Ex. 2 — Light from a star

What is the color of the sun? Looking at it, the answer seems "slightly yellow", but this is due the atmospheric filtering of the light[1]. Since it emits the entire visible region of the electromagnetic spectrum, its color should be "white". However, it does not emit all the electromagnetic frequencies with the same luminosity.

The spectral radiance of a star can be described as a "black-boby radiation". Therefore, the spectral brightness I_ν of the sun, and of a star in general, follows the Plancks's law for a spectral radiance of a black body. Moreover, its bolometric luminosity is, from the Stefan-Boltzmann law, $L_\odot = 4\pi\sigma R_\odot^2 T_{\rm eff}^4$, where $T_{\rm eff}$ is the effective temperature that characterize the sun[2].

1. — Find $T_{\rm eff}$ of the sun. Then, plot writing a short python code its black-body distribution as a function of the wavelength and extract through the code the wavelength and frequency

[1] The literature terminology "yellow dwarf star" used to classify stars as the sun is a misnomer.

[2] The temperature of a star is not well defined: it is a gradient from the nucleus to the surface and even in the surface fluctuates quite a lot. $T_{\rm eff}$ define a conventional temperature based on the black body radiance distribution. Whenever the temperature of a star is cited, it is a reference to this characteristic temperature, which is $\propto \sqrt[4]{L_\odot}$.

of its maximum spectral emission.
The spectrum of an astrophysical object is very important because its change ("redshift") gives hints about the movement of the object.

2. — At the frequency of the emission peak of the sun, give the spectral radiance as calculated from the Planck's law; calculate then the flux and the luminosity detected on Earth, considering $r_\odot$ as the average distance earth-sun. (**Suggestion:** consider the sun a disc-source perpendicular to the detector, i. e. the earth...)
3. — A star equivalent to the sun is moving with β in a direction with angle θ from the observer.
   a) Estimate the possible βs with which the object is moving with an apparent speed $\geq c$. (**Suggestion:** it will be useful to remind the trigonometric identities $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$ and $\cos 45 = \sin 45$...)
   b) With the minimum possible β, calculate the ranges of θs and the apparent brightness for the peak frequency of the sun.
   c) Still with the minimum β but with $\theta = 180^{\text{o}}$ (i. e. it is going far away from the earth), how much is the redshift?
   d) With the previous redshift and assuming a constant Hubble expansion rate H_0, how far is the star from the earth? And with a double redshift?

Answer (Ex. 2) —

1. — From the Stefan-Boltzmann law $L_\odot = 4\pi\sigma R_\odot^2 T_{\text{eff}}^4$, $T_{\text{eff}} = 5772$ K. Knowing the effective temperature of the sun and the Planck's law

$$B(\lambda, T) = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{\mathrm{e}^{\frac{hc}{\lambda k_{\mathrm{B}} T_{\text{eff}}}} - 1} \tag{10}$$

yield the black body distribution in Figure 3. The maximum of this distribution is easily calculated as $\lambda_{\text{peek}} \simeq 0.502$ μm that correspond to a frequency of $\nu_{\text{peek}} \simeq 597$ THz. The code used (in Python) is the following:

```
import numpy as np
import matplotlib.pyplot as pyp
from scipy.constants import h,c,k

Teff = 5772 # in K
l = np.linspace(0.005,3,1000000) # in um
L = l*10**(-6) # in m
B = ((2*h*c**2)/L**5 ) * 1/(np.e**((h*c)/(L*k*Teff))-1) # in W*sr^{-1}*m^{-3}

# Peak
Bpeak = np.max(B)
lpeak = l[np.where(B==Bpeak)[0][0]] # in um
fpeak = c*10**(-6)/lpeak # in THz
print Bpeak, lpeak, fpeak

# Plot
pyp.plot(l,B)
pyp.xlabel(r"$\lambda$ [$\mathbf{\lambda}$m]",weight='bold')
```

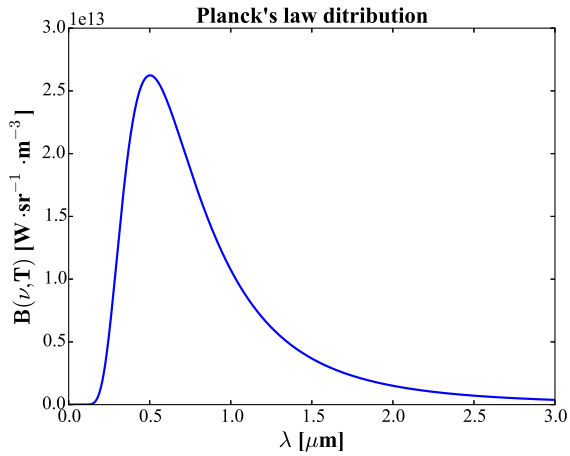


Figure 3: Black body spectral radiance distribution for the sun.

```
pyp.ylabel(r"$\mathbf{B(\nu,T)}$ "+\
           r"[W$\mathbf{\cdot}$sr$\mathbf{^{-1}\cdot}$m$\mathbf{^{-3}}$]",
           weight='bold')
pyp.title("Planck's law ditribution",weight='bold')
pyp.savefig("plancksBB.eps")
pyp.show()
```

2. — From the Planck's law as a function of the frequency

$$I_\nu = B\left(\nu, T\right) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T_\mathrm{eff}}} - 1}, \tag{11}$$

the spectral radiance is

$$I_\nu \simeq 2.2 \cdot 10^{-8}\ \mathrm{W} \cdot \mathrm{sr}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{Hz}^{-1}. \tag{12}$$

The angular radius of the Sun viewed from the Earth is

$$\theta_\odot = \arcsin\left(\frac{R_\odot}{r_\odot}\right) \approx 4.7 \cdot 10^{-3}\ \mathrm{rad}. \tag{13}$$

The sun is a disc-source perpendicular to the detector (i. e. the earth), therefore the flux

is calculated from the integral

$$\begin{aligned} S_\nu &= \int_{\text{sun}} I_\nu \cos\theta \mathrm{d}\theta \mathrm{d}\Omega = \\ &= I_\nu \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\theta_\odot} \cos\theta \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \\ &= 2\pi I_\nu \int_{\theta=0}^{\theta_\odot} \sin\theta \cos\theta \mathrm{d}\theta = \\ &= 2\pi I_\nu \int_0^{\sin\theta_\odot} x \mathrm{d}x = \\ &= \pi I_\nu \sin^2\theta_\odot, \end{aligned} \tag{14}$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ and the substitutions $x = \sin\theta$ and $\mathrm{d}x = \cos\theta \mathrm{d}\theta$ are used. Since $\theta_\odot \ll 1$,

$$\begin{aligned} S_\nu \approx \pi I_\nu \theta_\odot^2 &\approx 1.52 \cdot 10^{-12}\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{Hz}^{-1} = \\ &= 152\ \mathrm{TJy}. \end{aligned} \tag{15}$$

Finally, the luminosity is

$$L_\nu = 4\pi r_\odot^2 S_\nu \approx 4.2 \cdot 10^{11}\ \mathrm{W} \cdot \mathrm{Hz}^{-1} \tag{16}$$

3. —

a) The apparent β_{app} is yielded by the formula

$$\beta_{\text{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}, \tag{17}$$

where θ is the angle between the observer and the movement of the star. From the inequality $\beta_{\text{app}} \geq 1$:

$$\begin{aligned} & 1 - \beta\cos\theta \leq \beta\sin\theta \\ \Rightarrow\ & 1 \leq \beta(\sin\theta + \cos\theta) \\ \Rightarrow\ & \frac{1}{\beta\sqrt{2}} \leq (\sin\theta\cos 45 + \cos\theta\sin 45) \\ \Rightarrow\ & 0 \leq \frac{1}{\beta\sqrt{2}} \leq \sin(\theta + 45) \leq 1 \qquad (18) \\ \Rightarrow\ & \frac{1}{\sqrt{2}} \leq \beta \leq 1. \qquad (19) \end{aligned}$$

b) From Equation (19), the minimum possible β is $\beta_{\text{min}} = 1/\sqrt{2}$. From Equation (18), $\theta_{\text{min}} = 45^{\text{o}}$. For this angle, the only possible, the doppler factor is

$$\delta = \frac{\sqrt{1-\beta^2}}{1 - \beta\cos\theta} = \sqrt{2}. \tag{20}$$

Therefore, since I_ν/ν^3 is an invariant,

$$\frac{I_\nu^{\rm em}}{\nu_{\rm em}^3} = \frac{I_\nu^{\rm obs}}{\nu_{\rm obs}^3} = \frac{I_\nu^{\rm obs}}{(\delta\nu_{\rm em})^3}$$
$$\Rightarrow I_\nu^{\rm obs} = \delta^3 I_\nu^{\rm em} \simeq 6,2\cdot 10^{-8}\ {\rm W\cdot sr^{-1}\cdot m^{-2}\cdot Hz^{-1}}. \tag{21}$$

c) With $\theta = 180^{\rm o}$, the redshift is simply

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 = 1.41. \tag{22}$$

d) "Same redshift", means same β. Therefore

$$\beta c = H_0 d \tag{23}$$
$$\Rightarrow\ d = \frac{\beta c}{H_0} \simeq 3127\ {\rm Mpc}. \tag{24}$$

If the redshift is double, the β will be

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$
$$\Rightarrow \beta = \frac{(z+1)^2 - 1}{(z+1)^2 + 1} \simeq 0.87, \tag{25}$$

therefore

$$d = \frac{\beta c}{H_0} \simeq 3847\ {\rm Mpc}. \tag{26}$$

Series 3 – Interaction Lengths, Apparent Velocity and Luminosity

July 29, 2019

Ex. 1 — Interaction Length

1. — The analytic expression for the photoelectric absorption by the interstellar gas is:

$$\sigma(E) = 2 \times 10^{-22}\mathrm{cm}^2 \left(\frac{E}{\mathrm{keV}}\right)^{-\frac{8}{3}}, \tag{1}$$

   where E is the photon energy.
   Calculate the cross-section and mean-free-path for a photon in the interstellar medium with energies of 20 eV and 20 keV. Assume that the interstellar medium in our Galaxy contains an average density of 1 hydrogen atom per cubic centimeter.
2. — The intergalactic medium (the gas between galaxies) is ionized and has a mean electron density of about $2{\cdot}10^{-7}$ cm^{-3}. Calculate the mean-free-path of photons that do Thompson scattering in the intergalactic medium. The cross section for Thompson scattering is $\sigma_T = (8\pi/3)r_0^2$, where $r_0 = e^2/(m_e c^2) = 2.82 \cdot 10^{-13}$ cm is the classical electron radius. Express your result in megaparsecs.
3. —The size of the visible universe is about the speed of light times the time since the Big Bang, 13.8 Gyrs. Can we expect to see x-ray sources at cosmologically interesting distances (i.e. at a significant fraction of the size of the visible universe)?

Answer (Ex. 1) —

1. — The mean free path for a photon through a medium of hydrogen atoms with density of particles per unit volume ρ and cross-section σ is $l = 1/(\rho \cdot \sigma)$. Plugging in numbers yields:

$$l = \frac{1}{(1.0(\mathrm{H\ atom})\mathrm{cm}^{-3})(2 \cdot 10^{22}\mathrm{cm}^2)(e/(\mathrm{keV}))^{-8/3}} = \tag{2}$$

$$= (5.0 \cdot 10^{21}\mathrm{cm})\left(\frac{E}{\mathrm{keV}}\right)^{-\frac{8}{3}} = (1.6 \cdot 10^3\mathrm{pc})\left(\frac{E}{\mathrm{keV}}\right)^{-\frac{8}{3}} \tag{3}$$

The cross-sections and mean-free-paths are:

$$E = 20\,\mathrm{eV} : \sigma = 6.8 \cdot 10^{-18}\mathrm{cm}^2, l = 0.047\,\mathrm{pc} \tag{4}$$

$$E = 20\,\mathrm{keV} : \sigma = 6.8 \cdot 10^{-26}\mathrm{cm}^2, l = 4.7 \cdot 10^6\,\mathrm{pc} \tag{5}$$

2. — The Thompson scattering cross-section is $\sigma_T = 6.66 \cdot 10^{-25}$ cm^2 and using the average electron density, the mean-free-path is:

$$l = 1/(\rho \cdot \sigma) = 7.5 \cdot 10^{30}\mathrm{cm} = 2.4 \cdot 10^6\mathrm{Mpc}. \tag{6}$$

3. — The radius of the visible universe, i.e., the distance a photon can travel since the Big Bang, is about the speed of light multiplied by the time since the Big Bang. Since the age of the Universe is 13.8 Gyrs, the radius is 4231 Mpc (including the expansion of the universe would increase this by a factor of about 3). Thus, the intergalactic medium is transparent to X-rays.

Ex. 2 — Superluminar motion

1. — (1 for the speed, 1 for the distance)
Figure 1 below shows the jets emitted by GRS 1915+105, a Galactic X-ray binary source, during 1994. From the image, estimate the average speed at which each bright spot appears to be moving from the core (indicated by a cross). The source is distant 40000 ly. To which distance does the angular size of 1 arcsec correspond to?

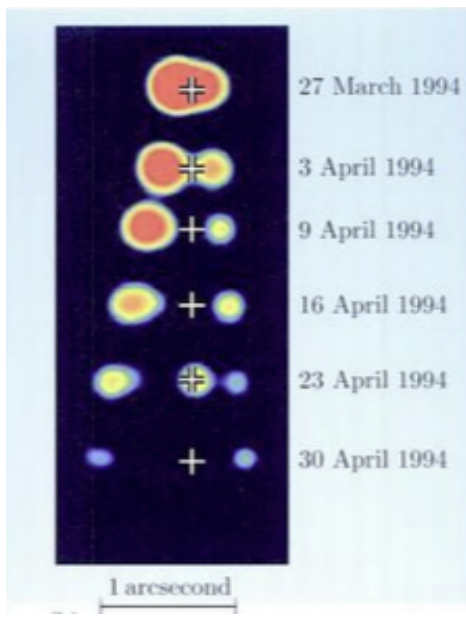


Figure 1: Example of apparent superluminar motion of the jet of GRS 1915+105.

2. — Let's now understand which is the reason that explains the apparent velocity to be larger then the velocity of the light. Consider an AGN, which is moving from position

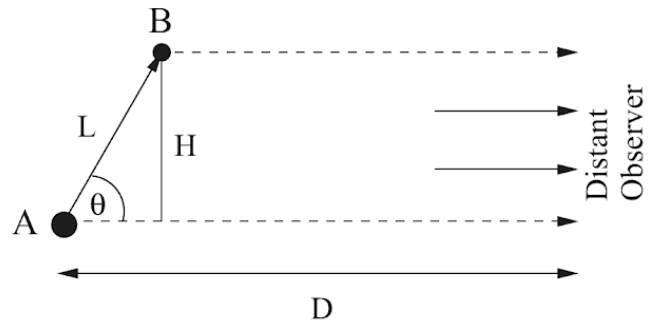


Figure 2

A to B in the sky in time δt, at an angle θ with respect to the line of sight of a distant observer, as shown in Figure 2. Assume that the AGN emits a first bunch of photons at time $t_0 = 0$ in A. At what time the photons will reach the observer?

3. — The AGN emits a second flare when located at position B. Assuming that the angle θ is small, which is the time at which this second flare will reach the observer?
4. — The observer sees the source moving along H. Which is the perceived velocity v_{app} of the radio jets from the observer?
5. — Determine the angle θ_{max} at which this velocity is maximal and determine its maximum value $v_{max,app}$. In which conditions the radio jets will appear as having superluminar motion?
6. — Write a small python script to plot the apparent velocity as a function of the orientation angle and the luminosity doppler boosting factor for the case $n = 3$, in polar coordinates. Choose at least 5 different values of β at which to plot the curves.

Answer (Ex. 2) —

1. — See solution on jupyter notebook
2. — Since the photon travels at the speed of light, it will take a time t_{AO} to cover distance D:

$$t_{AO} = \frac{D}{c}. \tag{7}$$

3. — The AGN moves along L. It will reach position B after a time δt and there will emit the second bunch of photons. Since the angle θ is assumed to be small, we can approximate the distance between B and the observer as $d_{BO} \simeq D - v\delta t\, cos\theta$, hence the photons will reach the observer at time:

$$t_{BO} = \delta t + \frac{D - v\delta t\, cos\theta}{c}. \tag{8}$$

4. — The observer perceives the source moving along H, hence covering the distance $d =$

$v\delta t\, sin\theta$, perpendicular to its sight of line with an apparent velocity:

$$v_{app} = \frac{v\delta t\, sin\theta}{t_{BO} - t_{AO}} = \frac{v\, sin\theta}{1 - \frac{v}{c}\, cos\theta}. \tag{9}$$

5. — The angle that maximizes the apparent jet velocity can be calculated by setting the derivative of $v_{app}(\theta)$ with respect to the angle θ to zero.

$$v\, cos\theta - \frac{v^2}{c}\, cos^2\theta - \frac{v^2}{c}\, sin^2\theta = 0, \tag{10}$$

so that θ_{max} is:

$$\theta_{max} = arccos\left(\frac{v}{c}\right). \tag{11}$$

For v close to the speed of light, this angle is very small. Introducing γ we can obtain an expression for the maximal observed jet velocity by inserting Eq. 11 into Eq. 9:

$$v_{max,app} = \gamma v. \tag{12}$$

If the jets have velocities close to c, this becomes:

$$v_{max,app} = \gamma c, \tag{13}$$

and this explains how the observed jets velocity can be larger than c even though the true jet speed does not exceed c.

6. — See solution on jupyter notebook

Series 4 – Shock waves and Fermi acceleration mechanism

July 29, 2019

Ex. 1 — Fermi acceleration mechanism
Fermi acceleration mechanisms. In the Fermi acceleration mechanism, charged particles increase considerably their energies crossing back and forth many times the border of a magnetic cloud (second-order Fermi mechanism) or of a shock wave (first-order Fermi mechanism). Compute the number of crossings that a particle must do in each of the mechanisms to have a ratio $E_{fin}/E_{in} = 10$, assuming:

1. — realistic values of $\beta = 10^{-4}$ for the magnetic cloud and $\beta = 10^{-2}$ for the shock wave;
2. — $\beta = 10^{-4}$ for both acceleration mechanisms.
3. — For a CR a realistic mean free path is $L \sim 0.1$ pc. Be Φ the pitch angle of the particle at the $i-$th scattering. Which is the mean time between collisions? Which would be the time needed to reach E_f for the magnetic cloud from point 1?

Answer (Ex. 1) —

1. —

$$\Delta E_1 = \frac{E_{f,1}}{E_0} \tag{1}$$

$$\Delta E_2 = \frac{E_{f,2}}{E_{f,1}} = \frac{E_{f,2}}{E_0 \cdot \Delta E_1} \rightarrow \frac{E_{f,2}}{E_0} = \Delta E_1 \cdot \Delta E_2 = (\Delta E)^2 \tag{2}$$

$$\Delta E_n = \frac{E_{f,2}}{E_0 \cdot \Delta E_1 ... \cdot \Delta E_n} \rightarrow \frac{E_{f,n}}{E_0} = (\Delta E)^n \tag{3}$$

   For the magnetic cloud ($\beta = 10^{-4}$) the number of cycles is $1.7 \cdot 10^8$ and for the shock wave ($\beta = 10^{-2}$) is $1.7 \cdot 10^2$.
2. — If using $\beta = 10^{-4}$ for the shock wave, the number of cycles becomes $1.7 \cdot 10^4$.
3. — Define L as mean free path and Φ as pitch angle, the time between collisions is $t_{coll} =\sim L/(c\, cos\Phi)$, which can be averaged to $2L/c$. Considering $L \sim 0.1$ pc and number of cycles calculated above for the magnetic cloud ($\beta = 10^{-4}$), the necessary time should be around 10^9 years.

Ex. 2 — SNR explosions and shock waves
In SNR explosion shock waves can accelerate particles. A supernova of mass $M = 10^L$ ($M^L = 2 \cdot 10^{33}$ g) has an ejected mass with typical kinetic energy $K \sim 10^{51}$ erg.

1. — How long does it take to extinguish the accelerator (time of free expansion of the ejecta)?

Answer (Ex. 2) —

1. — The speed of the shock front can be obtained as:

$$v = \sqrt{\frac{2K}{M}} \sim 3200 \text{ km/s}. \tag{4}$$

The shock gets extinguished when the mass of the ejecta reach a density equal to the average interstellar density $\rho_{IG} \sim 1$ p/cm$^3 = 1.6 \cdot 10^{-24}$ g/cm3:

$$\rho_{SN} = \text{M/Volume} = \text{M}/(4/3\pi R^3) = \rho_{IG} \rightarrow \text{Volume} \rightarrow R \sim 1.4 \cdot 10^{19} \text{ cm} \sim 5\text{pc}. \tag{5}$$

So the time can be calculated as:

$$T_{acc} = R/v \sim 1000 \text{ years} \tag{6}$$

Note that these are only typical order of magnitude numbers.

Ex. 3 — Boron-To-Carbon measurements

1. — Write an approximated Boron production rate due to the Carbon spallation process in the Galaxy, given its production cross-section $\sigma_{C\rightarrow B}$. Given that the Boron production rate is related to the Boron density by the lifetime of Boron in the Galaxy τ, express the ratio N_B/N_C.
2. — Fit the AMS measurements for the Boron-to-Carbon ratio available in the carbon-boron-AMS02.dat file. (**Suggestion:** use the log scales on the axes to perform the fit.)

3. — Which is the average interstellar gas number density? Express it in terms of the Boron average lifetime in the Galaxy.

Answer (Ex. 3) —

1. — Since we know the partial cross-section of spallation processes we can use the secondary-to-primary abundance ratios to infer the gas column density traversed by the average cosmic ray. Let us perform a simply estimate of the Boron-to-Carbon ratio. Boron is chiefly produced by Carbon and Oxygen with approximately conserved kinetic energy per nucleon, so we can relate the Boron source production rate, to the differential density of Carbon by this equation:

$$Q_B \sim n_H \cdot N_C \cdot \beta \cdot c \cdot \sigma_{C\rightarrow B} \tag{7}$$

where, n_H denotes the average interstellar gas number density, N_C is the Carbon density, β is the Carbon velocity and $\sigma_{C\rightarrow B}$ is the spallation cross-section of Carbon into Boron.

The Boron density is related to the production rate by the lifetime of Boron in the Galaxy, τ, before it escapes or losses its energy by spallation $Q_B = N_B/\tau$. So we can write:

$$\frac{N_B}{N_C} \sim n_H \cdot \beta \cdot c \cdot \sigma_{C \to B} \cdot \tau \tag{8}$$

2. — See Jupiter notebook
3. — The values from the are a = 0.44, and b = -0.34. Above about 10 GeV/nucleon the experimental data can be fitted to a test function, therefore the Boron-to-Carbon ratio can be expressed as:

$$\frac{N_B}{N_C} = 0.4\Big(\frac{\mathrm{E}}{\mathrm{GeV}}\Big)^{-0.3} \tag{9}$$

For energies above 10 GeV/nucleon we can approximate $\beta \sim 1$, which leads, using the values of the cross-section, to a life time gas density of:

$$n_H \cdot \tau \sim 10^{14}\Big(\frac{\mathrm{E}}{\mathrm{GeV}}\Big)^{-0.3} \mathrm{s\ cm}^{-3} \tag{10}$$