International Neutrino Summer School Group 2

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1.1) Show that only for a free massless fermion the chirality eigenstates are also helicity eigenstates.

Starting from the Dirac equation:

$$(\gamma^{\mu} - m)\,\psi = 0$$

$$\begin{pmatrix} E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -(E+m) \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\phi = \frac{\vec{p}\cdot\vec{\sigma}}{E-M}\chi \qquad\qquad \chi = \frac{\vec{p}\cdot\vec{\sigma}}{E+M}\psi$$

1.1) Show that only for a free massless fermion the chirality eigenstates are also helicity eigenstates.

Now we consider the left chiral state:

$$\psi_{\mathrm{L}} = \frac{1}{2}\left(1-\gamma^5\right)\psi = \frac{1}{2}\begin{pmatrix}\phi-\chi \\ \chi-\phi\end{pmatrix}$$

Considering only the top component:

$$\frac{1}{2}(\phi-\chi) = \frac{1}{2}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\phi = \frac{1}{2}\left(1-\frac{p_z\sigma_z}{E+m}\right)\begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix}$$

$$= \frac{1}{2}\left(1-\frac{p_z}{E+M}\right)\phi_+ + \frac{1}{2}\left(1+\frac{p_z}{E+M}\right)\phi_-$$

1.2) Show that the mass matrix M of a Majorana mass must be symmetric $M_{ij}=M_{ji}$.

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}\left(\left(\overline{\nu}\right)^{C} M\nu + \overline{\nu}M^{T}\left(\nu\right)^{C}\right)$$

$$\Longrightarrow \mathcal{C}\mathcal{L}_{\text{mass}} = \frac{1}{2}\left(\overline{\nu}M\left(\nu\right)^{C} + \left(\overline{\nu}\right)^{C} M^{T}\nu\right)$$

$$M^{T} = M$$

1.3) Show that if there are **N=3+s** massive neutrinos, the leptonic mixing matrix is dimension **3×N** and contains **3s+3** physical angles and **2s−1** phases for Dirac neutrinos and **3s+3** for Majorana neutrinos

$$\mathcal{L}_{CC} + \mathcal{L}_M = -\frac{g}{\sqrt{2}} \bar{\ell}^W_{L,i} \gamma^\mu \nu^W_i W^+_\mu - \bar{\ell}^W_{L,i} M_{\ell,ij} \ell^W_{R,j} - \frac{1}{2} \bar{\nu}^C_i M_{\nu,ij} \nu^W_j$$

$$\Longrightarrow U_{\mathrm{LEP}} \text{ must be } 3 \times N$$

In general, a $3 \times N$ matrix can be parametrized as:

$$U_{LEP} = \begin{pmatrix} a_{11} e^{i\phi_{11}} & \cdots & a_{1N} e^{i\phi_{1N}} \\ a_{21} e^{i\phi_{21}} & \ldots & a_{2N} e^{i\phi_{2N}} \\ a_{31} e^{i\phi_{31}} & \ldots & a_{3N} e^{i\phi_{3N}} \end{pmatrix}$$

1.3) Show that if there are **N=3+s** massive neutrinos, the leptonic mixing matrix is dimension **3×N** and contains **3s+3** physical angles and **2s−1** phases for Dirac neutrinos and **3s+3** for Majorana neutrinos

However, we know that $U_{\mathrm{LEP}}U_{\mathrm{LEP}}^{\dagger} = 1_3$

$\Longrightarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11}e^{i\phi_{11}} & \dots & a_{1N}e^{i\phi_{1N}} \\ a_{21}e^{i\phi_{21}} & \dots & a_{2N}e^{i\phi_{2N}} \\ a_{31}e^{i\phi_{31}} & \dots & a_{3N}e^{i\phi_{3N}} \end{pmatrix} \begin{pmatrix} a_{11}e^{-i\phi_{11}} & a_{21}e^{-i\phi_{21}} & a_{11}e^{-i\phi_{31}} \\ \vdots & \vdots & \vdots \\ a_{1N}e^{-i\phi_{1N}} & a_{2N}e^{-i\phi_{2N}} & a_{3N}e^{-i\phi_{3N}} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{j=N} a_{1j}^2 & a_{1j}a_{2j}e^{i(\phi_{1j}-\phi_{2j})} & a_{1j}a_{3j}e^{i(\phi_{1j}-\phi_{3j})} \\ a_{1j}a_{2j}e^{-i(\phi_{1j}-\phi_{2j})} & \sum_{j=1}^{j=N} a_{2j}^2 & a_{2j}a_{3j}e^{i(\phi_{2j}-\phi_{3j})} \\ a_{1j}a_{3j}e^{-i(\phi_{1j}-\phi_{3j})} & a_{2j}a_{3j}e^{-i(\phi_{2j}-\phi_{3j})} & \sum_{j=1}^{j=N} a_{3j}^2 \end{pmatrix}$$

1.4) The decay width for β decay $N \to Pe^- \bar{\nu}_e$ after integrating over the proton momentum is

$$d\Gamma = G_F^2 \cos^2\theta_C F(E,Z) 2\pi \sum_{spin} \sum_i |U_{ei}|^2 |\bar{u}_e(p)\gamma^0(1-\gamma^5)v_{\nu_i(k)}|^2 \frac{d^3p}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2w} \delta(E_0 - E - w)$$

G_F is the Fermi constant, θ_C is Cabbibo's angle, $F(E,Z)$ a Coulomb factor, E_0 is the mass difference between the initial and final nuclei, E is the electron energy and ω is the neutrino energy. Obtain the electron energy spectrum $d\Gamma/dE$ and show that the Kurie function

$$K(T) \equiv \left[\frac{\frac{d\Gamma}{dE}}{C\,F(E,Z)\,|\vec{p}|\,E}\right]^{1/2} = \sqrt{(Q-T)\sum_i |U_{ei}|^2 \sqrt{(Q-T)^2 - m_i^2}} \simeq \sqrt{(Q-T)\sqrt{(Q-T)^2 - \sum_i |U_{ei}|^2 m_i^2}}$$

with $Q=E_0 - m_e$, $T=E - m_e$, $C=G^2_F\cos^2\theta_C/\pi^3$. Show that the last equality holds only for $Q-T \gg m_i$ and $\sum_i |U_{ei}|^2 = 1.$

Plot $K(T)$ as a function of T for tritium ($Q = 18.6$ KeV) for $m_\beta = \sqrt{\sum_i |U_e i|^2 m_i^2} = 0$ and for $m_\beta = \sqrt{\sum_i |U_e i|^2 m_i^2} = 2.2$ eV.

$$d\Gamma = G_F^2 \cos^2\theta_C F(E,Z) 2\pi \sum_s \sum_i |U_{ei}|^2 \left|\overline{u}_e(p)\gamma^0\left(1-\gamma^5\right)v_\nu(k)\right|^2 \times$$

$$\times \frac{d^3p}{(2\pi)^3 2E}\frac{d^3k}{(2\pi)^3 2w}\delta\left(E_o - E - w\right)$$

Considering only the sum over spin:

$$\sum_s \left|\overline{u}_e(p)\gamma^0\left(1-\gamma^5\right)v_\nu(k)\right|^2 = \left[\overline{u}_e(p)\gamma^0\left(1-\gamma^5\right)v_\nu(k)\right]\left[\overline{u}_e(p)\gamma^0\left(1-\gamma^5\right)v_\nu(k)\right]^*$$

$$= \mathrm{Tr}\left[\gamma^0\left(1-\gamma^5\right)\left(\gamma^\mu k_\mu - m_v\right)\gamma^0\left(1-\gamma^5\right)\left(\gamma^\nu p_\nu + m_e\right)\right]$$

(Trace sadness...)

$$= 8Ew + 8\vec{k}\cdot\vec{p}$$

$$\mathrm{d}\Gamma = \frac{G_F^2\cos^2\theta_C}{\pi^3}\sum_i |U_{ei}|^2\left(Ew + p\cdot k\right)\frac{p^2\mathrm{d}p}{E}\frac{k^2\mathrm{d}k}{w}$$

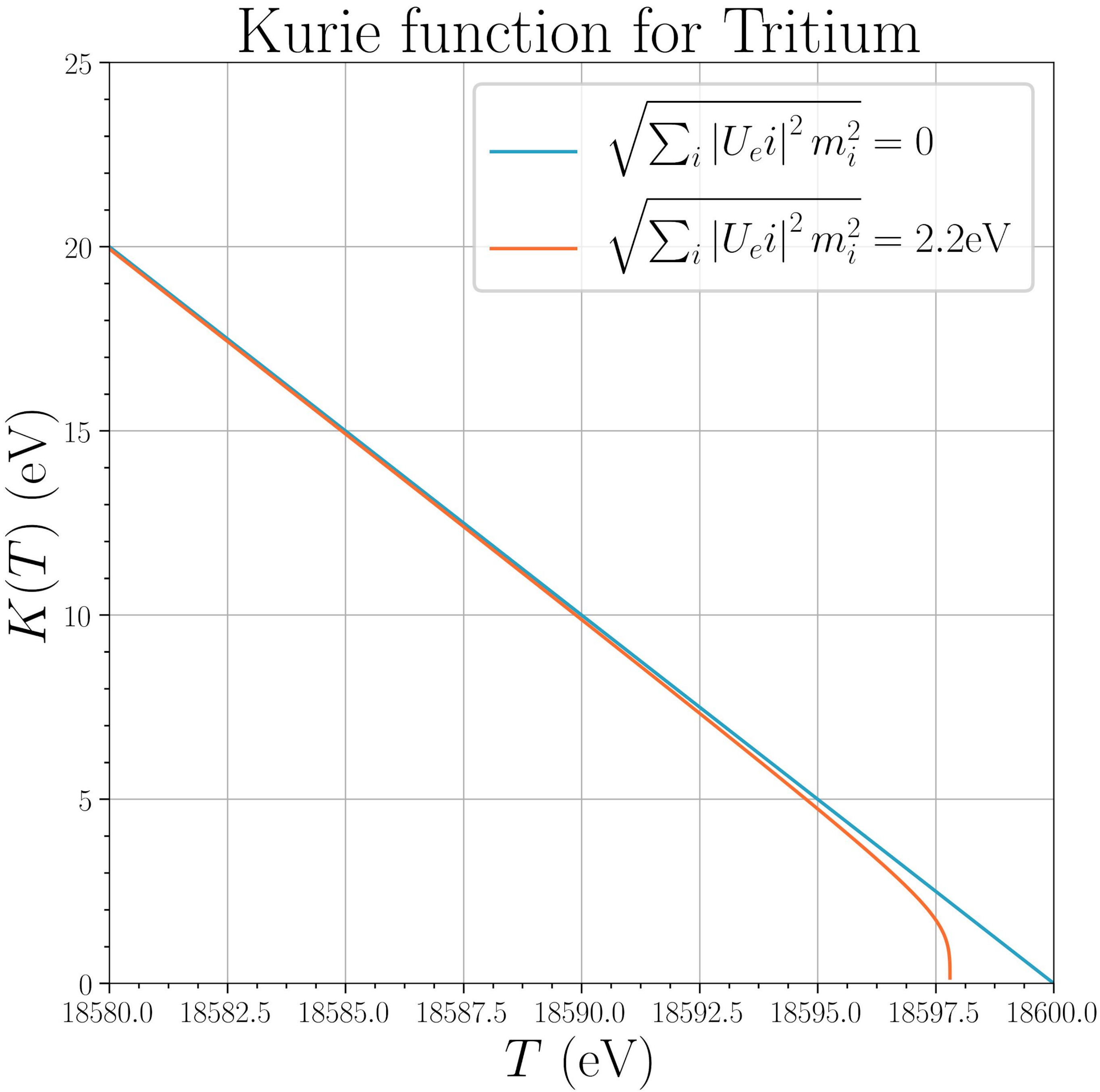
(Integral sadness...)

$$K(E) = \left[\frac{\frac{d\Gamma}{dE}}{CF(E,Z)|\vec{p}|E}\right] = \sqrt{(E_o - E)\sum_i \left|\vec{U}_{ei}\right|^2 \sqrt{(E_o - E)^2 - m_i^2}}$$

$$\simeq \sqrt{(E_0 - E)\sqrt{(E_0 - E)^2 - \sum_i |U_{ei}|^2 m_i^2}}$$

Now redefine variables:

$$K(T) = \left[\frac{\frac{d\Gamma}{dE}}{CF(E,Z)|\vec{p}|E}\right] = \sqrt{(Q - T)\sum_i \left|\vec{U}_{ei}\right|^2 \sqrt{(Q - T)^2 - m_i^2}}$$



Kurie function for Tritium
$\sqrt{\sum_i |U_e i|^2 m_i^2} = 0$
$\sqrt{\sum_i |U_e i|^2 m_i^2} = 2.2\text{eV}$
$K(T)$ (eV)
T (eV)
25
20
15
10
5
0
18580.0
18582.5
18585.0
18587.5
18590.0
18592.5
18595.0
18597.5
18600.0

1.5) The charged pion decays almost 100% of the time into a muon and a (muon-type) neutrino. In the reference frame where the parent pion is at rest, compute the muon energy as a function of the muon-mass (m_μ), the charged pion mass (m_π), and the neutrino mass (m_ν). What is the absolute value of the muon momentum (tri)vector? Numerically, what is the relative change of the muon momentum between $m_\nu=0$ and $m_\nu=0.1\ \mathrm{MeV}$? It is remarkable that the muon momentum from pion decay at rest has been measured at the 3.4×10^{-6} level (Phys. Rev. D53, 6065 (1996)). This provides the most stringent current constraint on the muon-neutrino mass.

$$|\vec{p}_\mu| = |\vec{p}_\nu| = \frac{\sqrt{\left(m_\pi^2 - (m_\mu + m_\nu)^2\right)\left(m_\pi^2 - (m_\mu - m_\nu)^2\right)}}{2m_\pi}$$

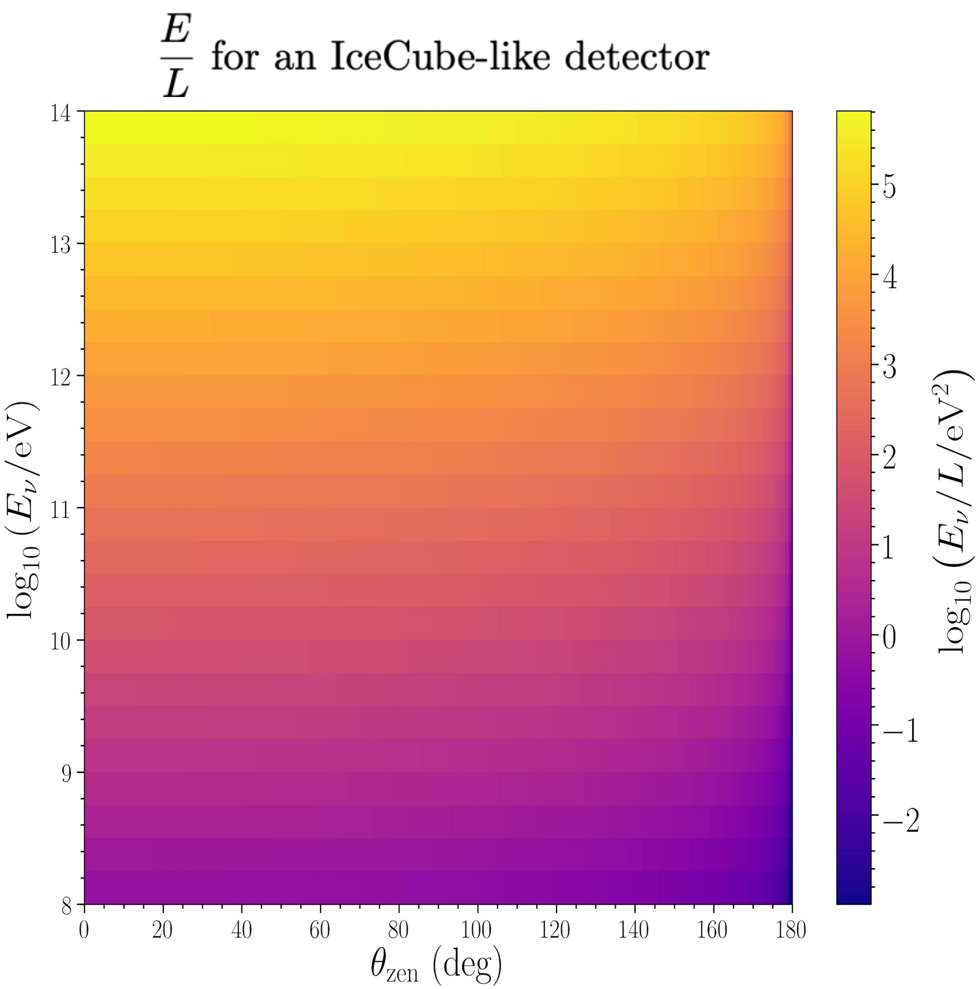
1.5) The charged pion decays almost 100% of the time into a muon and a (muon-type) neutrino. In the reference frame where the parent pion is at rest, compute the muon energy as a function of the muon-mass (m_μ), the charged pion mass (m_π), and the neutrino mass (m_ν). What is the absolute value of the muon momentum (tri)vector? Numerically, what is the relative change of the muon momentum between $m_\nu=0$ and $m_\nu=0.1\ \mathrm{MeV}$? It is remarkable that the muon momentum from pion decay at rest has been measured at the 3.4×10^{-6} level (Phys. Rev. D53, 6065 (1996)). This provides the most stringent current constraint on the muon-neutrino mass.

$$E_\mu = m_\mu^2 + \frac{\left(m_\pi^2 - (m_\mu + m_\nu)^2\right)\left(m_\pi^2 - (m_\mu - m_\nu)^2\right)}{4m_\pi^2}$$

$$\implies E_\mu = \frac{\sqrt{4m_\pi^2 m_\mu^2 + \left(m_\pi^2 - (m_\mu + m_\nu)^2\right)\left(m_\pi^2 - (m_\mu - m_\nu)^2\right)}}{2m_\pi}$$

$$E_\mu\Big|_{m_\mu=0} - E_\mu\Big|_{m_\mu=0.1\mathrm{MeV}} \equiv \Delta E_{m_\mu} = 3.584\times10^{-4}$$

1.6) Derive the characteristic E/L in eV^2 for atmospheric neutrinos, and for T2K and for Daya Bay.



Daya Bay:

$$\left.\frac{E}{L}\right|_{\mathrm{near}} = \frac{4\times 10^6\mathrm{eV}}{2.5\times 10^9\,\frac{1}{\mathrm{eV}}} = 1.6\times 10^{-3}\ \mathrm{eV}^2$$

$$\left.\frac{E}{L}\right|_{\mathrm{far}} = \frac{4\times 10^6\mathrm{eV}}{8\times 10^9\,\frac{1}{\mathrm{eV}}} = 5\times 10^{-4}\ \mathrm{eV}^2$$

T2K:

$$\left.\frac{E}{L}\right|_{\mathrm{near}} = \frac{6\times 10^8\mathrm{eV}}{1.4\times 10^9\,\frac{1}{\mathrm{eV}}} = 4.3\times 10^{-1}\ \mathrm{eV}^2$$

$$\left.\frac{E}{L}\right|_{\mathrm{far}} = \frac{6\times 10^8\ \mathrm{eV}}{1.475\times 10^{12}\,\frac{1}{\mathrm{eV}}} = 4\times 10^{-4}\ \mathrm{eV}^2$$

1.7) Show that in a disappearance experiment, the survival probability for neutrinos and antineutrinos is the same.

$$P_{\nu_\alpha \to \nu_\beta}(L,E) = \delta_{\alpha\beta} - 4\sum_{k>j} \Re e\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

$$+ 2\sum_{k>j} \mathfrak{Im}\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*\right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E}\right)$$

$$P_{\nu_\alpha \to \nu_\alpha}(L,E) = \delta_{\alpha\alpha} - 4\sum_{k>j} \mathfrak{Re}\left[U_{\alpha k}^* U_{\alpha k} U_{\alpha j} U_{\alpha j}^*\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

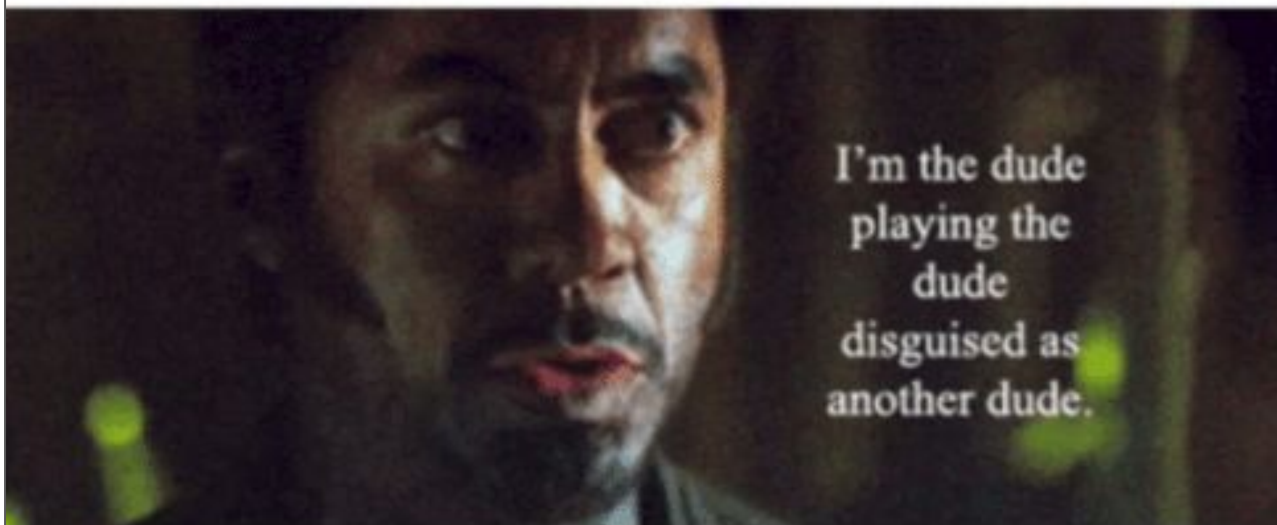
$$+ 2\sum_{k>j} \mathfrak{Im}\left[U_{\alpha k}^* U_{\alpha k} U_{\alpha j} U_{\alpha j}^*\right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E}\right)$$

$$= 1 - 4\sum_{k>j} \mathfrak{Re}\left[\left|U_{\alpha k}\right|^2 \left|U_{\alpha j}\right|^2\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

$$+ 2\sum_{k>j} \mathfrak{Ie}\left[\left|U_{\alpha k}\right|^2 \left|U_{\alpha j}\right|^2\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

$$\boxed{= 1 - 4\sum_{k>j} \left|U_{\alpha k}\right|^2 \left|U_{\alpha j}\right|^2 \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)}$$

An electron neutrino after it oscillates to a muon neutrino then oscillates again to a tau neutrino:



Identity theft is not a joke Jim! 1.2 x 10^89 neutrinos suffer every year!

Backup

1.8) a) What is the matter potential for $\nu_\tau \rightarrow \nu_s$? ($\nu_s$ is a sterile neutrino.) Compare with the potential for $\nu_e \rightarrow \nu_{\mu,\tau}$.

b) In the core of a supernova, the matter density is $\rho \sim 10^{14}$ g/cm^3. Obtain the characteristic value for the matter potential for $\nu_\tau \rightarrow \nu_s$ and $\overline{\nu}_\tau \rightarrow \nu_s$ in the core of the supernova.

c) For what characteristic mass differences (assume $E_\nu \sim 10\ MeV$) can the MSW effect occur in such supernova? In which of the two channels does it occur?

- Consider the case in which $\ell = \mu^-$ and the elastic scattering happens on a neutron at rest, *i.e.* the neutron quadri-momentum is given by $(m_n, 0)$ and the proton $(m_p + \omega, \mathbf{q})$. Show that the reconstructed neutrino energy reads

$$E_\nu = \frac{m_p^2 - m_n^2 - m_\mu^2 + 2m_n E_\mu}{2(m_n - E_\mu + p_\mu \cos\theta_\mu)}\,, \tag{1}$$

where θ_μ is the muon angle relative to the neutrino beam.

Solution: Considering the quadri-momentum conservation:

$$E_\nu + m_n = E_\mu + E_p \tag{2}$$

$$\mathbf{k}_\nu = \mathbf{k}_\mu + \mathbf{q} \tag{3}$$

where $\mathbf{k}_\nu$, $\mathbf{k}_\mu$ and $\mathbf{q}$ are the momentum of neutrino, muon and proton respectively. And $E_p = m_p + \omega$

Considering the invariant mass of proton,

$$m_p^2 = (E_p)^2 - (\mathbf{q})^2 \tag{4}$$

Combining (2), (3) and (4) we can have,

$$\begin{aligned} m_p^2 &= (E_\nu + m_n - E_\mu)^2 - (\mathbf{k}_\nu - \mathbf{k}_\mu)^2 \\ &= E_\nu^2 + E_\mu^2 + m_n^2 + 2m_n(E_\nu - E_\mu) - 2E_\nu E_\mu - |\mathbf{k}_\nu|^2 - |\mathbf{k}_\mu|^2 + 2\mathbf{k}_\nu \cdot \mathbf{k}_\mu \\ &= m_\mu^2 + m_n^2 - 2m_n E_\mu + 2E_\nu(m_n - E_\mu + |p_\mu| \cos\theta_\mu) \end{aligned}$$

$\Rightarrow$

$$E_\nu = \frac{m_p^2 - m_n^2 - m_\mu^2 + 2m_n E_\mu}{2(m_n - E\mu + |p_\mu| \cos\theta)}$$

Question: Write down the analogous expression for a moving neutron in the initial state, i.e $(E_n, \mathbf{p}_n)$. To derive the neutrino energy expression, define the angle between the neutron and the neutrino beam momentum as $\cos\theta_n = (\mathbf{k}_\nu \cdot \mathbf{p}_n)/(|\mathbf{k}_\nu||\mathbf{p}_n|)$.

Solution: Similarly that before we will consider the quadri-momentum conservation and the invariant mass of proton.

$$E_\nu + E_n = E_\mu + E_p \tag{1}$$

$$\mathbf{k}_\nu + \mathbf{p}_n = \mathbf{k}_\mu + \mathbf{q} \tag{2}$$

Where $\mathbf{k}_\nu$, $\mathbf{k}_\mu$, $\mathbf{p}_n$ and $\mathbf{q}$ are the momentum of neutrino, muon, neutron and proton respectively. Then

$$\begin{aligned}
m_p^2 &= (E_p)^2 - (\mathbf{q})^2 \\
&= (E_\nu + E_n - E_\mu)^2 - (\mathbf{k}_\nu + \mathbf{p}_n - \mathbf{k}_\mu)^2 \\
&= E_\nu^2 + E_\mu^2 + E_n^2 + 2E_n(E_\nu - E_\mu) - 2E_\nu E_\mu - |\mathbf{k}_\nu|^2 - |\mathbf{k}_\mu|^2 - |\mathbf{p}_n|^2 + 2\mathbf{k}_\nu \cdot \mathbf{k}_\mu - 2\mathbf{k}_\nu \cdot \mathbf{p}_n + 2\mathbf{k}_\mu \cdot \mathbf{p}_n \\
&= m_\mu^2 + m_n^2 - 2E_nE_\mu + 2\mathbf{k}_\mu \cdot \mathbf{p}_n + 2E_\nu(E_n - E_\mu + |p_\mu|\cos\theta_\mu - |p_n|\cos\theta_n)
\end{aligned}$$

$\Rightarrow$

$$E_\nu = \frac{m_p^2 - m_n^2 - m_\mu^2 + 2E_nE_\mu - 2\mathbf{k}_\mu \cdot \mathbf{p}_n}{2(E_n - E\mu + |p_\mu|\cos\theta) - |p_n|\cos\theta_n}$$

Lepton-Nucleus Cross Section Theory

The most general expression for the hadronic tensor is constructed out of $g^{\mu\nu}$ and the independent momentum of the initial nucleon p and the momentum transfer q, yielding

$$W^{\nu\mu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_N^2} p^\mu p^\nu + i\epsilon^{\mu\nu\alpha\beta} \frac{W_3}{2m_N^2} p_\alpha q_\beta + \frac{W_4}{m_N^2} q^\mu q^\nu + \frac{W_5}{m_N^2} (p^\mu q^\nu + p^\nu q^\mu) \tag{1}$$

where W_i are called structure function and m_N is the nucleon mass.

Question: In the electromagnetic case parity-violating effects are NOT present. The hadronic electromagnetic current matrix elements are polar-vectors and so the tensor must have specific properties under spatial inversion. In particular, in this case $W_3 = 0$. The current conservation condition at the hadronic vertex requires

$$q_\nu W^{\nu\mu} = q_\mu W^{\nu\mu} = 0 \tag{2}$$

As a result of this relation, verify that only two structure function are independent and the hadronic electromagnetic tensor reads

$$W^{\nu\mu} = W_1 \left(g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_N^2} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right) \tag{3}$$

Solution: Considering $q_\nu W^{\nu\mu} = 0$ and $W_3 = 0$ in equation (2), we have

$$\begin{aligned} q_\nu W^{\nu\mu} &= -W_1 q^\mu + \frac{W_2}{m_N^2} p^\mu q_\nu p^\nu + \frac{W_4}{m_N^2} q^\mu q^2 + \frac{W_5}{m_N^2} (p^\mu q^2 + q_\nu p^\nu q^\mu) \\ &= q^\mu (-W_1 + \frac{W_4}{m_N^2} q^2 + \frac{W_5}{m_N^2} q_\nu p^\nu) + p^\mu (\frac{W_2}{m_N^2} q_\nu p^\nu + \frac{W_5}{m_N^2} q^2) \end{aligned}$$

Lepton-Nucleus Cross Section Theory

Then we got

$$W_5 = -W_2 \frac{q \cdot p}{q^2} \tag{4}$$

$$W_4 = W_1 \frac{m_N^2}{q^2} + W_2 \left(\frac{q \cdot p}{q^2} \right)^2 \tag{5}$$

Finally using (4) and (5) in (1). Also considering W3=0,

$$\begin{aligned}
W^{\nu\mu} &= -W_1 g^{\mu\nu} + \frac{W_2}{m_N^2} p^\mu p^\nu + \frac{W_4}{m_N^2} q^\mu q^\nu + \frac{W_5}{m_N^2} (p^\mu q^\nu + p^\nu q^\mu) \\
&= -W_1 g^{\mu\nu} + \frac{W_2}{m_N^2} p^\mu p^\nu + \left(W_1 \frac{m_N^2}{q^2} + W_2 \left(\frac{q \cdot p}{q^2} \right)^2 \right) \frac{1}{m_N^2} q^\mu q^\nu - W_2 \frac{q \cdot p}{q^2} \frac{1}{m_N^2} (p^\mu q^\nu + p^\nu q^\mu) \\
&= W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_N^2} \left(p^\mu p^\nu + \left(\frac{q \cdot p}{q^2} \right)^2 q^\mu q^\nu - \frac{q \cdot p}{q^2} (p^\mu q^\nu + p^\nu q^\mu) \right) \\
&= W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_N^2} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right)
\end{aligned}$$