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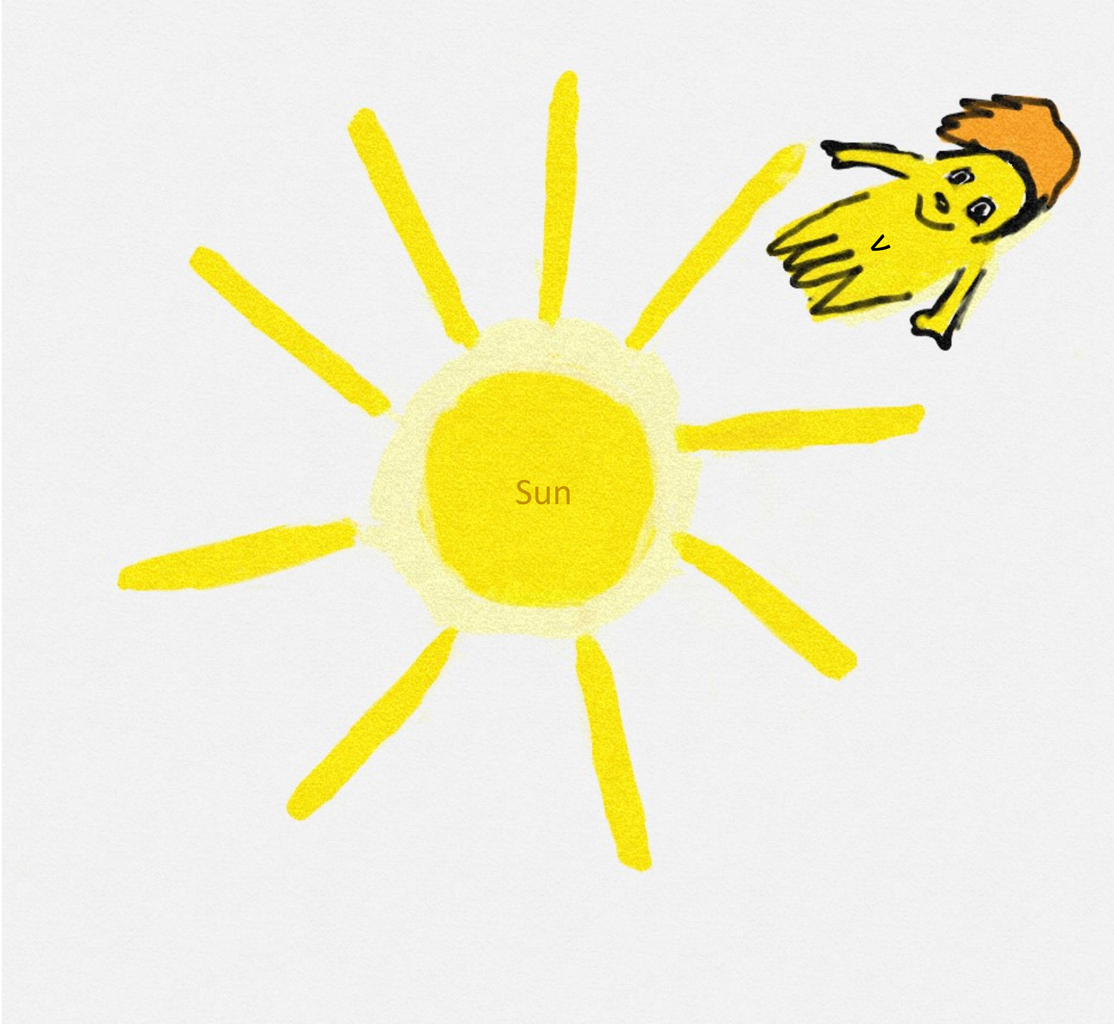
Relating quantum mechanics and kinetics of neutrino oscillations

Main findings

Quantum-mechanical and kinetic approaches to neutrino oscillations produce equivalent results for collisionless neutrino propagation. Kinetic equation also accounts for the uncertainty and Pauli principles if supplemented by initial conditions consistent with these fundamental principles.

What the quantum-mechanical approach is usually used for



What the kinetic approach is typically used for

Why use the kinetic approach to neutrino oscillations

Neutrino propagation in the Sun's interior is almost collisionless and can be described by the Schrödinger equation for the neutrino wave function. On the other hand, neutrino collisions with particles of the ambient medium can play a dominant role in certain phases of supernovae evolution. Due to the necessity of including also the scattering processes, flavor conversion of supernovae neutrinos is typically analyzed using kinetic equation for the matrix of densities. The latter is defined as the expectation value of bilinears of the creation and annihilation operators of the neutrino field:

$$\varrho_{ij}(t,\mathbf{x},\mathbf{p}) = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{i\mathbf{\Delta x}} \langle \hat{a}_j^\dagger(t,\mathbf{p}-\mathbf{\Delta}/2)\hat{a}_i(t,\mathbf{p}+\mathbf{\Delta}/2)\rangle$$

The kinetic equation follows from the Heisenberg equation of motion:

$$\partial_t \varrho + \tfrac{1}{2}\{\partial_{\mathbf{x}}\varrho, \partial_{\mathbf{p}}\mathsf{H}\} - \tfrac{1}{2}\{\partial_{\mathbf{p}}\varrho, \partial_{\mathbf{x}}\mathsf{H}\} = -i\,[\mathsf{H},\varrho] + \mathcal{C}\,[\varrho]$$

Because the density matrix is a function of the coordinate and momentum, it is believed to be in conflict with the uncertainty principle as well as unable to account for the effect of wave packet separation.

How is it related to the quantum-mechanical approach

Quantum-mechanical counterpart of the matrix of densities is the Wigner function. For a single-particle system:

$$\varrho_{ij}(t,\mathbf{x},\mathbf{p}) = \int d^3\mathbf{y}\, e^{-i\mathbf{p}\mathbf{y}} \psi_i(t,\mathbf{x}+\mathbf{y}/2)\psi_j^*(t,\mathbf{x}-\mathbf{y}/2)$$

Evolution equation for the Wigner function (describing collisionless neutrino propagation) follows from the Schrödinger equation and, to the first order in the gradient expansion, reads:

$$\partial_t \varrho + \tfrac{1}{2}\{\partial_{\mathbf{x}}\varrho, \partial_{\mathbf{p}}\mathsf{H}\} - \tfrac{1}{2}\{\partial_{\mathbf{p}}\varrho, \partial_{\mathbf{x}}\mathsf{H}\} = -i\,[\mathsf{H},\varrho]$$

Its form matches form of the (collisionless part of the) kinetic equation.

The first anticommutator on its left-hand side is the velocity term of the Liouville operator. Diagonals of the Wigner function propagate with velocities of the respective mass eigenstates. The off-diagonals propagate with the center of momentum velocity. The latter is well approximated by the average velocity of the mass eigenstates in the relativistic limit.

Initial conditions for the evolution equation are constructed from the initial conditions for the neutrino wave function.

Is it compatible with the Heisenberg uncertainty principle

Solutions of the Schrödinger equation are automatically consistent with the Heisenberg uncertainty principle. Solutions of the evolution equation constructed from the neutrino wave function posses this property as well. For e.g. Gaussian initial conditions:

$$\varrho_{ij}(t,\mathbf{x},\mathbf{p}) \propto \exp\left(-\frac{(\mathbf{p}-\mathbf{p}_w)^2}{2\sigma_p^2}\right) \exp\left(-\frac{(\mathbf{v}_{ij}(\mathbf{p})t-\mathbf{x})^2}{2\sigma_x^2}\right)$$

where σ_p and σ_x are the momentum and coordinate uncertainties. Unlike the Schrödinger equation, the evolution equation does not enforce the uncertainty principle and admits also classical solutions. On the other hand, it consistently evolves solutions satisfying the uncertainty principle if supplemented by respective initial conditions.

Because forms of the evolution and kinetic equations match, they produce identical solutions given identical initial conditions. Single-particle states accounting for the uncertainty principle and leading to the same initial conditions for the matrix of densities are constructed as:

$$|\psi\rangle = \sum_i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \psi_i(0,\mathbf{p})\hat{a}_i^\dagger(0,\mathbf{p})|0\rangle$$

Does it account for the effect of wave packet separation

The collision term that describes neutrino interactions in the detector contains integration over the neutrino momentum. Because momentum resolution of realistic detectors is far below the neutrino momentum uncertainty, they essentially integrate over all momentum components of the neutrino wave packet. For e.g. Gaussian initial conditions this results in the density matrix of the form:

$$\rho_{ij}(t,\mathbf{x}) \approx \varrho_{ij}(t,\mathbf{x},\mathbf{p}_w)\cdot \exp\left(-\frac{\Delta\mathbf{v}_{ij}^2(\mathbf{p}_w)t^2}{8\sigma_x^2}\right)$$

As is evident from this expression, the uncertainty principle is crucial to the onset of kinematical decoherence, which manifests itself through suppression of the off-diagonal (in the propagation basis) components of the density matrix. This picture is equivalent to the one of the wave packet separation, where the suppression of the off-diagonals emerges from growing spatial separation of wave packets of the neutrino mass eigenstates. This effect is crucial for explaining flavor composition of the day-time and night-time solar neutrinos.

Is it compatible with the Pauli exclusion principle

Neutrinos are produced copiously in the Sun. Hence, it is appropriate to describe them as an N-particle system. The N-particle wave function is antisymmetric under permutation of any two particles, that reflects the Pauli exclusion principle. The antisymmetry of the wave-function can be used to define a reduced single-particle Wigner function:

$$\varrho_{ij}(t,\mathbf{x},\mathbf{p}) = N \sum_{n_2\ldots n_N} \int d^3\mathbf{x}_2\ldots d^3\mathbf{x}_N \int \frac{d^3\mathbf{p}_2}{(2\pi)^3}\cdots\frac{d^3\mathbf{p}_N}{(2\pi)^3}$$
$$\times\, \varrho_{in_2\ldots n_N\, jn_2\ldots n_N}(t,\mathbf{x},\mathbf{x}_2,\ldots,\mathbf{x}_N,\mathbf{p},\mathbf{p}_2,\ldots,\mathbf{p}_N)$$

It satisfies the same evolution equation as the single-particle Wigner function and the matrix of densities:

$$\partial_t \varrho + \tfrac{1}{2}\{\partial_{\mathbf{x}}\varrho, \partial_{\mathbf{p}}\mathsf{H}\} - \tfrac{1}{2}\{\partial_{\mathbf{p}}\varrho, \partial_{\mathbf{x}}\mathsf{H}\} = -i\,[\mathsf{H},\varrho]$$

N-particle states accounting for the Pauli principle and leading to the same initial conditions for the matrix of densities are constructed as:

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \sum_{i_1\ldots i_N} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{p}_N}{(2\pi)^3}$$
$$\times\, \psi_{i_1\ldots i_N}(0,\mathbf{p}_1\ldots\mathbf{p}_N)\,\hat{a}_{i_1}^\dagger(0,\mathbf{p}_1)\ldots\hat{a}_{i_N}^\dagger(0,\mathbf{p}_N)|0\rangle$$