Five particle scattering at two loops in QCD

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in collaboration with: Brønnum-Hansen, Hartanto, Peraro, Chicherin, Gehrmann, Heinrich, Henn, Wasser; Zhang, Zoia

QCD@LHC, Buffalo NY, 18th July 2019
the NNLO frontier

new subtractions methods $\Rightarrow$ (almost) complete set of 2$\rightarrow$2 processes at NNLO!

~ 2000 - ~ 2010 -

NNLO QCD is becoming the new LHC standard for SM

<table>
<thead>
<tr>
<th>process</th>
<th>precision observables</th>
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</thead>
<tbody>
<tr>
<td>$pp \rightarrow 3j$</td>
<td>jet multiplicity ratios, $\alpha_s$ at high energies, 3-jet mass</td>
</tr>
<tr>
<td>$pp \rightarrow \gamma\gamma + j$</td>
<td>background to Higgs $p_T$, signal/background interference effects</td>
</tr>
<tr>
<td>$pp \rightarrow H + 2j$</td>
<td>Higgs $p_T$, Higgs coupling through vector boson fusion (VBF)</td>
</tr>
<tr>
<td>$pp \rightarrow V + 2j$</td>
<td>Vector boson $p_T$, $W^+/W^-$ ratios and multiplicity scaling</td>
</tr>
<tr>
<td>$pp \rightarrow VV + j$</td>
<td>backgrounds to $p_T$ spectra for new physics decaying via vector boson</td>
</tr>
</tbody>
</table>

e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\text{exp.}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$

e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\text{exp.}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$
outline

• challenges and latest results

• reconstructing analytic amplitudes using numerical evaluation over finite fields

• applications to some two loop QCD helicity amplitudes

• future outlook
planar amplitudes in QCD

<table>
<thead>
<tr>
<th></th>
<th>2 → 2</th>
<th>2 → 3</th>
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<tbody>
<tr>
<td></td>
<td>N = 4</td>
<td>QCD</td>
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<tr>
<td>one loop</td>
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<tr>
<td>integrand basis</td>
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<td>master integrals</td>
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<tr>
<td>master integrals</td>
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</table>

new basis of functions
reduction bottlenecks

large algebraic expressions

large systems of equations

numerical algebra and an analytic integral basis solved this issue at one-loop

Laporta-style integration-by-parts systems can easily run into millions for state-of-the-art applications

on-shell quantities can be extremely simple - when you find the right language!
e.g. unitarity (Bern, Dixon, Dunbar, Kosower), on-shell recursion (Britto, Cachazo, Feng, Witten)
five-point master integrals
using differential equations

planar

new! non-planar

compact analytic expressions
using ‘pentagon function’ basis

[Papadopoulos, Tommasini, Wever (2015)]
[Gehrmann, Henn, Lo Presti (2015)]
new! [Gehrmann, Henn, Lo Presti 1807.09812]

[Boehm et al. 1805.01873]
[Abreu et al. 1807.11522]
[Chicherin et al. 1809.06240]

[Chicherin et al. 1812.11057, 1812.11160, 1901.05932]
[Abreu et al. 1812.08941, 1901.08563]
five-point helicity amplitudes

combining of analytic integrals with analytic reduction to master integrals using finite field reconstruction methods

- planar five-gluon single-minus [SB et al. 1811.11699]
- planar five-gluon MHV [Abreu et al. 1812.04586]
- planar five-parton MHV [Abreu et al. 1904.00945]
- non-planar five-gluon $N=4$ [Abreu et al. 1812.08941, Chicherin et al. 1812.11057]
- non-planar five-gluon $N=8$ [Chicherin et al. 1901.05932, Abreu et al. 1901.08563]
- non-planar five-gluon all-plus [SB et al. 1905.03733]
- numerical evaluation for $W+2j$ [Hartanto et al. 1905.03733]

See Abreu’s talk later today
scattering amplitudes

\[ A^{(L)} = c_i(\{x\}) I_i(\{x\}) \]

algebraic part:
- extract rational functions using finite field arithmetic

analytic part:
- identify a suitable basis of integrals/special functions
why finite field arithmetic?

input variables $x_i$

algorithm

output $c_j(x_i)$
(rational function of input)

rational functions are reconstructed analytically from several numerical evaluations modulo a prime field

large intermediate expressions are avoided

applications in pQFT

combined with ‘dataflow’ graphs
Peraro (2018) and FINITEFLOW code
rational phase space

IBP reduction (Laporta)

numerators or cuts

‘reduced’ integrand

Laurent expansion of master integrals

subtract universal IR poles

independent coefficients of the finite remainders

e.g. momentum twistors [Hodges (2009)]

steps combined within numerical unitarity approach

e.g. analytic solutions to differential equations

numerical sampling with finite field arithmetic in FINITEFLOW

with finite field arithmetic in FINITEFLOW

sketch setup
analytic integrals and IR poles

\[ A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{ finite} \]

universal \( \epsilon \) pole structure

\[ MI_x = \sum_{y,z} \epsilon^y c_{xyz} m_{xyz}(f(W)) \]

monomials of \textbf{pentagon function} integrals. \( L_n \) etc. and one-fold integrals

\[ f_{3,4} - \frac{2}{3} \delta_{37,3} = \int_0^1 dt \partial_t \log W_{26}(t) \left( \begin{bmatrix} W_3 & W_2 \\ W_5 & W_{15} \end{bmatrix} (t) - \begin{bmatrix} W_5 & W_3 \\ W_7 & W_{12} \end{bmatrix} (t) - \zeta_2 \right) + \text{cyclic} \]
analytic integrals and IR poles

\[ A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{finite} \]

direct construction of finite remainder from (6d) tree-level amplitudes via **unitarity cuts, integrand reduction, IBP reductions** and reconstruction of rational coefficients of the **pentagon function basis**

- performance depends on the complexity (polynomial order) of the rational functions in the final answer
- rational parametrisation of external kinematics from momentum twistors
- extremely effective when cancellations are explicit
- numerical sampling is easily parallelised
Applications

Analytic helicity amplitudes for two-loop five-gluon scattering: the single-minus case

[SB, Brønnum-Hansen, Hartanto, Peraro]

Analytic form of the full two-loop five-gluon all-plus helicity amplitude

[SB, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia]

A numerical evaluation of planar two-loop helicity amplitudes for a W-boson plus four partons

[Hartanto, SB, Brønnum-Hansen, Peraro]
single-minus finite remainders

SB, Brønnum-Hansen, Hartanto, Peraro [1811.11699]

$$F^{(L),[i]}(1_g^{-}, 2_g^{+}, 3_g^{+}, 4_g^{+}, 5_g^{+}) = \frac{[25]^2}{[12][23][34][45][51]} \left( F^{(L),[i]}_{\text{sym}}(1, 2, 3, 4, 5) + F^{(L),[i]}_{\text{sym}}(1, 5, 4, 3, 2) \right)$$

leading order

$$F^{(1),[1]}_{\text{sym}}(1, 2, 3, 4, 5) = \frac{\text{tr}_+(2315)^2 \text{tr}_+(1243)}{3s_{25}^2s_{23}s_{34}s_{15}} - \frac{\text{tr}_+(2543)}{6s_{34}}$$

finite at one-loop:
compact expression [Bern Dixon Kosower (1993)]
single-minus finite remainders

- large cancellation between two-loop amplitude and universal poles
- only weight two functions remain

<table>
<thead>
<tr>
<th>$d_s - 2$ exponent</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>integrand coefficients ($\neq 0$)</td>
<td>4387</td>
<td>14565</td>
<td>4420</td>
</tr>
<tr>
<td>sample points</td>
<td>3</td>
<td>2214</td>
<td>22886</td>
</tr>
</tbody>
</table>

in this example - the run time is fast and additional optimisation was unnecessary

coefficients are found to be even simpler than this sampling suggests…

one-loop squared - very fast
single-minus finite remainders

\[ F_{\text{sym}}^{(2),[1]}(1, 2, 3, 4, 5) = c_{51}^{(2)} F_{\text{box}}^{(2)}(s_{23}, s_{34}, s_{15}) + c_{51}^{(1)} F_{\text{box}}^{(1)}(s_{23}, s_{34}, s_{15}) + c_{51}^{(0)} F_{\text{box}}^{(0)}(s_{23}, s_{34}, s_{15}) + c_{34}^{(2)} F_{\text{box}}^{(2)}(s_{12}, s_{15}, s_{34}) + c_{34}^{(1)} F_{\text{box}}^{(1)}(s_{12}, s_{15}, s_{34}) + c_{34}^{(0)} F_{\text{box}}^{(0)}(s_{12}, s_{15}, s_{34}) + c_{45} F_{\text{box}}^{(0)}(s_{12}, s_{23}, s_{45}) + c_{34;51} \hat{L}_1(s_{34}, s_{15}) + c_{51;23} \hat{L}_1(s_{15}, s_{23}) + c_{\text{rat}}, \]

compact 2-loop expressions - free of spurious singularities using one-loop-type (weight 2) function basis (c.f. Bern, Dixon, Kosower (1993, 1994))

\[ L_k(s, t) = \frac{\log(t/s)}{(s - t)^k}, \]

\[ \hat{L}_0(s, t) = 0, \]

\[ \hat{L}_1(s, t) = L_1(s, t), \]

\[ \hat{L}_2(s, t) = L_2(s, t) + \frac{1}{2(s - t)} \left( \frac{1}{s} + \frac{1}{t} \right), \]

\[ \hat{L}_3(s, t) = L_3(s, t) + \frac{1}{2(s - t)^2} \left( \frac{1}{s} + \frac{1}{t} \right). \]

\[ F_{\text{box}}^{(-1)}(s, t, m^2) = L_2 \left( 1 - \frac{s}{m^2} \right) + L_2 \left( 1 - \frac{t}{m^2} \right) + \log \left( \frac{s}{m^2} \right) + \log \left( \frac{t}{m^2} \right) - \frac{\pi^2}{6}, \quad (4.7a) \]

\[ F_{\text{box}}^{(0)}(s, t, m^2) = \frac{1}{u(s, t, m^2)} F_{\text{box}}^{(-1)}(s, t, m^2), \quad (4.7b) \]

\[ F_{\text{box}}^{(1)}(s, t, m^2) = \frac{1}{u(s, t, m^2)} \left[ F_{\text{box}}^{(0)}(s, t, m^2) + \hat{L}_1(s, m^2) + \hat{L}_1(m^2, t) \right], \quad (4.7c) \]

\[ F_{\text{box}}^{(2)}(s, t, m^2) = \frac{1}{u(s, t, m^2)} \left[ F_{\text{box}}^{(1)}(s, t, m^2) + \frac{s - m^2}{2t} \hat{L}_2(s, m^2) + \frac{m^2 - t}{2s} \hat{L}_2(m^2, t) - \left( \frac{1}{s} + \frac{1}{t} \right) \frac{1}{4m^2} \right], \quad (4.7d) \]

\[ F_{\text{box}}^{(3)}(s, t, m^2) = \frac{1}{u(s, t, m^2)} \left[ F_{\text{box}}^{(2)}(s, t, m^2) - \frac{(s - m^2)^2}{6t^2} \hat{L}_3(s, m^2) - \frac{(m^2 - t)^2}{6s^2} \hat{L}_3(m^2, t) - \left( \frac{1}{s} + \frac{1}{t} \right) \frac{1}{6m^2} \right], \quad (4.7e) \]

simple rational coefficients using \( s_{ij}, \text{tr+} \)
Numerical performance
interpret with care please! preliminary study!

How do analytic amplitudes perform for more general helicities?

Finite remainders for 4q1g
independent computation of [1904.00945]
evaluation in physical region
~10^5 points
Pentagon function basis
Evaluation ~30s per point
Plot of coefficient accuracy

\[
\log_{10} \left( \frac{F - F_{scaled}}{\frac{1}{2}(F + F_{scaled})} \right)
\]

Relatively stable overall - a few outlying points for spurious singularities
analytic form of the full two-loop five-gluon all-plus helicity amplitude

SB, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia [1905.03733]

compact analytic integrand
[SB, Mogull, Ochirov, O'Connell (2015)]

\[ A^{\gamma \alpha}(1^+, 2^+, 3^+, 4^+, 5^+) = \sum_{ \text{term} } [ C(\text{term}) (\frac{1}{2} A(\text{term}) + \frac{1}{2} A(\text{term}) ) + \frac{1}{2} A(\text{term}) ] \]

complete evaluation of non-planar integrals in the physical region

analytic evaluation of boundary terms

numerically verified with SecDec

explicit verification of collinear limits

IBP reduction (rank 5) with FINITEFLOW
all-plus finite remainders

- cancellation of all weight 1, 3 and 4 functions
- weight 2 described by box function

\[
A_1^{(1,0)} = \frac{\kappa}{5} \sum_{S_{T_1}} \left[ \frac{[24]^2}{\langle 13 \rangle \langle 35 \rangle \langle 51 \rangle} + 2 \frac{[23]^2}{\langle 14 \rangle \langle 45 \rangle \langle 51 \rangle} \right] \quad \kappa = (D_s - 2)/6
\]

\[
\mathcal{H}_1^{(2,0)} = \sum_{S_{T_1}} \left\{ -\kappa \frac{[45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} I_{123;45} + \kappa^2 \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \left[ 5 s_{12}s_{23} + s_{12}s_{34} + \frac{\text{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},
\]

\[
\mathcal{H}_{13}^{(2,1)} = \sum_{S_{T_{13}}} \left\{ \kappa \frac{[15]^2}{\langle 23 \rangle \langle 34 \rangle \langle 42 \rangle} \left[ I_{234;15} + I_{243;15} - I_{324;15} - 4 I_{345;12} - 4 I_{354;12} - 4 I_{435;12} \right] - 6 \kappa^2 \left[ \frac{s_{23} \text{tr}_- (1345)}{s_{34} \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} - \frac{3}{2} \frac{[12]^2}{\langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \right] \right\},
\]

conformally invariant building block
numerical evaluation of helicity amplitudes for a W boson and four partons

Hartanto, SB, Brønnum-Hansen, Peraro [1906.11862]

First benchmark numerical evaluation of the planar contributions in Euclidean region

```
finiteflow
```

- numerical phase space
- Feynman diagrams
- 'reduced' integrand
- IBP reduction (Laporta)
- amplitude in CDR

independent maximal cuts
Figure 2: Master integrals for leading colour $W^+4$ parton scattering at two loops with five external legs. $(a, b)$ represents the number of crossing of external legs $(a)$ and the number master integral for a given topology $(b)$. A massless (massive) external leg is indicated by a single (double) line external leg. The $\ast$ sign identifies master integral topologies that are not known analytically.

\[ N(\varepsilon) = e^E / (1 - e^\varepsilon) \]

where:

$$ H^{(2)}_{q\bar{q}q\bar{q}}(\varepsilon) = \frac{1}{16} \varepsilon^3 \left( \frac{\pi}{2} \right)^2 (\varepsilon) $$

$$ H^{(2)}_{qgg\bar{q}}(\varepsilon) = \frac{1}{16} \varepsilon^3 \left( \frac{\pi}{2} \right)^2 (\varepsilon) \left( C_F + C_A \right) $$

Note that the $H^{(2)}_{q\bar{q}q\bar{q}}(\varepsilon)$ and $H^{(2)}_{qgg\bar{q}}(\varepsilon)$ functions are given in the leading colour limit.

The function coefficients and anomalous dimensions without the contribution from closed fermion loops $N_f$ are:

$$ N_f = \frac{11}{3} C_A, \quad (3.12) $$

$$ g_0 = \frac{11}{3} C_A, \quad (3.14) $$

202 master integrals (including permutations) after IBP reduction.
evaluating the integrals

<table>
<thead>
<tr>
<th>$T$</th>
<th>$N(k_i, p_i, \mu_{ij})$</th>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 2, 1</td>
<td>$\Omega^{g}_{2:4</td>
<td>56} \mu_{11}$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>$\Omega^{s}_{2:4</td>
<td>56} \mu_{11}$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>$\Psi_{1:1</td>
<td>2</td>
<td>3} \Omega^{g}_{2:4</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>$\Phi_{2:2</td>
<td>3} \mu_{11}$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>$\Phi^{s}_{2:2</td>
<td>3} \mu_{11}$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>$\Psi_{1:4</td>
<td>56</td>
<td>1} \Phi_{2:2</td>
</tr>
</tbody>
</table>

$\Psi_{i;a|b|c} = \text{tr} - (a(k_i - p_a)(k_i - p_{ab})c)$,
$\Phi_{i;a|b} = \langle a | k_i | b \rangle$,
$\Omega_{i;a|b} = \langle a | k_i | p_b | a \rangle$.

many integrals known analytically (polylogs)

[Papadopoulos, Tommasini, Wever (2014-2015)]
[Gehrmann, von Mantueffel, Tancredi (2015)]
[Caola, Henn, Melnikov, Smirnov (2014)]
[Gehrmann, Remiddi (2001)]

stabilise sector decomposition (FIESTA/pySecDec) numerics using local numerators

choose a set of master integrals to make sure divergence structure of numerical sector is simple
### numerical results

<table>
<thead>
<tr>
<th>$qgq^l\bar{\nu}l$</th>
<th>$\epsilon^{-4}$</th>
<th>$\epsilon^{-3}$</th>
<th>$\epsilon^{-2}$</th>
<th>$\epsilon^{-1}$</th>
<th>$\epsilon^{0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{A}^{(2)}_{-++++-}$</td>
<td>4.50000</td>
<td>-3.63577(3)</td>
<td>-277.2182(7)</td>
<td>-344.56(1)</td>
<td>2051.1(2)</td>
</tr>
<tr>
<td>$P^{(2)}_{-++++-}$</td>
<td>4.5</td>
<td>-3.63576</td>
<td>-277.2186</td>
<td>-344.569(6)</td>
<td>—</td>
</tr>
<tr>
<td>$\hat{A}^{(2)}_{-++++-}$</td>
<td>4.50000</td>
<td>-3.63581(9)</td>
<td>-13.6826(2)</td>
<td>6.143(5)</td>
<td>66.21(7)</td>
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<tr>
<td>$P^{(2)}_{-++++-}$</td>
<td>4.5</td>
<td>-3.63576</td>
<td>-13.6824</td>
<td>6.145(1)</td>
<td>—</td>
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<tr>
<td>$\hat{A}^{(2)}_{-++++-}$</td>
<td>4.50000</td>
<td>-3.63579(5)</td>
<td>-18.79219(7)</td>
<td>-6.633(6)</td>
<td>79.02(4)</td>
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<tr>
<td>$P^{(2)}_{-++++-}$</td>
<td>4.5</td>
<td>-3.63576</td>
<td>-18.79212</td>
<td>-6.6303(5)</td>
<td>—</td>
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</table>

<table>
<thead>
<tr>
<th>$qQQq^l\bar{\nu}l$</th>
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<th>$\epsilon^{-3}$</th>
<th>$\epsilon^{-2}$</th>
<th>$\epsilon^{-1}$</th>
<th>$\epsilon^{0}$</th>
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</thead>
<tbody>
<tr>
<td>$\hat{A}^{(2)}_{-++++-}$</td>
<td>2.00000</td>
<td>-7.16949(9)</td>
<td>-9.9055(2)</td>
<td>39.922(6)</td>
<td>154.79(7)</td>
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<tr>
<td>$P^{(2)}_{-++++-}$</td>
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<td>-9.9054</td>
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<td>—</td>
</tr>
<tr>
<td>$\hat{A}^{(2)}_{-++++-}$</td>
<td>2.00000</td>
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<td>-12.9371(1)</td>
<td>41.432(8)</td>
<td>189.53(6)</td>
</tr>
<tr>
<td>$P^{(2)}_{-++++-}$</td>
<td>2</td>
<td>-7.16944</td>
<td>-12.9370</td>
<td>41.4353(6)</td>
<td>—</td>
</tr>
</tbody>
</table>

Good agreement with universal poles

First step towards analytic reconstruction
outlook

• numerical algorithms and finite field arithmetic can provide powerful tools for analytic amplitude computations

• traditional bottlenecks can be avoided by direct reconstruction of on-shell physical quantities (e.g. finite remainders)

• surge of progress on two-loop amplitudes! major challenge to combine amplitudes into physical cross sections at NNLO