Five particle scattering at two loops in QCD

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in collaboration with: Brønnum-Hansen, Hartanto, Peraro, Chicherin, Gehrmann, Heinrich, Henn, Wasser, Zhang, Zoia

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Science & Technology Facilities Council

IP3

European Research Council
Established by the European Commission

the NNLO frontier

new subtractions methods ~ 2000 - $\Rightarrow$ (almost) complete set of 2→2 processes at NNLO! ~ 2010 -

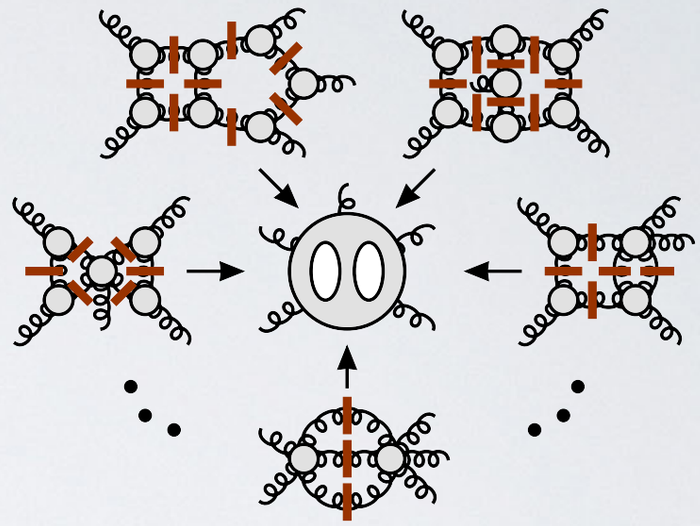
NNLO QCD is becoming the new LHC standard for SM

process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

example: 3j/2j ratio at the LHC can probe of the running of α_s in a new energy regime

e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\text{exp.}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$

outline



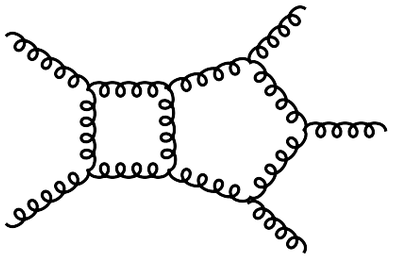
- challenges and latest results

- reconstructing analytic amplitudes using
  numerical evaluation over finite fields

- applications to some two loop QCD helicity amplitudes

- future outlook

planar amplitudes in QCD



planar gluon scattering

		$2 \to 2$		$2 \to 3$	
		$\mathcal{N}=4$	QCD	$\mathcal{N}=4$	QCD
one loop	integrand basis	1	65	5	175
	master integrals	1	2	1	2
two loops	integrand basis	2	15360	15	55580
	master integrals	1	7	3	61

new basis of functions

reduction bottlenecks

large algebraic expressions

numerical algebra and an analytic integral basis solved this issue at one-loop

large systems of equations

Laporta-style integration-by-parts systems can easily run into millions for state-of-the-art applications

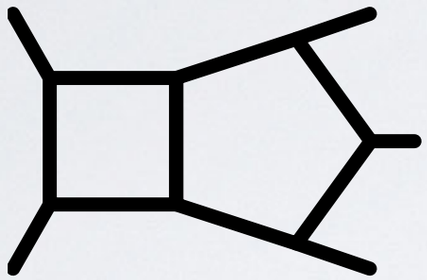
on-shell quantities can be extremely simple - when you find the right language!

e.g. unitarity (Bern, Dixon, Dunbar, Kosower), on-shell recursion (Britto, Cachazo, Feng, Witten)

five-point master integrals

using differential equations

planar



[Papadopoulos, Tommasini, Wever (2015)]

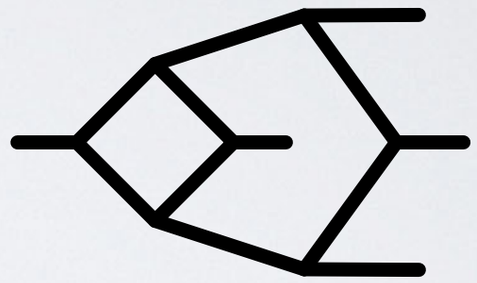
[Gehrmann, Henn, Lo Presti (2015)]

new! [Gehrmann, Henn, Lo Presti 1807.09812]

↓

compact analytic expressions
using 'pentagon function' basis

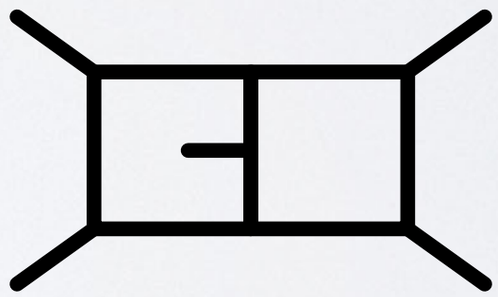
new! non-planar



[Boehm et al. 1805.01873]

[Abreu et al.1807.11522]

[Chicherin et al.1809.06240]



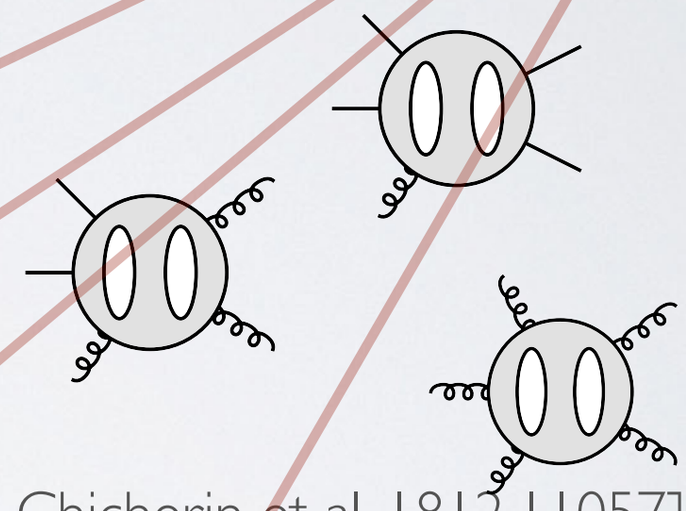
[Chicherin et al. 1812.11057, 1812.11160, 1901.05932]

[Abreu et al. 1812.08941,1901.08563]

five-point helicity amplitudes

combining of analytic integrals with analytic reduction to master integrals using finite field reconstruction methods

See Abreu's talk later today



★ planar five-gluon single-minus [SB et al. 1811.11699]

planar five-gluon MHV [Abreu et al. 1812.04586]

planar five-parton MHV [Abreu et al. 1904.00945]

non-planar five-gluon N=4 [Abreu et al. 1812.08941, Chicherin et al. 1812.11057]

non-planar five-gluon N=8 [Chicherin et al. 1901.05932, Abreu et al. 1901.08563]

★ non-planar five-gluon all-plus [SB et al. 1905.03733]

★ **focus for this talk**

★ *numerical* evaluation for W+2j [Hartanto et al. 1905.03733]

scattering amplitudes

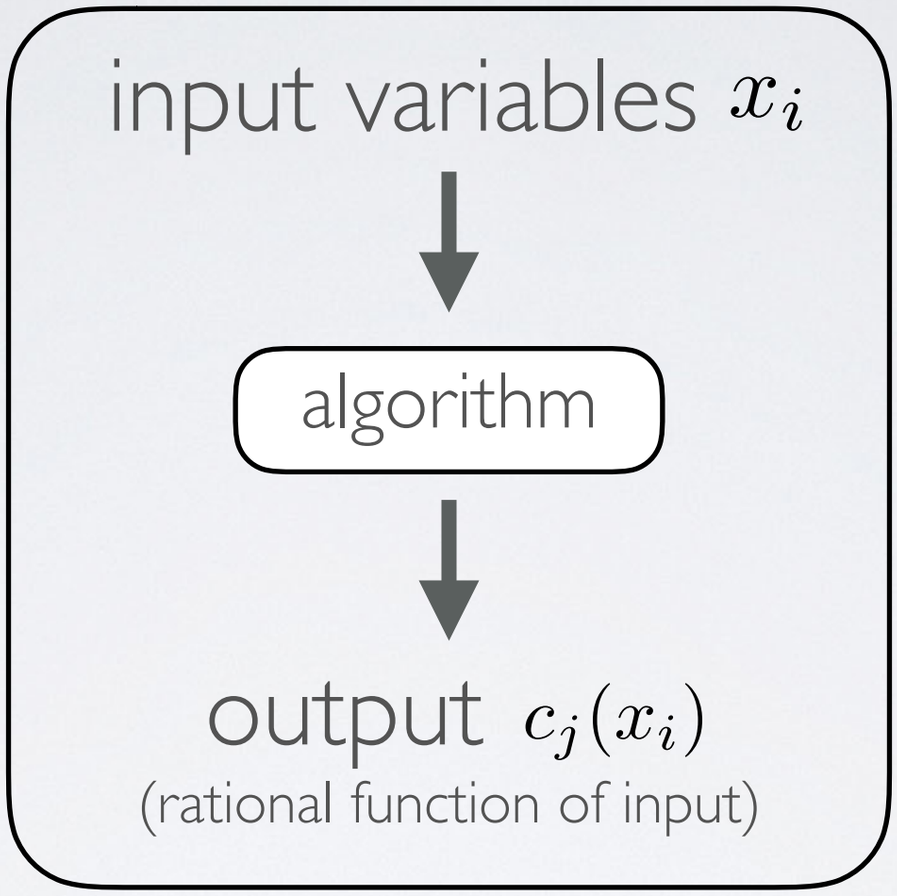
$$A^{(L)} = c_i(\{x\})\, I_i(\{x\})$$

algebraic part:
extract rational functions
using finite field arithmetic

analytic part:
identify a suitable basis of
integrals/special functions

why finite field arithmetic?

rational functions are reconstructed analytically from several numerical evaluations modulo a prime field



large intermediate expressions are avoided

applications in pQFT

von Mantueffel, Schabinger (2015), Peraro (2016, 2018)

combined with 'dataflow' graphs

Peraro (2018) and FINITEFLOW code

sketch setup

numerical sampling with finite field arithmetic in FINITEFLOW

rational phase space

e.g. momentum twistors [Hodges (2009)]

↓

numerators or cuts

↓

'reduced' integrand

IBP reduction (Laporta)

} steps combined within numerical unitarity approach [Gluza, Kajda, Kosower (2011)][Ita (2015)][Larsen Zhang (2015)] [Kosower (2018)]

↓

Laurent expansion of master integrals

e.g. analytic solutions to differential equations

↓

subtract universal IR poles

↓

independent coefficients of the finite remainders

analytic integrals and IR poles

$$A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{finite}$$

universal **ε** pole structure
e.g. [Catani (1998)], [Sterman, Tejeda-Yeomans (2003)]
[Becher,Neubert (2009)], [Gardi, Magnea (2009)]

higher order terms in **ε** needed

$$MI_x = \sum_{y,z} \epsilon^y c_{xyz} m_{xyz}(f(W))$$

alphabet of 31 letters

monomials of **pentagon function** integrals. Li_n etc. and one-fold integrals

[Gehrmann, Henn, Lo Presti (2018)]

$$I^{[6-2\epsilon]}\left(\quad\right) \propto f_{3,4}$$

$$f_{3,4} - \frac{2}{3} d_{37,3} = \int_0^1 dt\, \partial_t \log W_{26}(t) \left(\left[\frac{W_3}{W_5}, \frac{W_2}{W_{15}}\right](t) - \left[\frac{W_5}{W_2}, \frac{W_3}{W_{12}}\right](t) - \zeta_2 \right) + \text{cyclic}$$

analytic integrals and IR poles

$$A^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + \text{finite}$$

direct construction of finite remainder from (6d) tree-level amplitudes via **unitarity cuts, integrand reduction, IBP reductions** and reconstruction of rational coefficients of the **pentagon function basis**

- performance depends on the complexity (polynomial order) of the rational functions in the final answer
- rational parametrisation of external kinematics from momentum twistors
- extremely effective when cancellations are explicit
- numerical sampling is easily parallelised

Applications

Analytic helicity amplitudes for two-loop five-gluon scattering: the single-minus case

[SB, Brønnum-Hansen, Hartanto, Peraro]

Analytic form of the full two-loop five-gluon all-plus helicity amplitude

[SB, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia]

A numerical evaluation of planar two-loop helicity amplitudes for a W-boson plus four partons

[Hartanto, SB, Brønnum-Hansen, Peraro]

single-minus finite remainders

SB, Brønnum-Hansen, Hartanto, Peraro [1811.11699]

exponent of d_s-2
where $d_s = g^\mu{}_\mu$

$$F^{(L),[i]}\left(1_g^-, 2_g^+, 3_g^+, 4_g^+, 5_g^+\right) = \frac{[25]^2}{[12]\langle 23\rangle\langle 34\rangle\langle 45\rangle[51]}\left(F^{(L),[i]}_{\rm sym}(1,2,3,4,5) + F^{(L),[i]}_{\rm sym}(1,5,4,3,2)\right)$$

leading order

$$F^{(1),[1]}_{\rm sym}(1,2,3,4,5) = \frac{{\rm tr}_+(2315)^2\,{\rm tr}_+(1243)}{3s_{25}^2 s_{23}s_{34}s_{15}} - \frac{{\rm tr}_+(2543)}{6s_{34}}$$

finite at one-loop:
compact expression [Bern Dixon Kosower (1993)]

single-minus finite remainders

- large cancellation between two-loop amplitude and universal poles
- only weight two functions remain

$d_s - 2$ exponent	0	1	2
integrand coefficients ($\neq 0$)	4387	14565	4420
sample points	3	2214	22886

one-loop squared
- very fast

in this example - the run time is fast and additional optimisation was unnecessary

coefficients are found to be even simpler than this sampling suggests…

single-minus finite remainders

$$F^{(2),[1]}_{\rm sym}(1,2,3,4,5) = c^{(2)}_{51} F^{(2)}_{\rm box}(s_{23},s_{34},s_{15}) + c^{(1)}_{51} F^{(1)}_{\rm box}(s_{23},s_{34},s_{15}) + c^{(0)}_{51} F^{(0)}_{\rm box}(s_{23},s_{34},s_{15})$$
$$+ c^{(2)}_{34} F^{(2)}_{\rm box}(s_{12},s_{15},s_{34}) + c^{(1)}_{34} F^{(1)}_{\rm box}(s_{12},s_{15},s_{34}) + c^{(0)}_{34} F^{(0)}_{\rm box}(s_{12},s_{15},s_{34})$$
$$+ c_{45} F^{(0)}_{\rm box}(s_{12},s_{23},s_{45}) + c_{34;51}\hat{L}_1(s_{34},s_{15}) + c_{51;23}\hat{L}_1(s_{15},s_{23}) + c_{\rm rat},$$

compact 2-loop expressions - free of spurious singularities using one-loop-type (weight 2) function basis (c.f. Bern, Dixon, Kosower (1993,1994))

$$L_k(s,t) = \frac{\log(t/s)}{(s-t)^k},$$

$$\hat{L}_0(s,t) = L_0(s,t),$$
$$\hat{L}_1(s,t) = L_1(s,t),$$
$$\hat{L}_2(s,t) = L_2(s,t) + \frac{1}{2(s-t)}\left(\frac{1}{s}+\frac{1}{t}\right),$$
$$\hat{L}_3(s,t) = L_3(s,t) + \frac{1}{2(s-t)^2}\left(\frac{1}{s}+\frac{1}{t}\right).$$

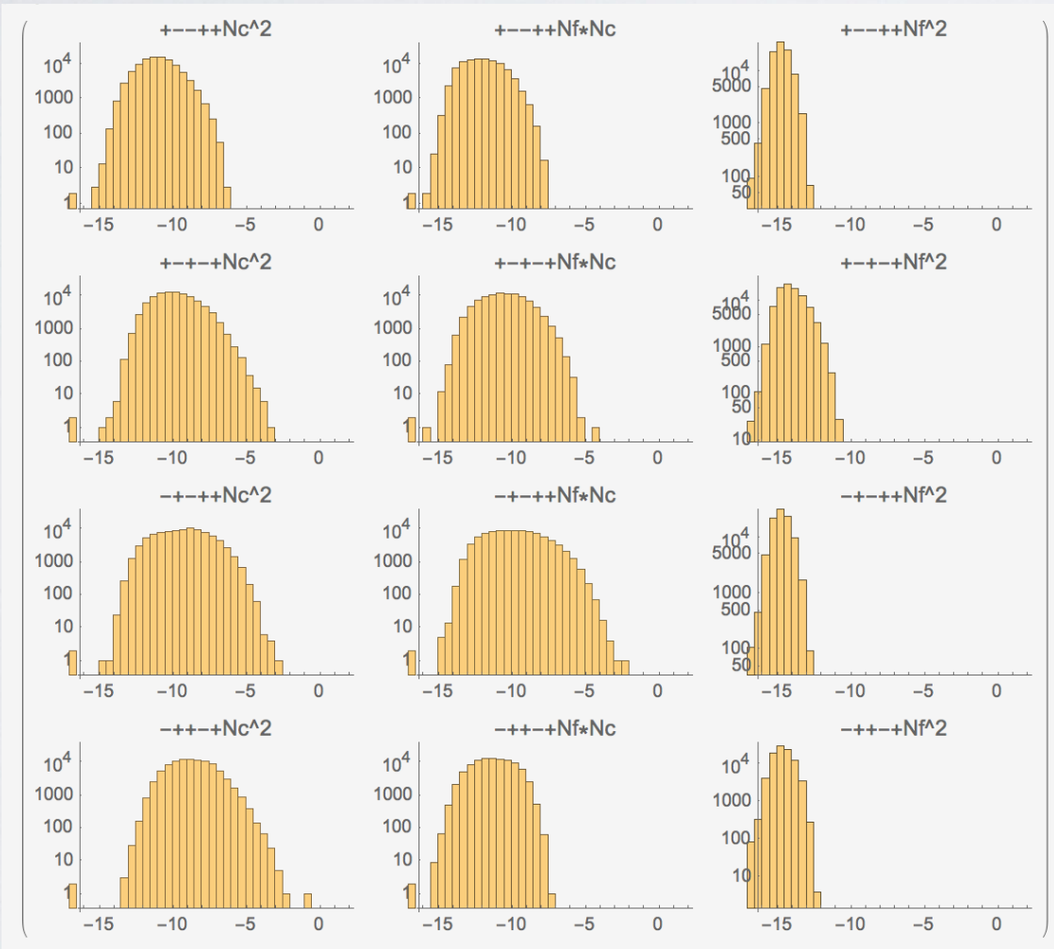
simple rational coefficients using s_{ij}, tr_+

$$F^{(-1)}_{\rm box}(s,t,m^2) = {\rm Li}_2\left(1-\frac{s}{m^2}\right) + {\rm Li}_2\left(1-\frac{t}{m^2}\right) + \log\left(\frac{s}{m^2}\right)\log\left(\frac{t}{m^2}\right) - \frac{\pi^2}{6}, \quad (4.7a)$$
$$F^{(0)}_{\rm box}(s,t,m^2) = \frac{1}{u(s,t,m^2)} F^{(-1)}_{\rm box}(s,t,m^2), \quad (4.7b)$$
$$F^{(1)}_{\rm box}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(0)}_{\rm box}(s,t,m^2) + \hat{L}_1(s,m^2) + \hat{L}_1(m^2,t)\right], \quad (4.7c)$$
$$F^{(2)}_{\rm box}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(1)}_{\rm box}(s,t,m^2) + \frac{s-m^2}{2t}\hat{L}_2(s,m^2) + \frac{m^2-t}{2s}\hat{L}_2(m^2,t) - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{4m^2}\right], \quad (4.7d)$$
$$F^{(3)}_{\rm box}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(2)}_{\rm box}(s,t,m^2) - \frac{(s-m^2)^2}{6t^2}\hat{L}_3(s,m^2) - \frac{(m^2-t)^2}{6s^2}\hat{L}_3(m^2,t) - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{6m^4}\right], \quad (4.7e)$$

numerical performance

interpret with care please! preliminary study!

how do analytic amplitudes perform for more general helicities?



finite remainders for 4q1g

independent computation of [1904.00945]

evaluation in physical region

~10^5 points

pentagon function basis

evaluation ~30s per point

plot of coefficient accuracy

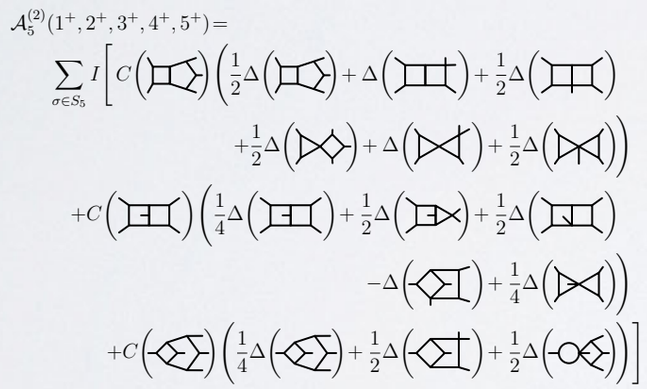
$$\log_{10}\left(\frac{F - F^{\text{scaled}}}{\frac{1}{2}(F + F^{\text{scaled}})}\right)$$

relatively stable overall - a few outlying points for spurious singularities

analytic form of the full two-loop five-gluon all-plus helicity amplitude

SB, Chicherin, Gehrmann, Heinrich, Henn,
Peraro, Wasser, Zhang, Zoia [1905.03733]

compact analytic integrand
[SB, Mogull, Ochirov, O'Connell (2015)]



explicit verification of collinear limits

complete evaluation of non-planar integrals in the physical region

analytic evaluation of boundary terms

numerically verified with SecDec

IBP reduction (rank 5) with FINITEFLOW

all-plus finite remainders

- cancellation of all weight 1,3 and 4 functions
- weight 2 described by box function

$$A_1^{(1,0)} = \frac{\kappa}{5} \sum_{S_{\tau_1}} \left[\frac{[24]^2}{\langle 13\rangle\langle 35\rangle\langle 51\rangle} + 2\,\frac{[23]^2}{\langle 14\rangle\langle 45\rangle\langle 51\rangle} \right] \qquad \kappa = (D_s - 2)/6$$

$$\mathcal{H}_1^{(2,0)} = \sum_{S_{\tau_1}} \left\{ -\kappa\,\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\, I_{123;45} + \kappa^2\,\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},$$

$$\mathcal{H}_{13}^{(2,1)} = \sum_{S_{\tau_{13}}} \left\{ \kappa\,\frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4\, I_{345;12} - 4\, I_{354;12} - 4\, I_{435;12} \right] \right.$$
$$\left. - 6\,\kappa^2 \left[\frac{s_{23}\,\mathrm{tr}_-(1345)}{s_{34}\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - \frac{3}{2}\,\frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle} \right] \right\},$$

conformally invariant building block

numerical evaluation of helicity amplitudes for a W boson and four partons

Hartanto, SB, Brønnum-Hansen, Peraro [1906.11862]

First benchmark numerical evaluation of the planar contributions in Euclidean region

FINITEFLOW

numerical phase space

↓

Feynman diagrams

↓

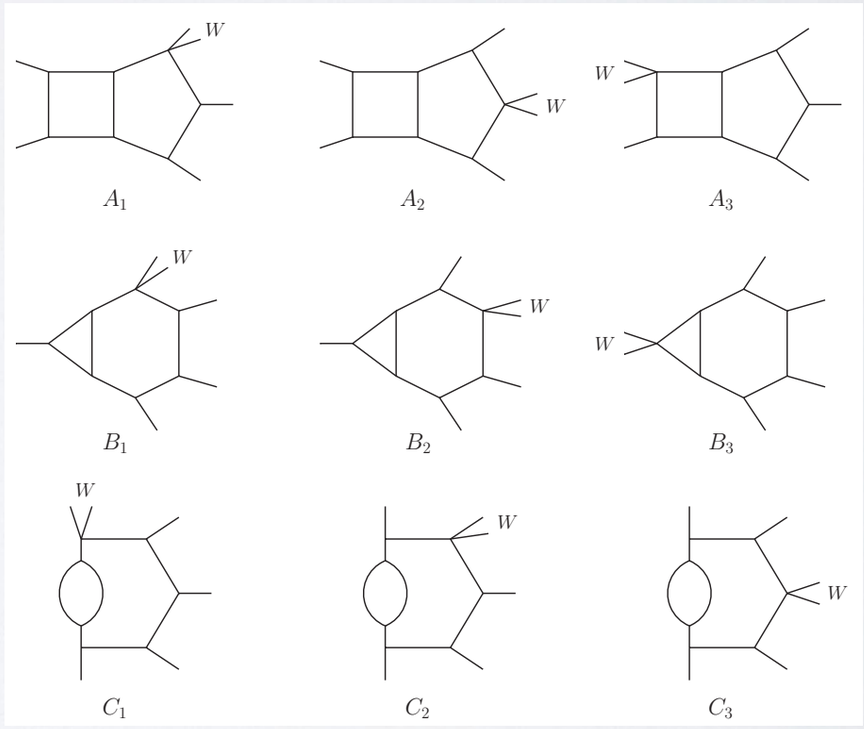
'reduced' integrand

↓

IBP reduction (Laporta)

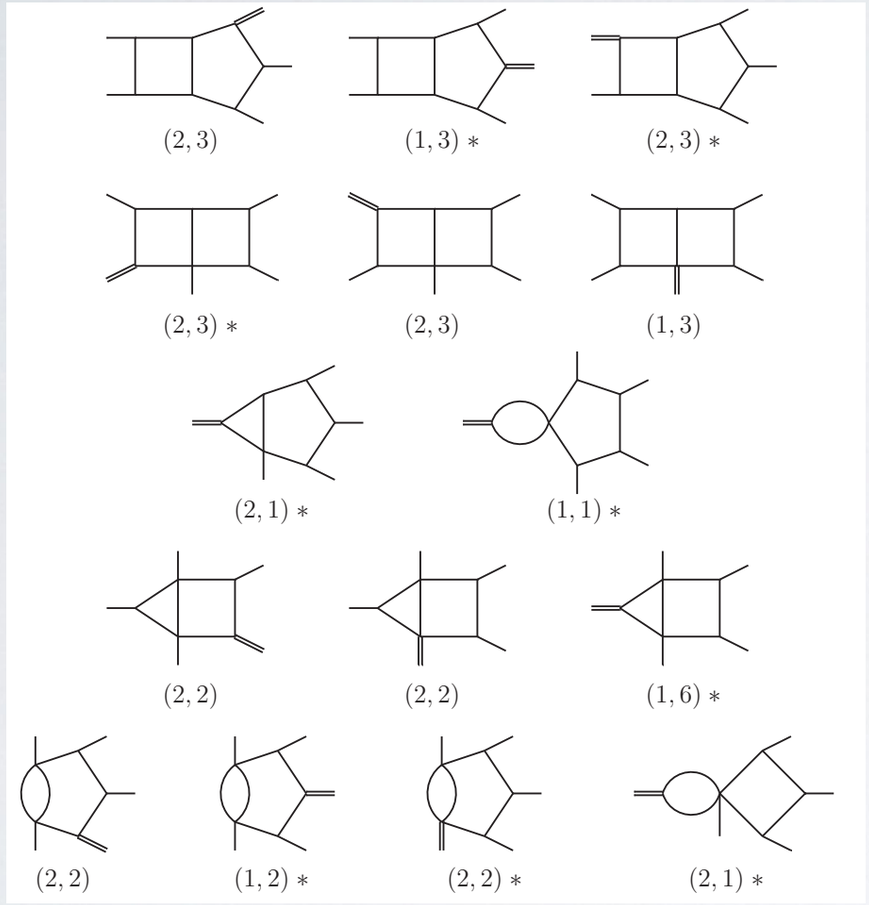
↓

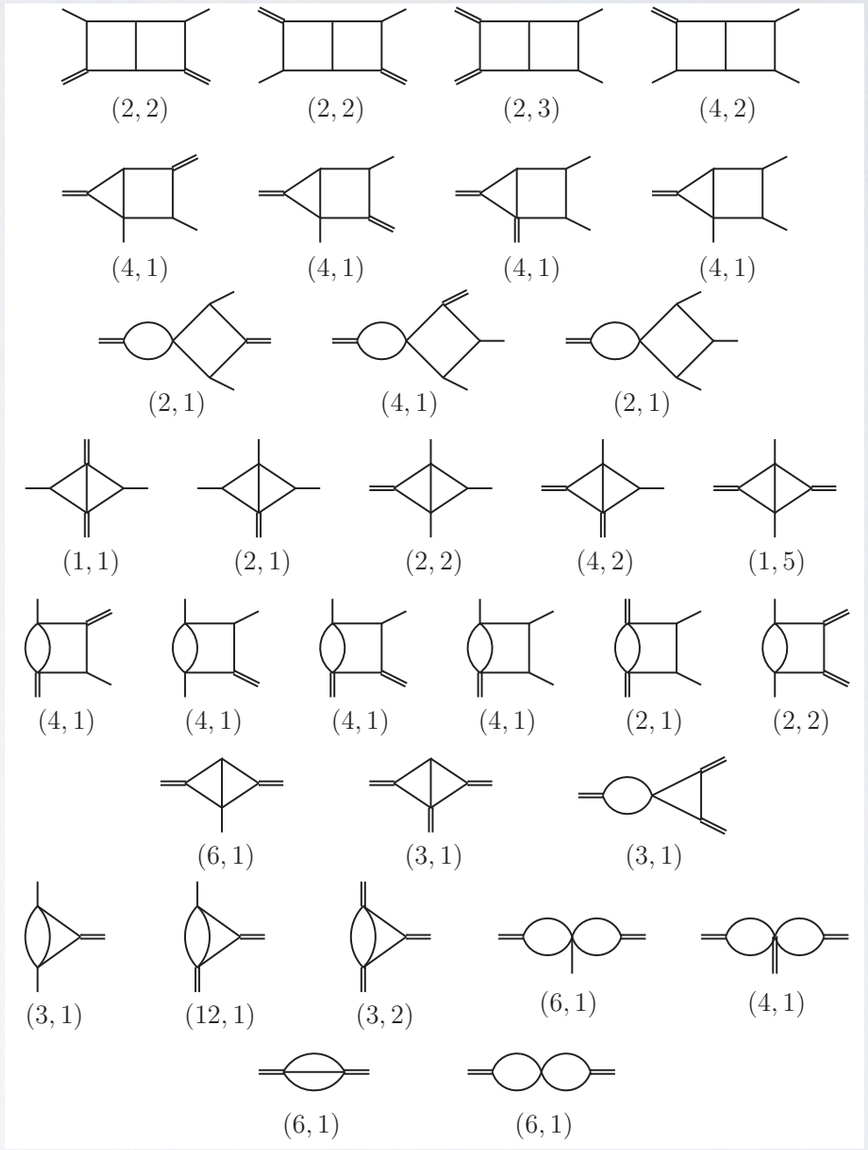
amplitude in CDR



independent maximal cuts

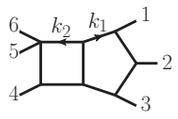
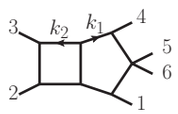
master integral basis





202 master integrals (including permutations) after IBP reduction

evaluating the integrals

T	$N(k_i, p_i, \mu_{ij})$	δ_1	δ_2
	$\Omega_{2;4\|56}\ \mu_{11}$	ϵ	1
	$\Omega^*_{2;4\|56}\ \mu_{11}$	ϵ	1
	$\Psi_{1;1\|2\|3}\ \Omega_{2;4\|56}$	1	1
	$\Phi_{2;2\|3}\ \mu_{11}$	ϵ	1
	$\Phi^*_{2;2\|3}\ \mu_{11}$	ϵ	1
	$\Psi_{1;4\|56\|1}\ \Phi_{2;2\|3}$	1	1

$$\Psi_{i;a|b|c} = \mathrm{tr}_-(a(k_i - p_a)(k_i - p_{ab})c),$$
$$\Phi_{i;a|b} = \langle a|k_i|b],$$
$$\Omega_{i;a|b} = \langle a|k_i|p_b|a\rangle.$$

many integrals known analytically (polylogs)

[Papadopoulos, Tommasini, Wever (2014-2015)]
[Gehrmann, von Mantueffel, Tancredi (2015)]
[Caola, Henn, Melnikov, Smirnov (2014)]
[Gehrmann, Remiddi (2001)]

stabilise sector decomposition (FIESTA/ pySecDec) numerics using **local numerators**

choose a set of master integrals to make sure divergence structure of numerical sector is simple

numerical results

$qgg\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-++++-}$	4.50000	-3.63577(3)	-277.2182(7)	-344.56(1)	2051.1(2)
$P^{(2)}_{-++++-}$	4.5	-3.63576	-277.2186	-344.569(6)	—
$\widehat{A}^{(2)}_{-+-++-}$	4.50000	-3.63581(9)	-13.6826(2)	6.143(5)	66.21(7)
$P^{(2)}_{-+-++-}$	4.5	-3.63576	-13.6824	6.145(1)	—
$\widehat{A}^{(2)}_{--+++-}$	4.50000	-3.63579(5)	-18.79219(7)	-6.633(6)	79.02(4)
$P^{(2)}_{--+++-}$	4.5	-3.63576	-18.79212	-6.6303(5)	—

$q\bar{Q}Q\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-+-++-}$	2.00000	-7.16949(9)	-9.9055(2)	39.922(6)	154.79(7)
$P^{(2)}_{--+++-}$	2	-7.16944	-9.9054	39.9245(8)	—
$\widehat{A}^{(2)}_{--+++-}$	2.00000	-7.16948(8)	-12.9371(1)	41.432(8)	189.53(6)
$P^{(2)}_{--+++-}$	2	-7.16944	-12.9370	41.4353(6)	—

good agreement with universal poles

first step towards analytic reconstruction

outlook

- numerical algorithms and finite field arithmetic can provide powerful tools for analytic amplitude computations

- traditional bottlenecks can be avoided by direct reconstruction of on-shell physical quantities (e.g. finite remainders)

- surge of progress on two-loop amplitudes! major challenge to combine amplitudes into physical cross sections at NNLO