QCD@LHC —University of Buffalo, New York, 16 May 2019

IR divergences of scattering amplitudes at 3 and 4 loops

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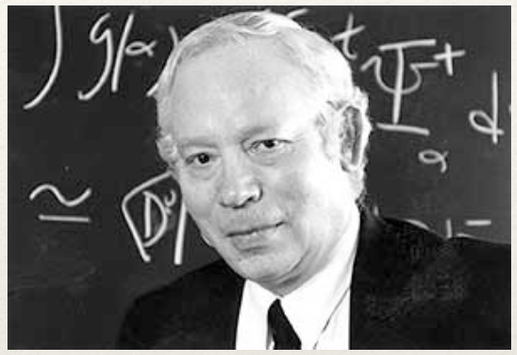
continuation on 0903.1126 [JHEP], 1208.4847 [JHEP] with T. Becher, V. Ahrens, L. Vernazza

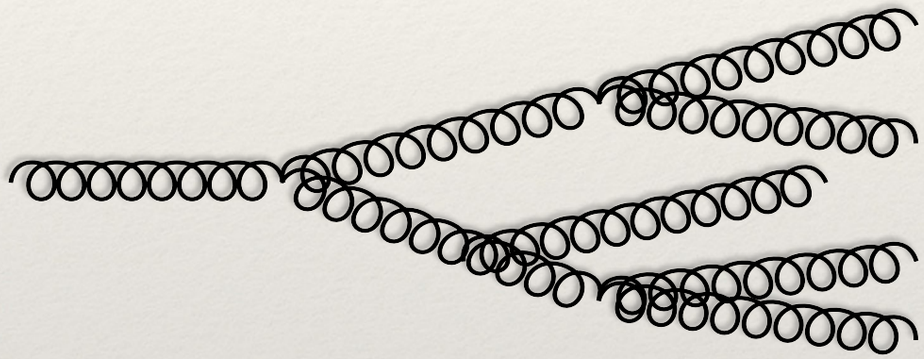
Introduction

- Understanding the IR structure of scattering amplitudes in non-abelian gauge theories is an important task:
  - construction of IR subtraction terms
  - resummation of large (double and single) logarithms in cross sections
  - insights about the perturbative structure of quantum field theories

Introduction

- Difficulty of the problem noted already in the pioneering work of Weinberg:





"... In [Yang-Mills theory] a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence. **The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible.**" [Phys. Rev. 140B, 516 (1965)]

- Since then, lots of progress has been made!

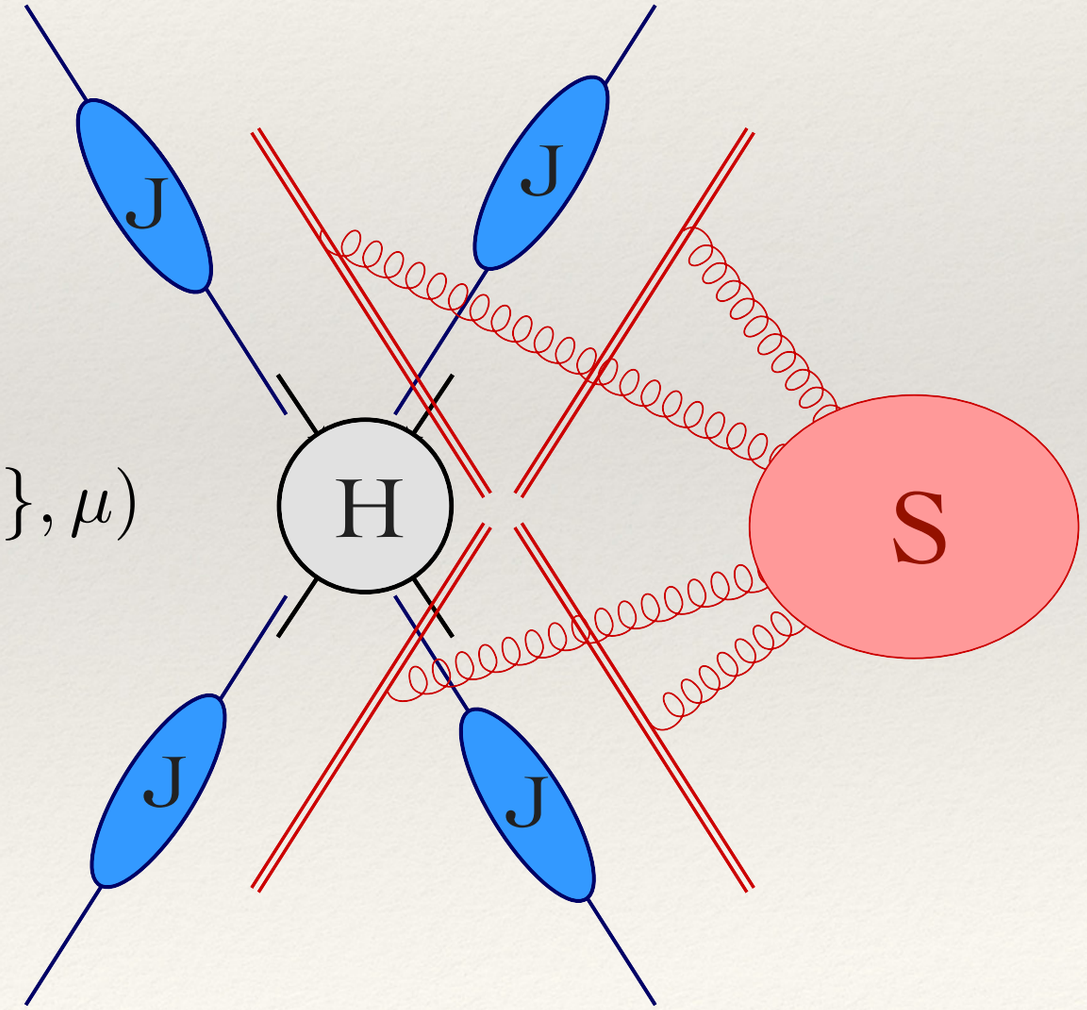
Introduction

- Our goal is to resolve some pertinent contradictions and exploit recent multi-loop results
  - careful re-evaluation of 2-particle collinear limits
  - generalized Casimir scaling relation for cusp anomalous dimension
  - derive 3- and 4-loop contributions to anomalous dimension required for resummation of any (global) *n*-particle cross section at N^3LL order

Factorization and SCET

- Cross sections for processes involving energetic, massless partons can be factorized in SCET into hard, jet and soft functions:

$$d\sigma \sim H(\{s_{ij}\},\mu) \prod_i J_i(M_i^2,\mu) \otimes S(\{\Lambda_{ij}^2\},\mu)$$



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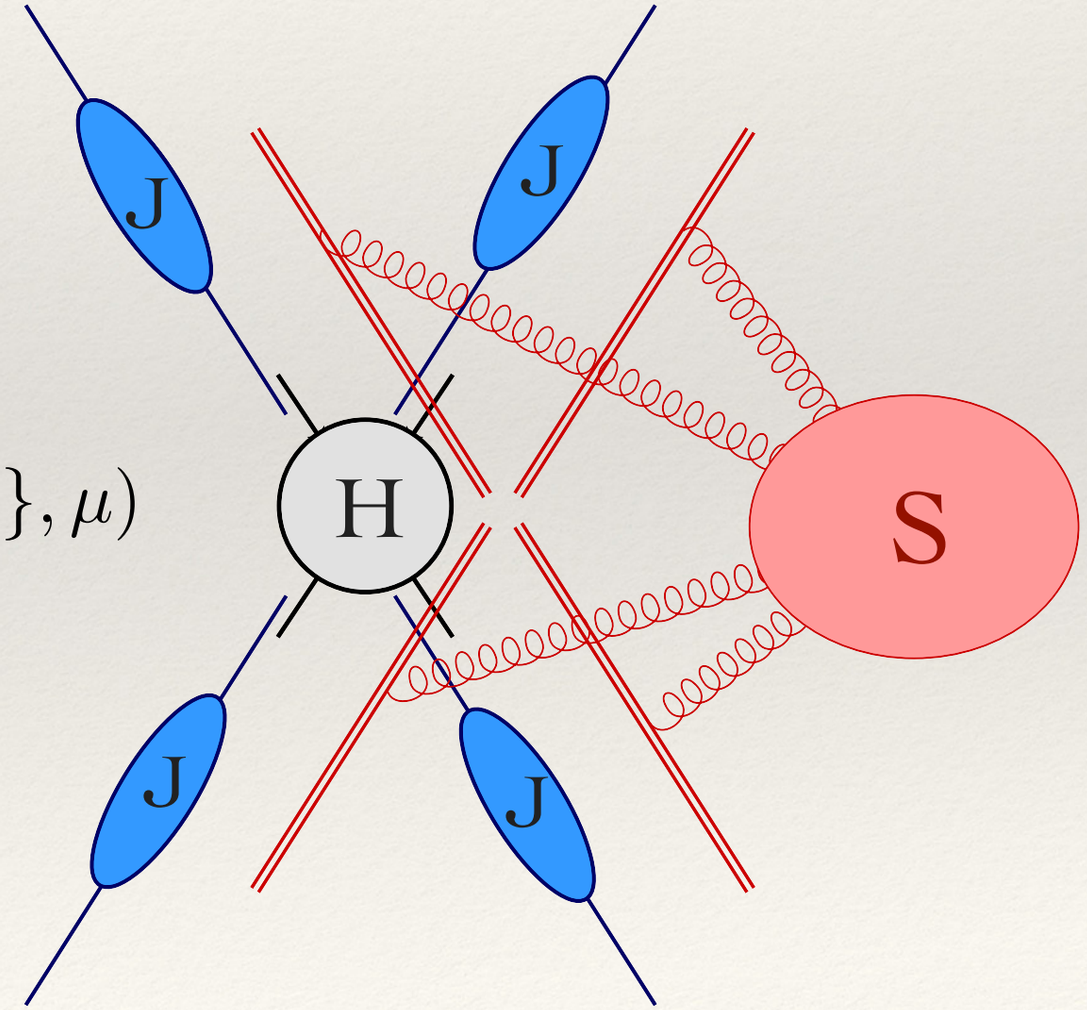

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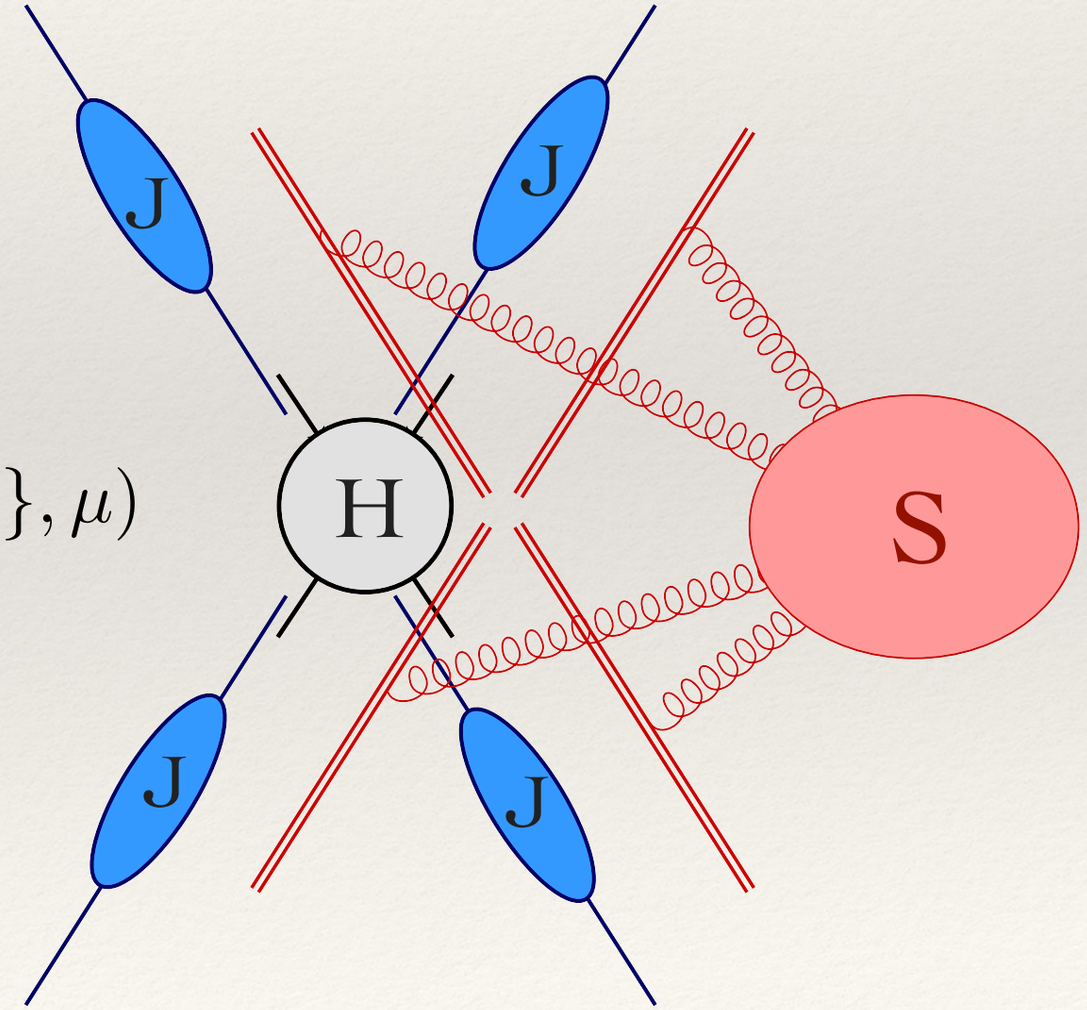
imaginary parts of 2-point functions



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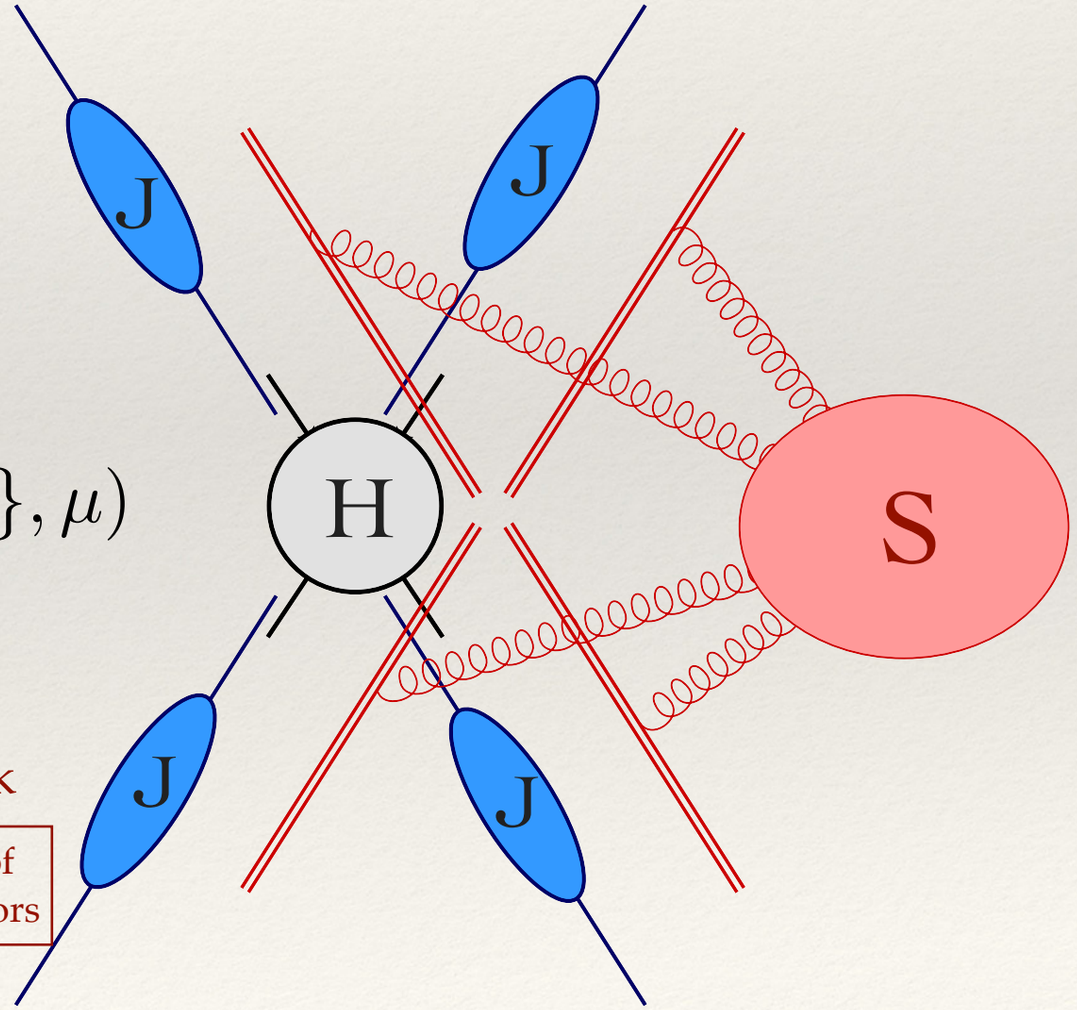
soft cross-talk

matrix elements of Wilson-line operators



Factorization and SCET

- IR divergences of partonic scattering amplitudes are in one-to-one correspondence with UV divergences of jet and soft functions in SCET

[Becher, MN 2009]

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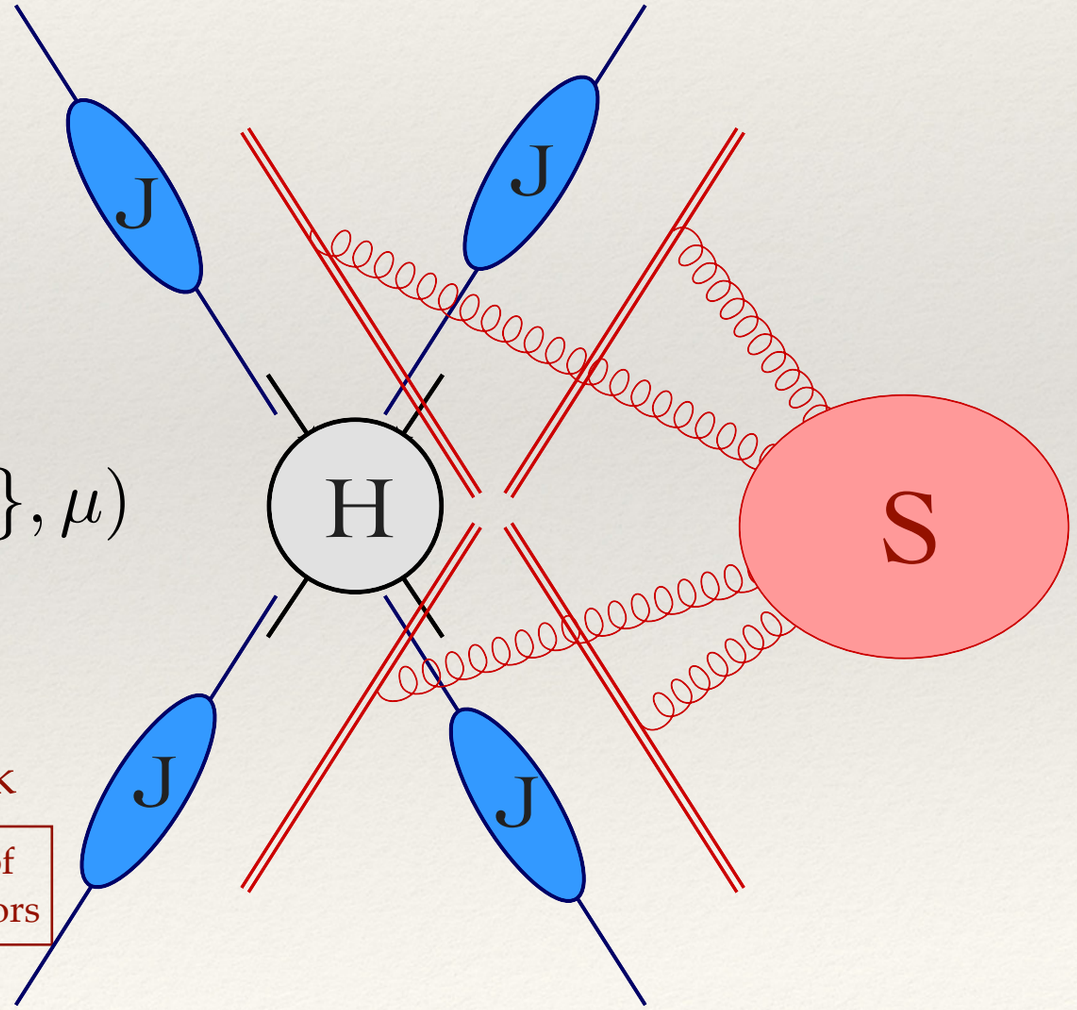


Factorization and SCET

- IR divergences of partonic scattering amplitudes are in one-to-one correspondence with UV divergences of jet and soft functions in SCET [Becher, MN 2009]

- They are governed by an anomalous dimension, which is a matrix in color space:

$$\mathbf{\Gamma} \equiv \mathbf{\Gamma}_{J\otimes J\cdots\otimes S} = \sum_{i=1}^{n} \gamma_{J_i}(p_i,\mu) + \boldsymbol{\gamma}_s(\{p_1,\ldots p_n\},\mu)$$

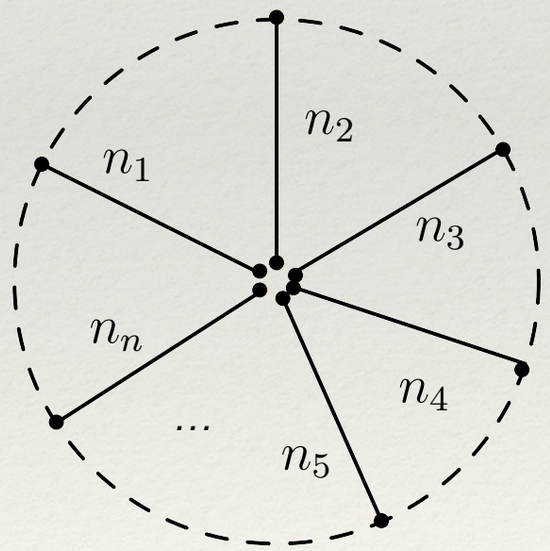
- The IR poles can then be removed by the multiplicative factor:

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_{\mu}^{\infty}\frac{\mathrm{d}\mu'}{\mu'}\,\mathbf{\Gamma}(\{\underline{p}\},\mu')\right]$$

1. Soft-collinear factorization

- Additive structure of $\mathbf{\Gamma}$ follows from the fact that different collinear sectors in SCET interact only via the exchange of soft gluons

- SCET decoupling transformation [Bauer, Pirjol, Stewart 2001] shows that soft anomalous dimension γ_s governs the UV divergences of an operator consisting of n Wilson lines in the (light-like) directions of the external particles



1. Soft-collinear factorization

- Jet anomalous dimensions have a simple structure:

$$\gamma_{J_i}(p_i,\mu) = -\Gamma^i_{\rm cusp}(\alpha_s)\ln\frac{\mu^2}{-p_i^2} + \gamma^i_c(\alpha_s)$$

- All nontrivial dynamics described by soft anomalous dimension γ_s, which depends on the soft variables:

$$\beta_{ij} = \ln\frac{-2\sigma_{ij}\,p_i\cdot p_j\,\mu^2}{(-p_i^2)(-p_j^2)}$$

- The fact that the sum of all pieces must be independent of the IR regulators $(-p_i^2)$ requires: [Gardi, Magnea 2009; Becher, MN 2009]

$$\sum_{j\neq i}\frac{\partial\mathbf{\Gamma}_s(\{\underline{\beta}\},\mu)}{\partial\beta_{ij}} = \Gamma^i_{\rm cusp}(\alpha_s)$$

1. Soft-collinear factorization

- The partial differential equation:

$$\sum_{j\neq i} \frac{\partial \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu)}{\partial \beta_{ij}} = \Gamma^i_{\text{cusp}}(\alpha_s)$$

implies that the soft anomalous dimension γ_s can only depend **linearly** on the β_{ij} variables, but it can be an **arbitrary function** of the conformal cross ratios:

$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

(solution to the homogeneous DEQ)

[Gardi, Magnea 2009; Becher, MN 2009]

1. Soft-collinear factorization

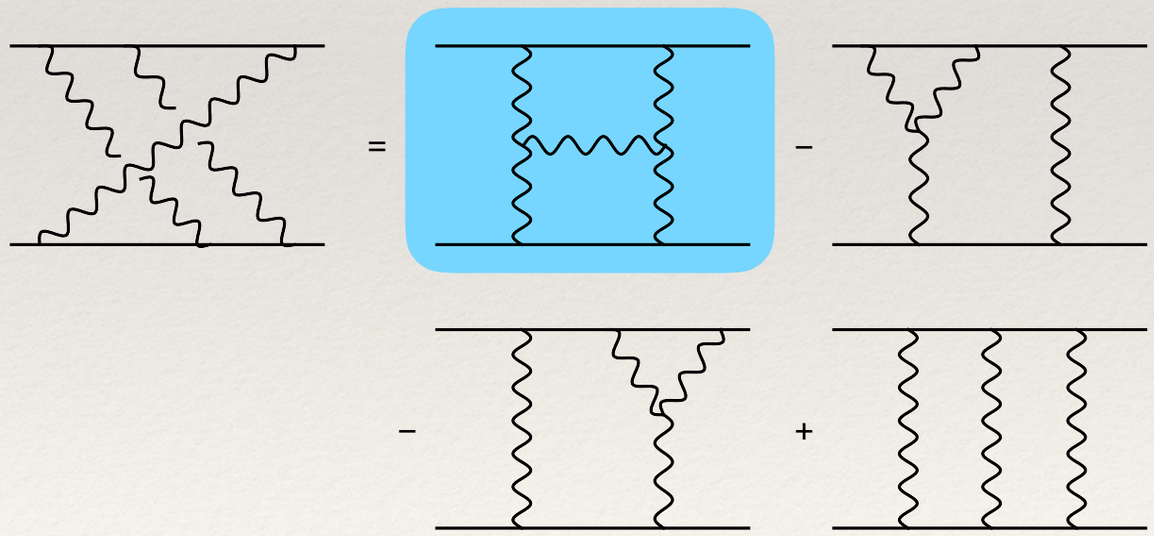
- For a systematic resummation of large Sudakov double logarithms, one needs:

Log. approx.	Accuracy $\sim \alpha_s^n L^k$	$\ln(\mu^2)$ terms	other terms
LL	$n+1 \le k \le 2n$ (α_s^{-1})	1-loop	tree-level
NLL	$n \le k \le 2n$ (α_s^0)	2-loop	1-loop
NNLL	$n-1 \le k \le 2n$ (α_s)	3-loop	2-loop
NNNLL	$n-2 \le k \le 2n$ (α_s^2)	4-loop	3-loop

- Note that N^{3}LL resummation is needed for a consistent improvement of NNLO fixed-order calculations!

2. Non-abelian exponentiation

- IR divergences of Wilson-line operators "exponentiate" in the sense that only color structures associated with **single connected gluon webs** contribute to the anomalous dimension (follows from the fact that soft-gluon attachments are eikonal)

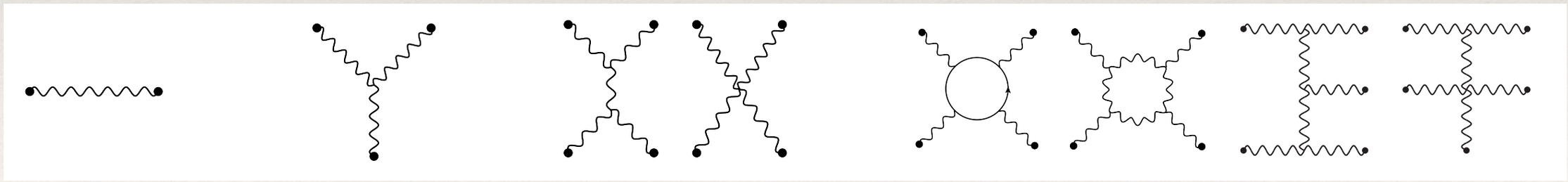


2. Non-abelian exponentiation

- For n>2 Wilson lines, the identification of webs is a non-trivial problem, which can be solved using the replica trick [Gardi, Laenen, Starvenga, White 2010]

- We have shown that (at least to 4-loop order) this gives rise to the same webs as for two Wilson lines [Becher, MN (in prep.)]

- They are:



1-loop order 2-loop order 3-loop order 4-loop order

2. Non-abelian exponentiation

- Using the group-theory identities:

$$\mathrm{tr}_{\mathrm{adj.}}\left(\boldsymbol{T}^a\boldsymbol{T}^b\right) = f^{acd}\,f^{bcd} = C_A\,\delta^{ab}\,,$$

$$\mathrm{tr}_{\mathrm{adj.}}\left(\boldsymbol{T}^a\boldsymbol{T}^b\boldsymbol{T}^c\right) = i f^{ade}\,f^{beg}\,f^{cgd} = \frac{iC_A}{2}\,f^{abc}\,,$$

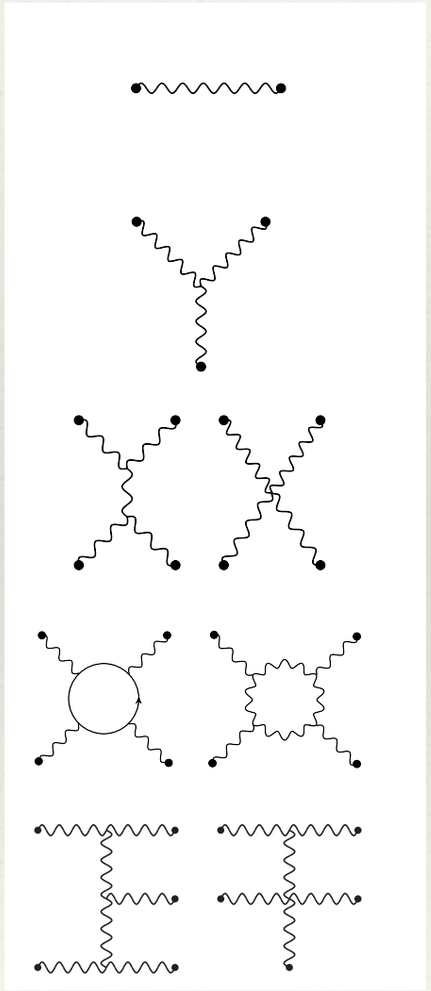
$$\mathrm{tr}_{\mathrm{adj.}}\left(\boldsymbol{T}^a\boldsymbol{T}^b\boldsymbol{T}^c\boldsymbol{T}^d\right) = f^{aef}\,f^{bfg}\,f^{cgh}\,f^{dhe} = d_A^{abcd} + \frac{C_A}{6}\left(f^{ade}\,f^{bce} - f^{abe}\,f^{cde}\right)$$

  we have shown that it suffices to consider symmetrized color structures for these webs

- Terms with commutators can be reduced to lower-order structures

2. Non-abelian exponentiation

- They are:



$$\mathcal{D}_{ij} = \boldsymbol{T}_i^a \boldsymbol{T}_j^a \,, \qquad \text{starting at one-loop order}$$

$$\mathcal{T}_{ijk} = i f^{abc} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c\right)_+ , \qquad \text{starting at two-loop order}$$

$$\mathcal{T}_{ijkl} = f^{ade} f^{bce} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d\right)_+ , \qquad \text{starting at three-loop order}$$

$$\mathcal{D}^R_{ijkl} = d_R^{abcd} \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \,, \qquad \text{starting at four-loop order}$$

$$\mathcal{T}_{ijklm} = f^{adf} f^{bcg} f^{efg} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \boldsymbol{T}_m^e\right)_+ , \qquad \text{starting at four-loop order}$$

- *D*-symbols with an odd number of indices do not arise

3. Collinear factorization

- When two parton momenta become collinear (assume both are outgoing), the n-particle anomalous dimension must be related to the (n-1)-particle anomalous dimension, such that:

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\ldots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\ldots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

with $P = p_1 + p_2$ depends on the momenta and color generators of particles 1 and 2 only [Becher, MN 2009]

General form of Γ

- Using the soft-collinear factorization and non-abelian exponentiation, one finds that **up to 2-loop order**:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

[Becher, MN 2009; Gardi, Magnea 2009]

- The 3-gluon web $\mathcal{T}_{ijk} = i f^{abc} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c\right)_+$ does not contribute, because it is impossible to construct an antisymmetric function of the three soft variables β_{ij}, β_{jk} and β_{ik}
- Anomalous dimensions γ_{cusp} and γ^{i} are known to 4- and 3-loop order, respectively

General form of Γ

- Recent calculations of contributions to the 4-loop coefficient of γ_{cusp}:

[Henn, Smirnov, Smirnov, Steinhauser, Lee 2016]

$$\gamma^3_{\text{cusp}} = C_A^3 \left(\frac{84278}{81} - \frac{44416\pi^2}{243} + \frac{20992\zeta_3}{27} + \frac{902\pi^4}{45} - 352\zeta_5 - \frac{292\pi^6}{315} - \frac{176\pi^2\zeta_3}{9} - 32\zeta_3^2 \right)$$

[Grozin 2018] [Moch, Ruijl, Ueda, Vermaseren, Vogt 2017]

$$+ 2T_F n_f \left[C_F^2 \left(\frac{572}{9} + \frac{592\zeta_3}{3} - 320\zeta_5 \right) + C_F C_A \, (38.75 \pm 0.2) \right.$$

[Henn, Smirnov, Smirnov, Steinhauser 2016]

$$\left. + C_A^2 \left(-\frac{39883}{81} + \frac{13346\pi^2}{243} - \frac{16252\zeta_3}{27} - \frac{44\pi^4}{27} + 224\zeta_5 + \frac{128\pi^2\zeta_3}{9} - \frac{38.75 \pm 0.2}{2} \right) \right]$$

$$+ (2T_F n_f)^2 \left[C_F \left(\frac{2392}{81} - \frac{640\zeta_3}{9} + \frac{16\pi^4}{45} \right) + C_A \left(\frac{923}{81} - \frac{304\pi^2}{243} + \frac{2240\zeta_3}{27} - \frac{56\pi^4}{135} \right) \right]$$

[Lee, Smirnov, Smirnov, Steinhauser 2017; von Manteuffel, Schabinger 2019]

$$+ (2T_F n_f)^3 \left(-\frac{32}{81} + \frac{64\zeta_3}{27} \right)$$

[Lee, Smirnov, Smirnov, Steinhauser 2017]

General form of Γ

- Using the soft-collinear factorization and non-abelian exponentiation, one finds that **up to 2-loop order**:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

[Becher, MN 2009; Gardi, Magnea 2009]

- The fact that "dipole correlations" are all there is at 1- and 2-loop order has led several groups (including ours) to speculate that this may be true to all orders:

"dipole conjecture"

General form of Γ

- Using the soft-collinear factorization and non-abelian exponentiation, one finds that **up to 2-loop order**:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

[Becher, MN 2009; Gardi, Magnea 2009]

- However, starting from **3-loop order** nontrivial functions of conformal cross ratios can and do arise!

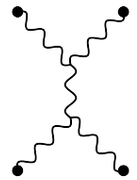
[Almelid, Duhr, Gardi 2015]

General form of Γ

- General structure at 3 and 4 loops:

$$\mathbf{\Delta\Gamma}_3 = -\bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\ \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

[Becher, MN 2009]



- Note that for given indices *i,j,k,l* there are only two independent cross ratios, since $\beta_{ijkl} + \beta_{iklj} + \beta_{iljk} = 0$

General form of Γ

- General structure at 3 and 4 loops:

$$\mathbf{\Delta\Gamma}_3 = -\bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\ \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

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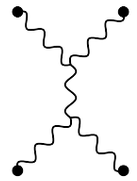
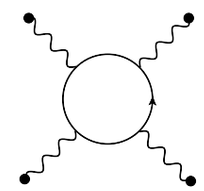
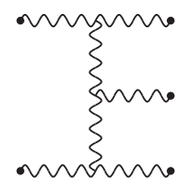


$$\begin{aligned}
\mathbf{\Delta\Gamma}_4 = & -\sum_R g_1^R(\alpha_s) \left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \right) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk}\ \ln \frac{\mu^2}{-s_{ij}} \right] \\
& + \sum_R g_5^R(\alpha_s) \sum_{(i,j)} \mathcal{D}^R_{iijj} + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G_1^R(\beta_{ijkl}, \beta_{iklj}, \beta_{iljk}) \\
& + \sum_{(i,j,k,l)} \left[\mathcal{T}_{iijkl}\, G_2(\beta_{ijkl}, \beta_{iklj}) + \mathcal{T}_{ijkli}\, G_3(\beta_{ijkl}, \beta_{iklj}) \right] \\
& + \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, G_4(\beta_{ijkl}, \beta_{iklj}, \beta_{ijkm}, \beta_{ikmj}, \beta_{ijml})
\end{aligned}$$

[Ahrens, MN, Vernazza 2012]



R: repr. of internal particles



- Note that for given indices *i,j,k,l* there are only two independent cross ratios, since $\beta_{ijkl} + \beta_{iklj} + \beta_{iljk} = 0$

2-particle collinear limit

- We have carefully re-evaluated the contributions to the anomalous dimension of the 2-particle splitting amplitude from the above results: [Becher, MN (in prep.)]

$$\boldsymbol{\Gamma}_{\mathrm{Sp}}(\{p_1,p_2\},\mu) = \boldsymbol{\Gamma}(\{p_1,\ldots,p_n\},\mu) - \boldsymbol{\Gamma}(\{P,p_3\ldots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

2-particle collinear limit

- We find:

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \gamma_{\rm cusp}\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\,\ln\frac{\mu^2}{-s_{12}} + \boldsymbol{T}_1\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln z + \boldsymbol{T}_2\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln(1-z)\right] + \gamma^1 + \gamma^2 - \gamma^P$$

$$+\ \bar{f}_2(\alpha_s)\left[2\mathcal{T}_{1122} - 4\sum_{i\neq 1,2}\mathcal{T}_{12ii} + \frac{C_A^2}{4}\,\boldsymbol{T}_1\cdot\boldsymbol{T}_2\right]$$

$$+\sum_{(i,j)\neq 1,2}\left[8\mathcal{T}_{12ij}\,\hat{F}(\omega_{ij},\epsilon_{ij}) + 4\mathcal{T}_{1ij2}\,\hat{F}(-\omega_{ij},-\omega_{ij})\right]$$

$$+\sum_R g_1^R(\alpha_s)\left\{-\left[6\mathcal{D}^R_{1122} + 4\left(\mathcal{D}^R_{1112}+\mathcal{D}^R_{1222}\right)\right]\left[\ln\frac{\mu^2}{-s_{12}} + \ln z(1-z)\right]\right.$$

$$\left. - 2\left[\mathcal{D}^R_{1111}\ln z + \mathcal{D}^R_{2222}\ln(1-z)\right] - 2\sum_{2<i<j}\mathcal{D}^R_{12ij}\,\beta_{12ij}\right\}$$

$$+\sum_R g_5^R(\alpha_s)\left[2\mathcal{D}^R_{1122} - 4\sum_{i>2}\mathcal{D}^R_{12ii}\right] + 24\sum_{2<i<j}\mathcal{D}^R_{12ij}\,G_1^R(\omega_{ij},\epsilon_{ij},-\omega_{ij})$$

$$+\ \text{contributions from } \mathcal{T}_{ijklm} \text{ structures} \quad \longleftarrow \quad \text{must vanish!}$$

$$\omega_{ij} = \beta_{12ij} \to -\infty$$
$$\epsilon_{ij} = \beta_{1ij2} \to 0$$

2-particle collinear limit

- In Becher, MN 2009 and Dixon, Gardi, Magnea 2010 it was assumed that the functions of conformal cross ratios vanish in all 2-particle collinear limits, in which case it would follow that:

$$\bar{f}_2(\alpha_s) = g_1^R(\alpha_s) = g_5^R(\alpha_s) = 0$$

- However, a more careful analysis shows that there is a **more general solution**, in which the functions of conformal cross ratios have nontrivial collinear limits!

2-particle collinear limit

- Let us assume that:

$$\hat{F}(\omega_{ij}, \epsilon_{ij}) \to \hat{F}(-\infty, 0) = -\frac{\bar{f}_2(\alpha_s)}{2}$$

$$F(-\omega_{ij}, -\omega_{ij}) \to F(\infty, \infty) = c_2(\alpha_s) \text{ arbitrary}$$

$$G_1^R(\omega_{ij}, \epsilon_{ij}, -\omega_{ij}) \to \frac{g_1^R(\alpha_s)}{12}\,\omega_{ij} - \frac{g_5^R(\alpha_s)}{3}$$

- Then there are no further restrictions on the coefficient functions!

2-particle collinear limit

- We obtain: [Becher, MN (in prep.)]

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \gamma_{\rm cusp}\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\,\ln\frac{\mu^2}{-s_{12}} + \boldsymbol{T}_1\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln z + \boldsymbol{T}_2\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln(1-z)\right] + \gamma^1 + \gamma^2 - \gamma^P$$

$$+ 6\bar{f}_2(\alpha_s)\left(\mathcal{T}_{1122} + \frac{C_A^2}{8}\,\boldsymbol{T}_1\cdot\boldsymbol{T}_2\right)$$

$$- \sum_R \left[6\mathcal{D}^R_{1122} + 4\left(\mathcal{D}^R_{1112} + \mathcal{D}^R_{1222}\right)\right]\left\{g_1^R(\alpha_s)\left[\ln\frac{\mu^2}{-s_{12}} + \ln z(1-z)\right] + g_5^R(\alpha_s)\right\}$$

$$- \sum_R 2g_1^R(\alpha_s)\left[\mathcal{D}^R_{1111}\ln z + \mathcal{D}^R_{2222}\ln(1-z)\right]$$

Higher-order coefficient functions

- The 3-loop coefficient $\bar{f}_2(\alpha_s)$ and the 3-loop function $\hat{F}(x,y)$ have been obtained in a "heroic" calculation of the 3-loop 3-particle amplitude [Almelid, Duhr, Gardi 2015]

- One finds:

$$\bar{f}_2(\alpha_s) = -16\,(\zeta_5 + 2\zeta_2\zeta_3)\left(\frac{\alpha_s}{4\pi}\right)^3$$

$\hat{F}(x,y) = -\hat{F}(y,x)$ beautifully simple expression in terms of Brown's single-valued harmonic polylogs

Higher-order coefficient functions

- By specializing our general formula for the case of the quark and gluon form factors (n=2; i=F,A) we obtain:

$$\mathbf{\Gamma} = -\Big[C_i\,\gamma_{\rm cusp}(\alpha_s) - 2\sum_R C_4(R,R_i)\,g_1^R(\alpha_s)\Big]\,\ln\frac{\mu^2}{-s_{12}} + 2\gamma^i(\alpha_s) + 2\sum_R C_4(R,R_i)\,g_5^R(\alpha_s)$$

"cusp anomalous dimension $\Gamma_{\rm cusp}{}^i$"

where:

$$\mathcal{D}^R_{iiii} = \frac{d^{abcd}_R d^{abcd}_{R_i}}{N_{R_i}} \equiv \frac{d^{(4)}_{RR_i}}{N_{R_i}} \equiv C_4(R,R_i)$$ quartic Casimir invariant

- From recent 4-loop results for the cusp anomalous dimension, we derive: [Moch, Ruijl, Ueda, Vermaseren, Vogt 2018]

$$g_1^F(\alpha_s) = 2T_F n_f\,(61.95 \pm 0.1)\left(\frac{\alpha_s}{4\pi}\right)^4 + \ldots\,,$$

$$g_1^A(\alpha_s) = (253.5 \pm 0.1)\left(\frac{\alpha_s}{4\pi}\right)^4 + \ldots\,.$$

Generalized Casimir scaling of $\Gamma_{\rm cusp}{}^{i}$

- Cusp anomalous dimension (i=F,A):

$$\Gamma^i_{\rm cusp}(\alpha_s) = C_i\,\gamma_{\rm cusp}(\alpha_s) - 2\sum_R C_4(R,R_i)\,g_1^R(\alpha_s) + \mathcal{O}(\alpha_s^5)$$

- Up to 3-loop order, one finds the simple (quadratic) Casimir scaling:

$$\Gamma^g_{\rm cusp}(\alpha_s)/\Gamma^q_{\rm cusp}(\alpha_s) = C_A/C_F$$

- The terms proportional to $g_1^R(\alpha_s)$ instead scale with the ratio of **quartic Casimir invariants** (independently of R):

$$C_4(F,A)/C_4(F,F)$$

"generalized Casimir scaling"

[see also: Moch, Ruijl, Ueda, Vermaseren, Vogt 2018]

Resummation at N^{3}LL order

- Final result for the anomalous dimension:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

$$-\bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

$$-\sum_R g_1^R(\alpha_s) \left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij}\right) \ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln\frac{\mu^2}{-s_{ij}} \right]$$

$$+\sum_R g_5^R(\alpha_s) \sum_{(i,j)} \mathcal{D}^R_{iijj} + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G_1^R(\beta_{ijkl}, \beta_{iklj}, \beta_{iljk})$$

$$+\sum_{(i,j,k,l)} \left[\mathcal{T}_{iijkl}\, G_2(\beta_{ijkl}, \beta_{iklj}) + \mathcal{T}_{ijkli}\, G_3(\beta_{ijkl}, \beta_{iklj}) \right]$$

$$+\sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, G_4(\beta_{ijkl}, \beta_{iklj}, \beta_{ijkm}, \beta_{ikmj}, \beta_{ijml})$$

Resummation at N^{3}LL order

- Final result for the anomalous dimension:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

known at 4-loop order

$$- \bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

$$- \sum_R g_1^R(\alpha_s) \left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \right) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln \frac{\mu^2}{-s_{ij}} \right]$$

$$+ \sum_R g_5^R(\alpha_s) \sum_{(i,j)} \mathcal{D}^R_{iijj} + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G_1^R(\beta_{ijkl}, \beta_{iklj}, \beta_{iljk})$$

$$+ \sum_{(i,j,k,l)} \left[\mathcal{T}_{iijkl}\, G_2(\beta_{ijkl}, \beta_{iklj}) + \mathcal{T}_{ijkli}\, G_3(\beta_{ijkl}, \beta_{iklj}) \right]$$

$$+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, G_4(\beta_{ijkl}, \beta_{iklj}, \beta_{ijkm}, \beta_{ikmj}, \beta_{ijml})$$

Resummation at N³LL order

- Final result for the anomalous dimension:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

$$- \bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

$$- \sum_R g_1^R(\alpha_s) \left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \right) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln \frac{\mu^2}{-s_{ij}} \right]$$

$$+ \sum_R g_5^R(\alpha_s) \sum_{(i,j)} \mathcal{D}^R_{iijj} + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G_1^R(\beta_{ijkl}, \beta_{iklj}, \beta_{iljk})$$

$$+ \sum_{(i,j,k,l)} \left[\mathcal{T}_{iijkl}\, G_2(\beta_{ijkl}, \beta_{iklj}) + \mathcal{T}_{ijkli}\, G_3(\beta_{ijkl}, \beta_{iklj}) \right]$$

$$+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, G_4(\beta_{ijkl}, \beta_{iklj}, \beta_{ijkm}, \beta_{ikmj}, \beta_{ijml})$$

known at 4-loop order

known at 3-loop order

Resummation at N^{3}LL order

- Final result for the anomalous dimension:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

$$- \bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} \hat{F}(\beta_{ijlk}, \beta_{iklj})$$

$$- \sum_R g_1^R(\alpha_s) \left[\sum_{(i,j)} \left(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \right) \ln \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln \frac{\mu^2}{-s_{ij}} \right]$$

$$+ \sum_R g_5^R(\alpha_s) \sum_{(i,j)} \mathcal{D}^R_{iijj} + \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl} G_1^R(\beta_{ijkl}, \beta_{iklj}, \beta_{iljk})$$

$$+ \sum_{(i,j,k,l)} \left[\mathcal{T}_{iijkl} G_2(\beta_{ijkl}, \beta_{iklj}) + \mathcal{T}_{ijkli} G_3(\beta_{ijkl}, \beta_{iklj}) \right]$$

$$+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm} G_4(\beta_{ijkl}, \beta_{iklj}, \beta_{ijkm}, \beta_{ikmj}, \beta_{ijml})$$

known at 4-loop order

known at 3-loop order

unknown 4-loop terms

Resummation at N^{3}LL order

- Final result for the anomalous dimension:

$$\begin{aligned}
\mathbf{\Gamma}(\{\underline{p}\},\mu) = &\sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) \\
&-\bar{f}_2(\alpha_s)\sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\,\hat{F}(\beta_{ijlk},\beta_{iklj}) \\
&-\sum_R g_1^R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\ln\frac{\mu^2}{-s_{ij}}\right] \\
&+\sum_R g_5^R(\alpha_s)\sum_{(i,j)}\mathcal{D}^R_{iijj} + \sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,G_1^R(\beta_{ijkl},\beta_{iklj},\beta_{iljk}) \\
&+\sum_{(i,j,k,l)}\left[\mathcal{T}_{iijkl}\,G_2(\beta_{ijkl},\beta_{iklj}) + \mathcal{T}_{ijkli}\,G_3(\beta_{ijkl},\beta_{iklj})\right] \\
&+\sum_{(i,j,k,l,m)}\mathcal{T}_{ijklm}\,G_4(\beta_{ijkl},\beta_{iklj},\beta_{ijkm},\beta_{ikmj},\beta_{ijml})
\end{aligned}$$

known at 4-loop order

known at 3-loop order

unknown 4-loop terms

Conclusions

- Using methods from SCET, combined with existing multi-loop calculations of the 2- and 3-parton amplitudes in QCD, we have derived the anomalous dimensions needed to perform N^{3}LL resummations of arbitrary (global) n-parton scattering cross sections in QCD

- Up to 4 loops, all contributions to the cusp anomalous dimension of quarks and gluons scale with either the quadratic or the quartic Casimir invariants of SU(N_c)