Quantum Chromodynamics

& LHC Phenomenology

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Oxford Theoretical Physics & STFC

2008 CERN-Fermilab Hadron Collider Physics Summer School

Some bibliography

- Text books
  - QCD and Collider physics, Ellis, Stirling, Webber (a.k.a. The pink book)
  - Basics of Perturbative QCD, Dokshitzer, Khoze, Mueller, Troyan
  - Foundations of Quantum Chromodynamics, Muta
- Summaries of recent research activity
  - Les Houches writeups
    - The NLO multi-leg working group: summary report [0803.0494]
    - Standard Model handles and candles WG: summary report [0803.0678]
    - Physics at the TeV colliders, SM & Higgs WG: summary report [hep-ph/0604120]
  - Hera-LHC writeups: Part A & B [hep-ph/0601012 & hep-ph/0601013]
- Various papers cited on the way

Today's high energy colliders

Collider	Process	status
Hera	$e^{\pm}p$	ended 2006
Tevatron	$p\bar{p}$	running
LHC	pp	starts this year

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$\Rightarrow$ *Aim: make sure that our ability to discover new particles and determine their properties will not be limited by the quality of our understanding of QCD*

LHC early physics

Tevatron: 1.8 TeV [run I], 1.96 TeV [runII]
LHC: 10 TeV [2008], 14TeV [afterwards]

Tevatron: 4 fb^{-1} [summer 2008] & 6-10 fb^{-1} [2009] ?
LHC: 10-20 fb^{-1} ['09-'10], 100 fb^{-1} ['11-'14], 1000 fb^{-1} [after '16, SLHC?] ?

LHC: Highest energy & luminosity
operating regime such that even early data often sufficient for discoveries

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Out of ≈ 680 questions ≈ 40 asked what one can do with early LHC data

The answer to this question very much depends on beam control, detector understanding/performance and control over QCD

LHC early physics

Previous fake discovery claims

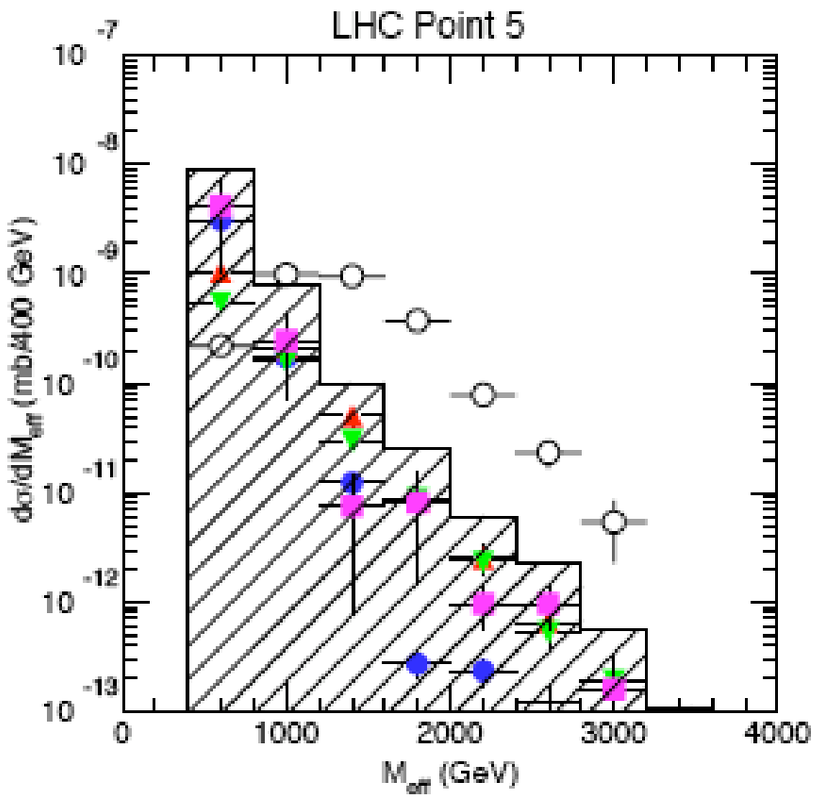
- UA1 monojets (top? SUSY?) 1984
- sbottom production at the Tevatron 2002

Crucial to

1. understand how much a given approximation can be trusted

2. know how to improve on it if necessary

NB: same mistakes already done at the LHC ...
(come back to this later)



So why care about QCD

Essentially the driving motivation here will be that

only a solid understanding of QCD will guarantee that we won't make unjustified discovery claims & will be essential in identifying BSM physics

Of course QCD is interesting for a variety of other reasons

hadron masses, protons & neutrons, confinement, hadron decays, lattice QCD, nuclear physics, star formation and evolution, early universe formation, heavy ion, quark-gluon plasma, chiral perturbaton theory, connections to string theory, AdS/CFT ...

Outline

<u>This course: 4 × (1h +15')</u> ⇒ not much time!

- will only review/recall some basic QCD concepts (will not discuss in detail things common to other QFTs)
- will focus on peculiarities of QCD (asymptotic freedom, soft divergences, ...)
- will try to give you an idea/overview of where QCD is now (parton densities, NLO, NNLO, parton showers, ...), i.e. last two lectures will focus on active research topics

Outline

First lecture

- why do we care about QCD today
- experimental evidence for color $\Rightarrow$ SU(3)
- QCD Lagrangian, gauge invariance, Feynman rules, gauge fixing & ghosts
- the color algebra
- isospin symmetry
- R-ratio & UV divergences $\Rightarrow$ renormalization
- the running coupling & the beta function
- asymptotic freedom & confinement
- soft & collinear divergences & infrared safety
- Sterman-Weinberg jets

Motivations for QCD

Satisfactory model for strong interactions: non-abelian gauge theory SU(3)

$$U^\dagger U = U U^\dagger = 1 \quad \det(U) = 1$$

Hadron spectrum fully classified with the following assumptions

- hadrons (barions,mesons): made of spin 1/2 quarks
- each quark of a given flavour comes in N_c=3 colors
- SU(3) is an exact symmetry
- hadrons are colour neutral, i.e. colour singlet under SU(3)

$$\sum_i \psi_i^* \psi_i \to \sum_{ijk} U_{ij}^* U_{ik} \psi_j \psi_k = \sum_k \psi_k^* \psi_k$$

$$\sum_{ijk} \epsilon_{ijk} \psi_i \psi_j \psi_k \to \sum_{ii'jj'kk'} \epsilon_{ijk} U_{ii'} U_{jj'} U_{kk'} \psi_{i'} \psi_{j'} \psi_{k'} = \sum_{i'j'k'} \epsilon_{i'j'k'} \det(U) \psi_{i'} \psi_{j'} \psi_{k'}$$

- observed hadrons are colour neutral ⇒ hadrons have integer charge

First experimental evidence for colour

I. Existance of Δ^{++} particle: particle with three up quarks of the same spin and with symmetric spacial wave function. Without an additional quantum number Pauli's principle would be violated ⇒ color

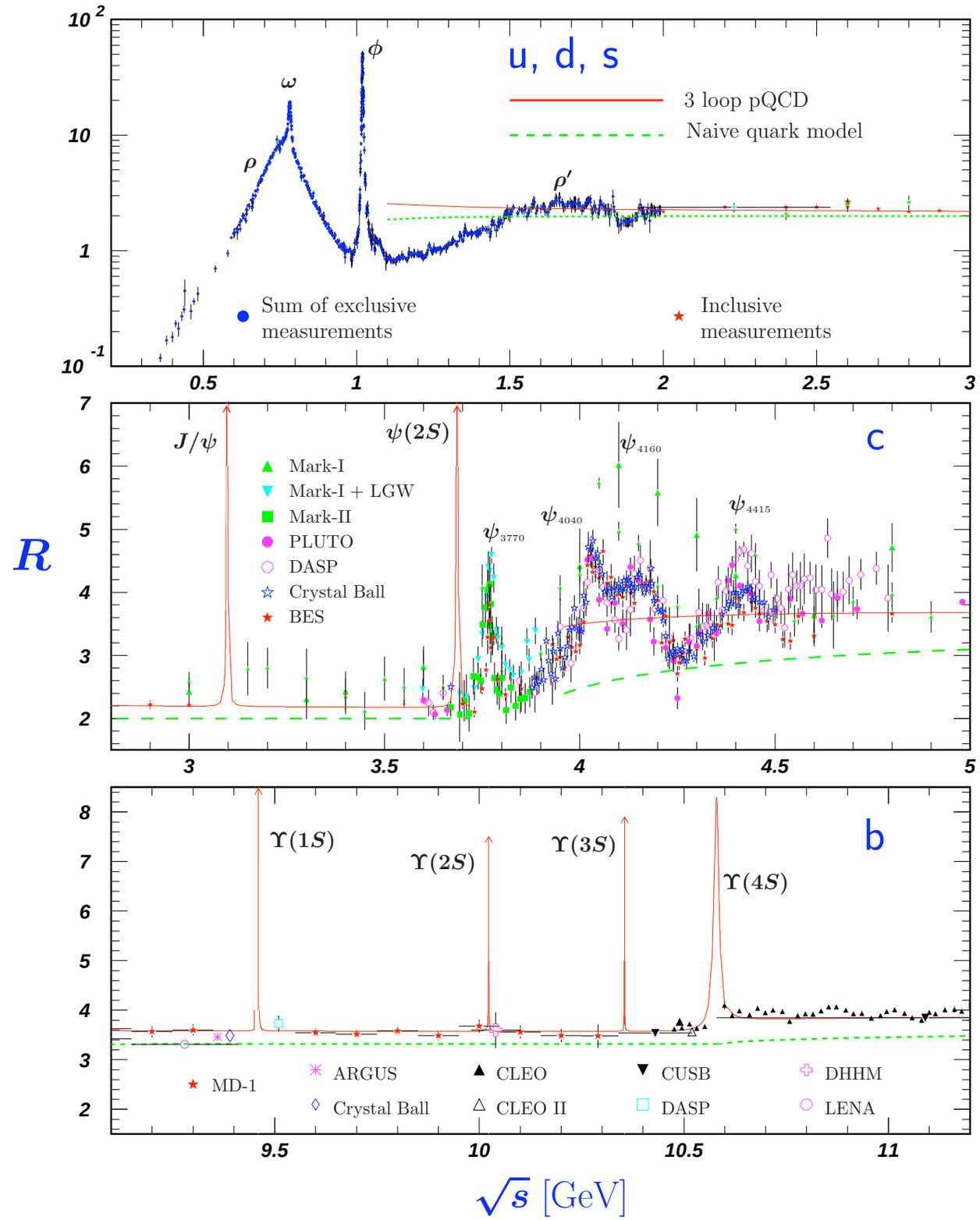
II. R-ratio: ratio of ($e^+e^- \to$ hadrons)/($e^+e^- \to \mu^+\mu^-$)

$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

Data compatible with N_c=3 (r, g, b). Will come back to R later.

charge 2/3 mass=	up few MeV	charm ~1.25 GeV	top ~172 GeV
charge -1/3 mass =	down few MeV	strange ~100 MeV	bottom ~5 GeV

The R-ratio: comparison to data



The QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_f \bar{\psi}_i^{(f)} \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)}$$

$$D_{ij}^\mu \equiv \partial^\mu \delta_{ij} + i g_s t_{ij}^a A_a^\mu \,, \qquad F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc} A_\mu^b A_\nu^c$$

⇒ covariant derivative ⇒ field strength

- only one QCD parameter g_s regulating the strength of the interaction (quark masses have EW origin)
- setting $g_s = 0$ one obtains the free Lagrangian (free propagation of quarks and gluons without interaction)
- terms proportional to g_s in the field strength cause self-interaction between gluons (makes the difference wrt QED)
- color matrices t_{ij}^a are the generators of SU(3)
- QCD flavour blind (differences only due to EW)

The generators of SU(N_c)

The fundamental representation of SU(N_c) has N_c^2-1 generators t^a_{ij} :
$N_c \times N_c$ traceless hermitian matrices $\Rightarrow$ N_c^2-1 gluons

$$U = e^{i\theta_a(x)t^a}$$

f_{abc} are the structure constants of the SU(N_c) algebra, they generate a represention of the algebra called adjoint representation

Definition:

$$[t_a, t_b] = i f_{abc}\, t^c \qquad\qquad f_{abc} = -f_{acb} = -f_{bac}$$

One explicit representation: t^A = 1/2 λ^A (λ^A : Gell-mann matrices)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

Local gauge transformations

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

$$\psi \to \psi' = U(x)\psi$$

$$t^a A_a \to t^a A'_a = U(x) t^a A_a U^{-1}(x) + \frac{i}{g_s} \left(\partial U(x)\right) U^{-1}(x)$$

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It follows that

$$D_\mu \psi \to D'_\mu \psi' = U(x) D_\mu \psi$$

(covariant = transform "with" the field)

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} U^\dagger(x)$$

$$t^a F^a_{\mu\nu} \to t^a F^{a'}_{\mu\nu} = U(x) t^a F^a_{\mu\nu} U^{-1}(x)$$

e.g. because $i\, g_s\, t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$

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Therefore

$$-\frac{1}{4} F'^{\mu\nu}_a F'^a_{\mu\nu} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} \qquad \sum_f \bar{\psi}'^{(f)}_i \left(i \not{D}'_{ij} - m_f \delta_{ij}\right) \psi'^{(f)}_j = \sum_f \bar{\psi}^{(f)}_i \left(i \not{D}_{ij} - m_f \delta_{ij}\right) \psi^{(f)}_j$$

$\Rightarrow$ the QCD Lagrangian is gauge invariant

Local gauge transformations

Exercise: show that a gluon mass term ($m^2 A_a A^a$) is not gauge invariant

Exercise: show that if one replaces the covariant with an ordinary partial derivative the Lagrangian is invariant only under global gauge transformations (global = same transformation in every point)

Isospin symmetry

Isospin SU(2) symmetry: invariance under $u \leftrightarrow d$

Particles in the same isospin multiplet have very similar masses (proton and neutron, neutral and charged pions)

The QCD Lagrangian has isospin symmetry if $m_u = m_d$ or $m_u, m_d \to 0$

The fermionic Lagrangian becomes

$$\mathcal{L}_F = \sum_f \left(\bar{\psi}_L^{(f)} \not{D} \psi_L^{(f)} + \bar{\psi}_R^{(f)} \not{D} \psi_R^{(f)} \right) - \sum_f m_f \left(\bar{\psi}_R^{(f)} \psi_L^{(f)} + \bar{\psi}_L^{(f)} \psi_R^{(f)} \right)$$

$$\psi_L = P_L \psi \,, \quad \psi_R = P_R \psi \,, \quad P_{L/R} = \frac{1}{2} \left(1 \mp \gamma_5 \right)$$

So neglecting fermion masses the Lagrangian has the larger symmetry

$$SU_L(N_f) \times SU_R(N_f) \times U_L(1) \times U_R(1)$$

Feynman rules: propagators

Obtain quark/gluon propagators from free piece of the Lagrangian

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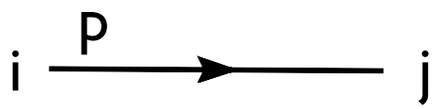
<u>Quark propagator</u>: replace i$\partial \rightarrow$p and take the i $\times$ inverse

$$\mathcal{L}_{\mathrm{q,free}} = \sum_f \bar{\psi}_i^{(f)} \left(i\partial\!\!\!/ - m_f \right) \delta_{ij} \psi_j^{(f)}$$



$$\frac{i}{p\!\!\!/ - m} \delta_{ij}$$

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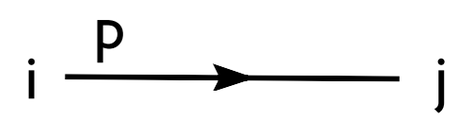

$$\frac{i}{p\!\!\!/ - m}\delta_{ij}$$

<u>Gluon propagator</u>: replace i$\partial \rightarrow$p and take the i $\times$ inverse ?

$$\mathcal{L}_{\mathrm{g,free}} = -\frac{1}{2} A_\mu \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu\right) A_\nu$$

➥ inverse does not exist. How can one to define the propagator ?

Gauge fixing

<u>Solution</u>: add to the Lagrangian a gauge fixing term which depends on an arbitrary parameter λ

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$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda}\left(\partial^\mu A^A_\mu\right)^2$$

λ=1 Feynman gauge
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<u>Gluon propagator:</u>

$$\frac{i}{k^2+i\epsilon}\left(-g_{\mu\nu}+\left(1-\frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{k^2}\right)\delta_{ab}$$

Check that the λ dependence fully cancels at the end of your calculation

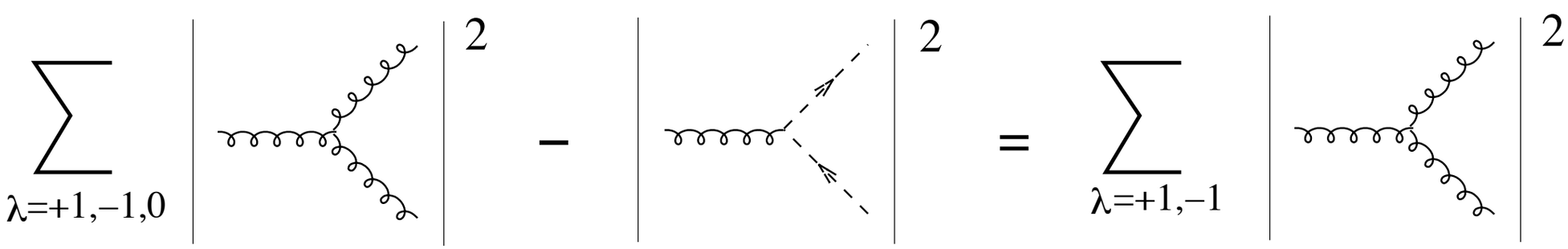
Ghosts

In covariant gauges gauge fixing term must be supplemented with ghost term to cancel unphysical longitudinal degrees of freedom which should not propagate

$$\mathcal{L}_{\text{ghosts}} = \partial_\mu \eta^{a\dagger} D^\mu_{ab} \eta^b$$

$$\frac{i}{k^2}\delta_{ab}$$

η: complex scalar field which obeys Fermi statistics



Axial gauges

Alternative: choose an axial gauge (introduce an arbitrary direction n)

$$\mathcal{L}_{\text{axial gauge-fixing}} = -\frac{1}{2\lambda}\left(n^\mu A^a_\mu\right)^2$$

The gluon propagator becomes

$$d_{\mu\nu} = \frac{i}{k^2 + i\epsilon}\left(-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} - \frac{(n^2+\lambda k^2)k_\nu k_\mu}{(n\cdot k)^2}\right)\delta_{ab}$$

Light cone gauge: $n^2 = 0$ and $\lambda = 0$

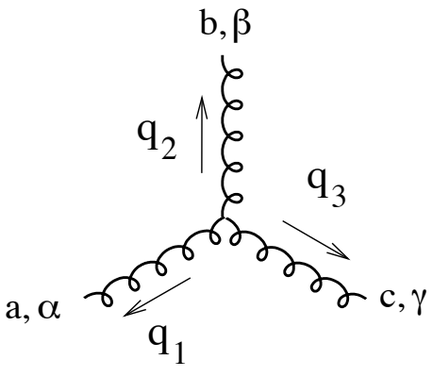
Axial gauges for $k^2 \to 0$

$$d_{\mu\nu}k^\mu = d_{\mu\nu}n^\mu = 0$$

i.e. only two physical polarizations propagate, that's why often the term physical gauge is used

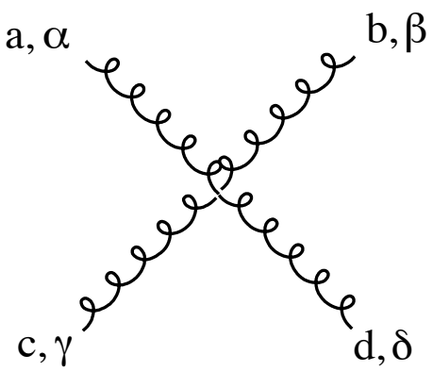
QCD Feynman rules: the vertices

3-gluon: $\mathcal{L} = \frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^\mu_b A^\nu_c$



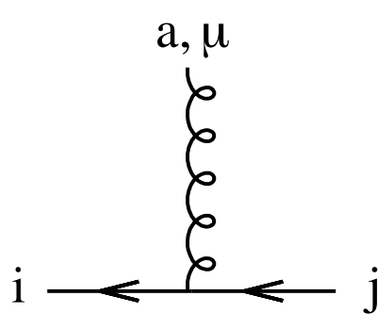
$$-ig\,(if_{abc})[g_{\alpha\beta}(q_1 - q_2)_\gamma + g_{\beta\gamma}(q_2 - q_3)_\alpha + g_{\gamma\alpha}(q_3 - q_1)_\beta]$$

4-gluon: $\mathcal{L} = -\frac{g^2}{4} f^{abe} f^{cde} A^\mu_a A^\nu_b A^c_\mu A^d_\nu$



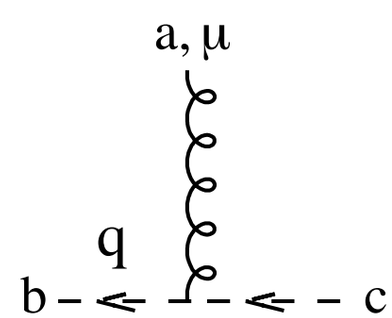
$$\begin{aligned} -ig^2 \,[\; & f^{abe} f^{cde}(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \\ & + f^{ace} f^{bde}(g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma}) \\ & + f^{ade} f^{cbe}(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\beta} g_{\gamma\delta})\Big] \end{aligned}$$

Quark-gluon: $\mathcal{L} = -g\bar{\psi} \not{A}^a\, t^a\, \psi$



$$-\,ig\, t^a_{ij}\, \gamma_\mu$$

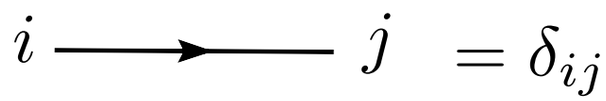
Ghost-gluon: $\mathcal{L} = -g f_{abc}\, \partial\bar{\eta}^a A^b_\mu\, \eta^c$



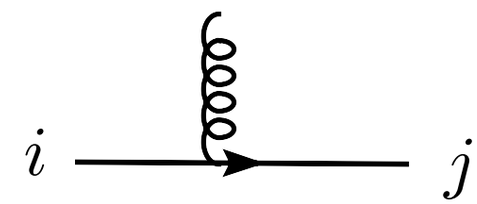
$$-\,ig\, q_\mu(-if_{abc})$$

Color algebra: fundamental identities

Fundamental representation 3:



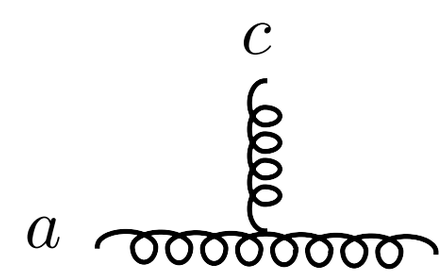
$= \delta_{ij}$



$= t^a_{ij}$

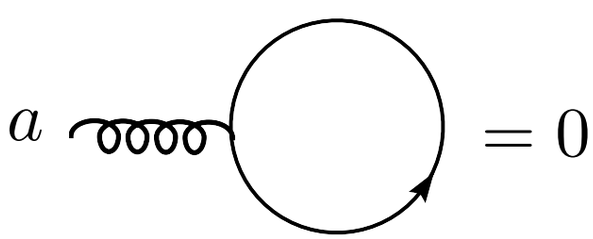
Adjoint representation 8:

$= \delta_{ab}$



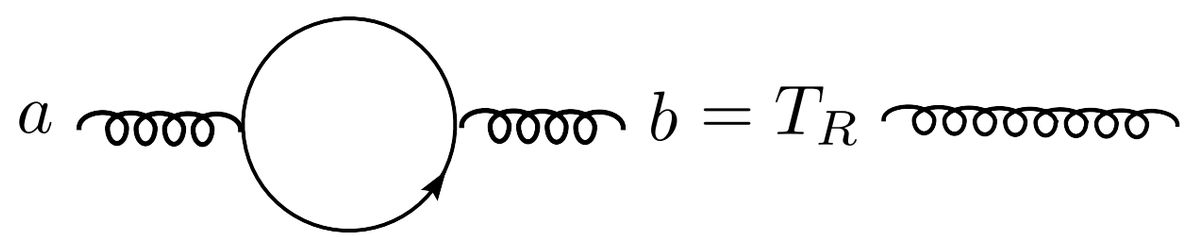
$= i f_{abc}$

Trace identities:



$= 0$

$\mathrm{Tr}(t^a) = 0$



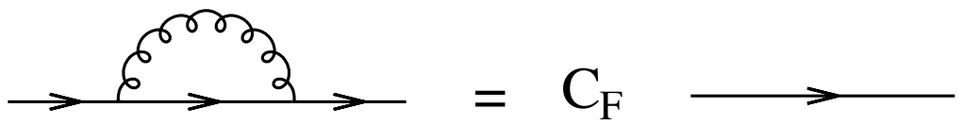
$= T_R$

$\mathrm{Tr}(t^a\, t^b) = \mathrm{T_R}\delta^{ab}$

Color algebra: Casimirs & Fierz identity

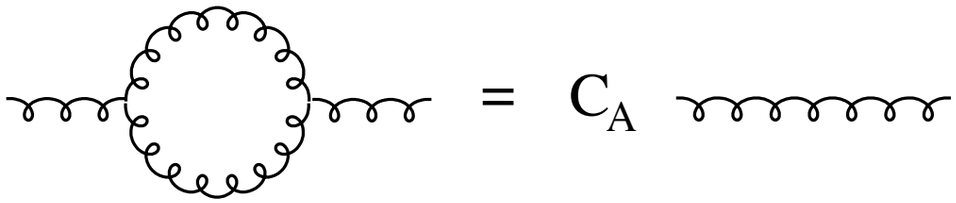
Fundamental representation 3:

$$\sum_a (t^a_{ij})(t^a_{kj}) = C_F \delta_{ij} \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$



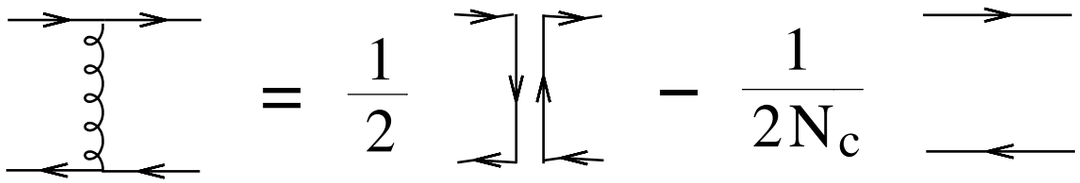
Adjoint representation 8:

$$\sum_{cd} f^{acd} f^{bdc} = C_A \delta^{ab} \qquad C_A = N_c$$



Fierz identity:

$$(t^a)^i_k (t^a)^l_j = \frac{1}{2}\delta^i_j \delta^l_k - \frac{1}{2N_c}\delta^i_k \delta^l_j$$



Color algebra: Casimirs & Fierz identity

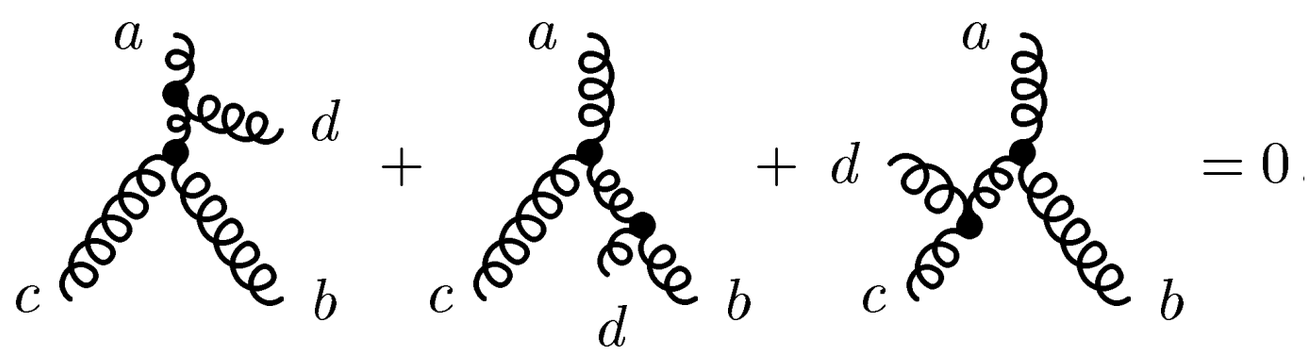
Exercise: from the fact that the identity and t^a form a complete set and using the trace relations for t^a prove the Fierz identity

Exercise: in a similar way show that

$$i f_{abc} = 2\mathrm{Tr}\left([t_a, t_b] t_c\right)$$

More on Color algebra

Jacobi identity: $f^{abe} f^{cde} + f^{bce} f^{ade} + f^{cae} f^{bde} = 0$

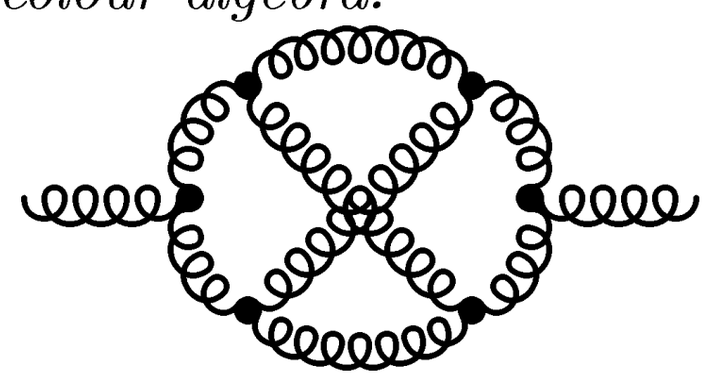


Exercise: using cyclic identity for commutators prove the Jacobi identity

Exercise:

This one is a masterpiece. Giving a proof of the following identity by mere reflection, without performing any calculations *(and even drawings), will show that you have mastered the graphic colour algebra:*

Yu. Dokshitzer

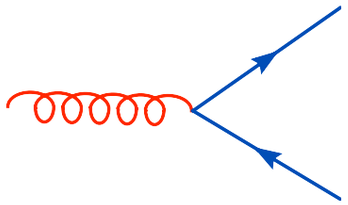


$= 0 .$

The R-ratio: perturbative expansion

To lowest order the R-ratio is given by

$$R_0 = \frac{\sigma_0(\gamma^* \to \text{hadrons})}{\sigma_0(\gamma^* \to \mu^+\mu^-)} = N_c \sum_f q_f^2$$



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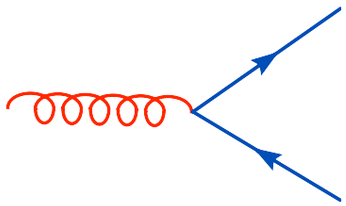

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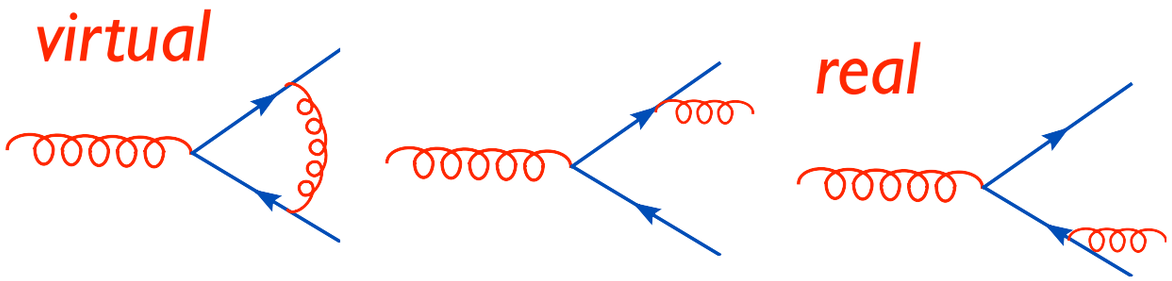
First order correction



Real and virtual do not interfere: different # of legs

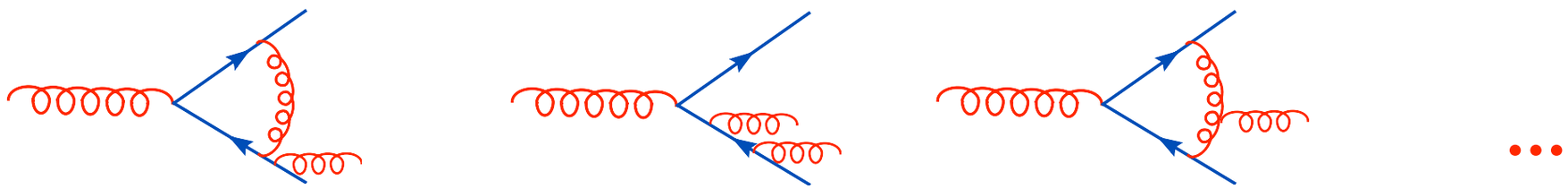
$$|A_1|^2 = |A_0|^2 + \alpha_s \left(|A_{1,r}|^2 + 2\mathrm{Re}\{A_0 A_{1,v}^*\}\right) + \mathcal{O}(\alpha_s^2) \qquad \alpha_s = \frac{g_s^2}{4\pi}$$

First order result reads

$$R_1 = R_0 \left(1 + \frac{\alpha_s}{\pi}\right)$$

R-ratio and UV divergences

Including the second order correction one gets



$$R_2 = R_0 \left(1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi} \right)^2 \left(c + \pi b_0 \ln \frac{M_{\rm UV}^2}{Q^2} \right) \right)$$

$$b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

Ultra-violet divergences do not cancel. Result depends on UV cut-off.

Renormalization and running coupling

The divergence is dealt with by renormalizing the coupling constant

$$\alpha_s(\mu) = \alpha_s^{\rm bare} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\rm bare}\right)^2$$

R expressed in terms of the renormalized coupling is finite

$$R = R_0 \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{\mu^2}{Q^2}\right) + \mathcal{O}(\alpha_s^3(\mu))\right)$$

Renormalizability of the theory guarantees that *the same redefinition of the coupling removes all UV divergences from all physical quantities (massless case)*

Will not cover renomalization in these lectures, but it suffices to know that renormalization of S-matrix elements is achieved by replace bare masses and bare coupling with renormalized ones

- the coupling $\Rightarrow$ β function
- the masses $\Rightarrow$ anomalous dimensions γ_m

Regularization

Regularization: a way to handle divergences, i.e. make divergent quantities meaningful

In QCD dimensional regularization is today the standard procedure

Integrals which would diverge in D=4 are continued to $D=4-2\varepsilon$

Divergence show up us poles in ε

This procedure works both for UV divergences and IR divergences (see later)

Alternative regularization schemes: photon mass (EW), cut-offs, Pauli-Villard ...

The beta-function

$$\beta(\alpha_s^{\rm ren}) \equiv \mu^2 \frac{\alpha_s^{\rm ren}(\mu^2)}{d\mu^2}$$

We immediately get $\beta = -b_0 \alpha_s^2(\mu) + \ldots$

So at lowest order $\frac{1}{\alpha_s(\mu)} = b_0 \ln \frac{\mu^2}{\mu_0^2} + \frac{1}{\alpha_s(\mu_0)} \quad \Longrightarrow \quad \alpha_s(\mu) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$

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The coupling becomes large before and perturbation theory is unreliable

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Technically: Λ is the integration constant in the above formula for α_s.
If one changes the formula, one must change Λ (e.g. if one goes from one to two-loops or if one changes the number of active flavours)

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Question: what is different in QED? Why does nobody talk about Λ_{QED} ?

Renormalization Group Equation

The renormalization scale is arbitrary. Dependence on it must cancel in physical observables up to the order to which one does the calculation.

So, for any observable A one can write a renormalization group equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0 \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \qquad \alpha_s = \alpha_s(\mu^2)$$

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Introduce

$$t = \ln \frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_a(Q^2)} \frac{d\alpha}{\beta(\alpha)} \qquad \left[-\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right] A(e^t, \alpha_s) = 0$$

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Hence $A(1, \alpha_s(Q^2))$ is solution to the RGE equation

All scale dependence of A enters only through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q if the beta-function is known

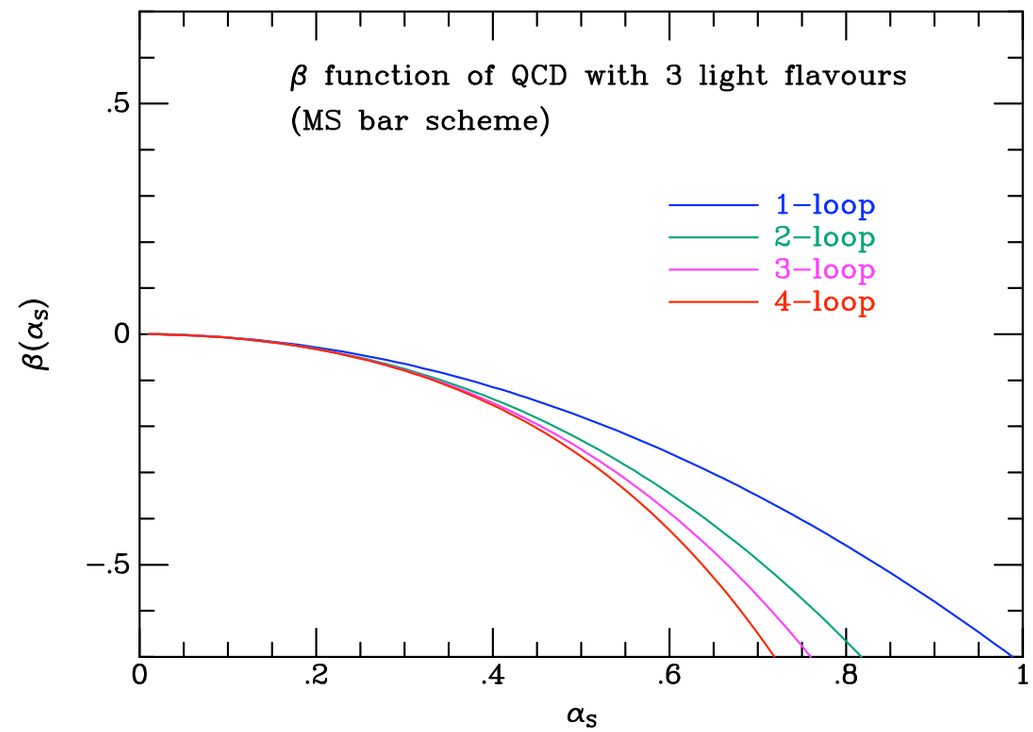
More on the beta-function

Expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu) \sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11N_c - 4n_f T_R}{12\pi}$$

$$b_1 = \frac{17N_c^2 - 5N_c n_f - 3C_F n_f}{24\pi^2}$$



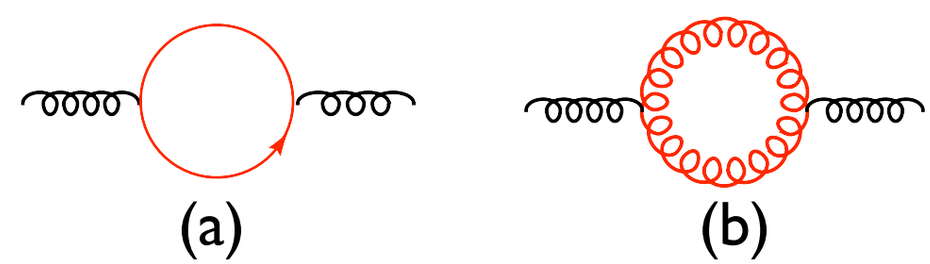
Today, the beta-function known up to four loops, but only first two coefficients are independent of the renormalization scheme

<u>NB</u>:

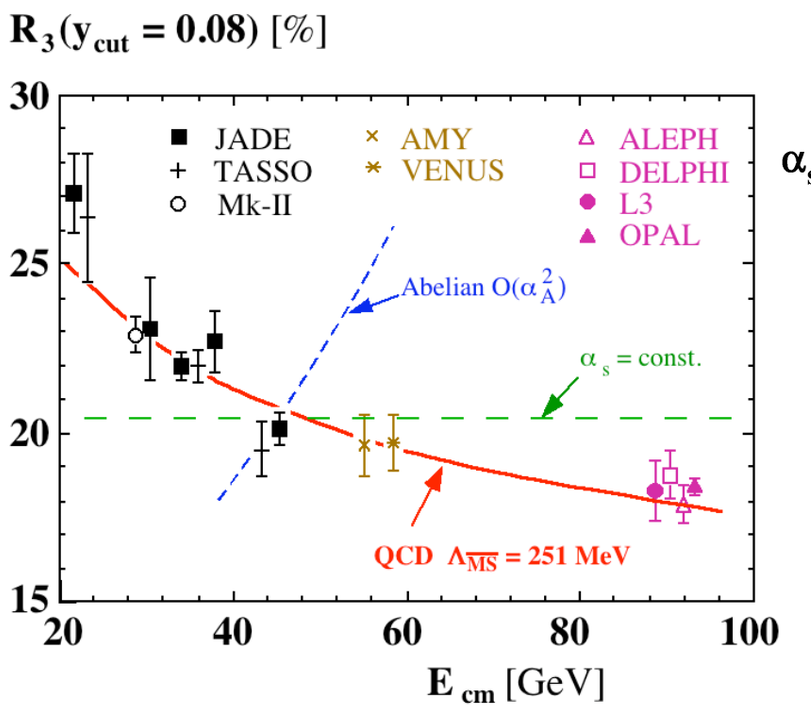
(a) negative contribution to $b_0 \sim n_f$

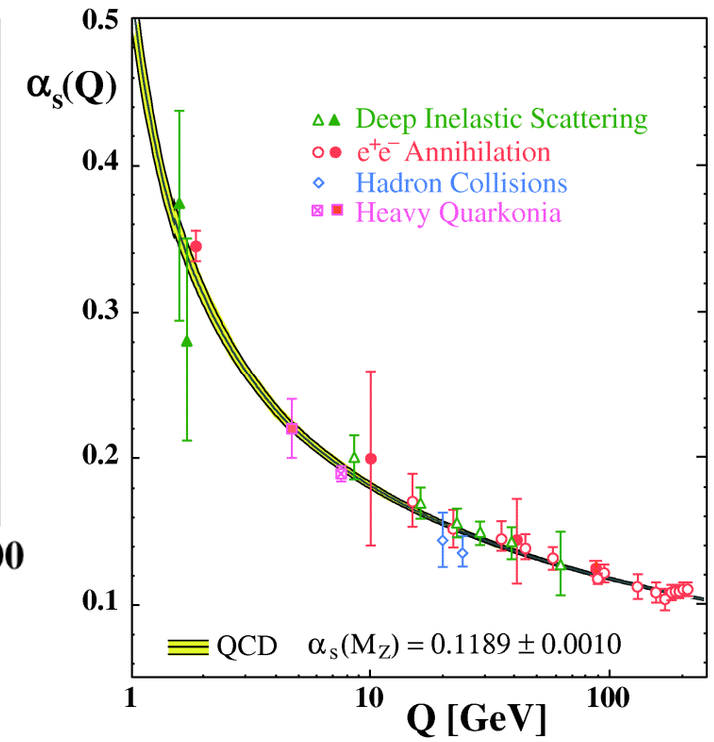
(b) positive contribution to $b_0 \sim N_c$

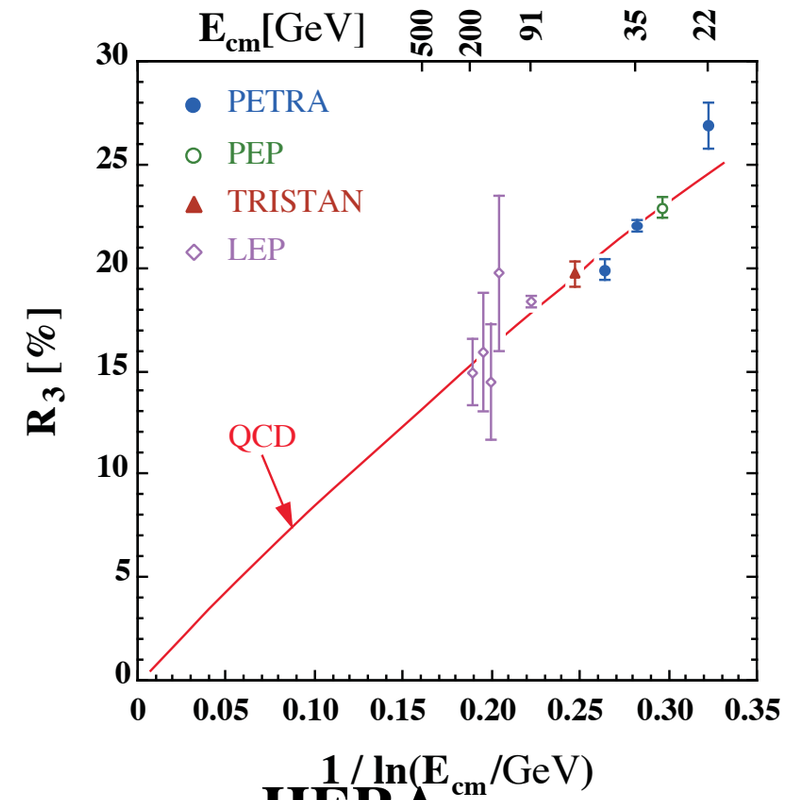
(b) > (a) $\Rightarrow$ $b_0 > 0$
while in QED (b) = 0 $\Rightarrow$ $b_{0,QED} < 0$

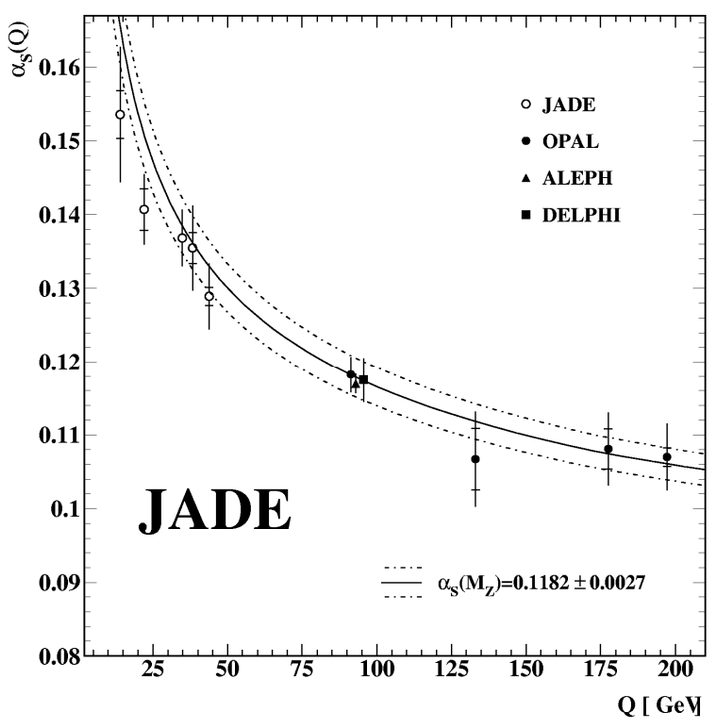


Measurements of the running coupling



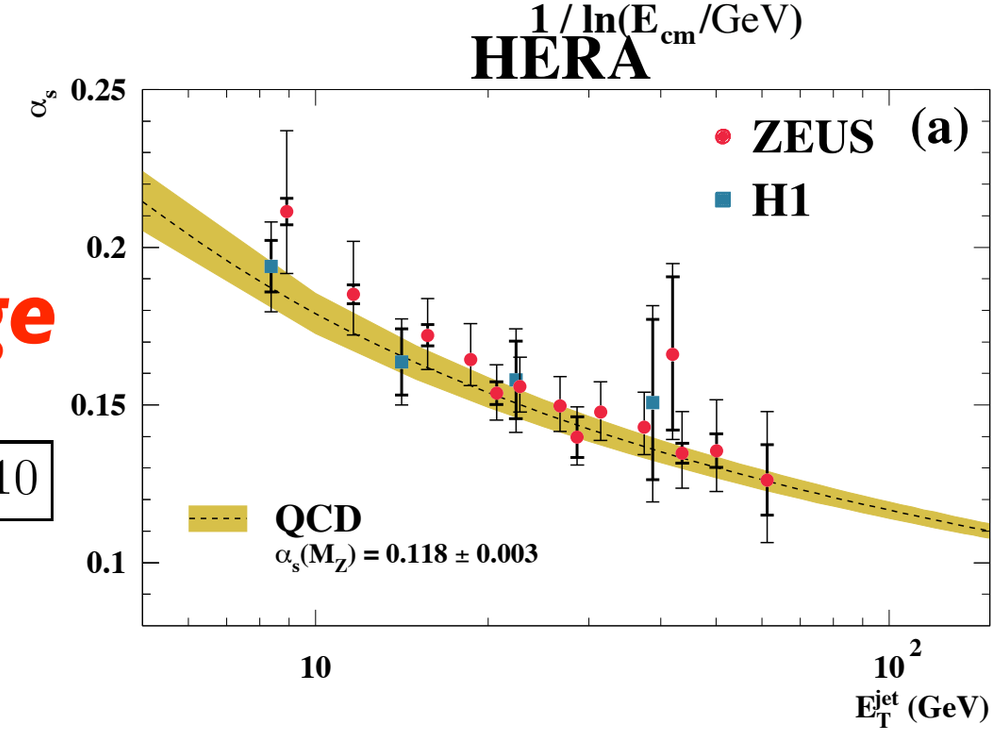






World average

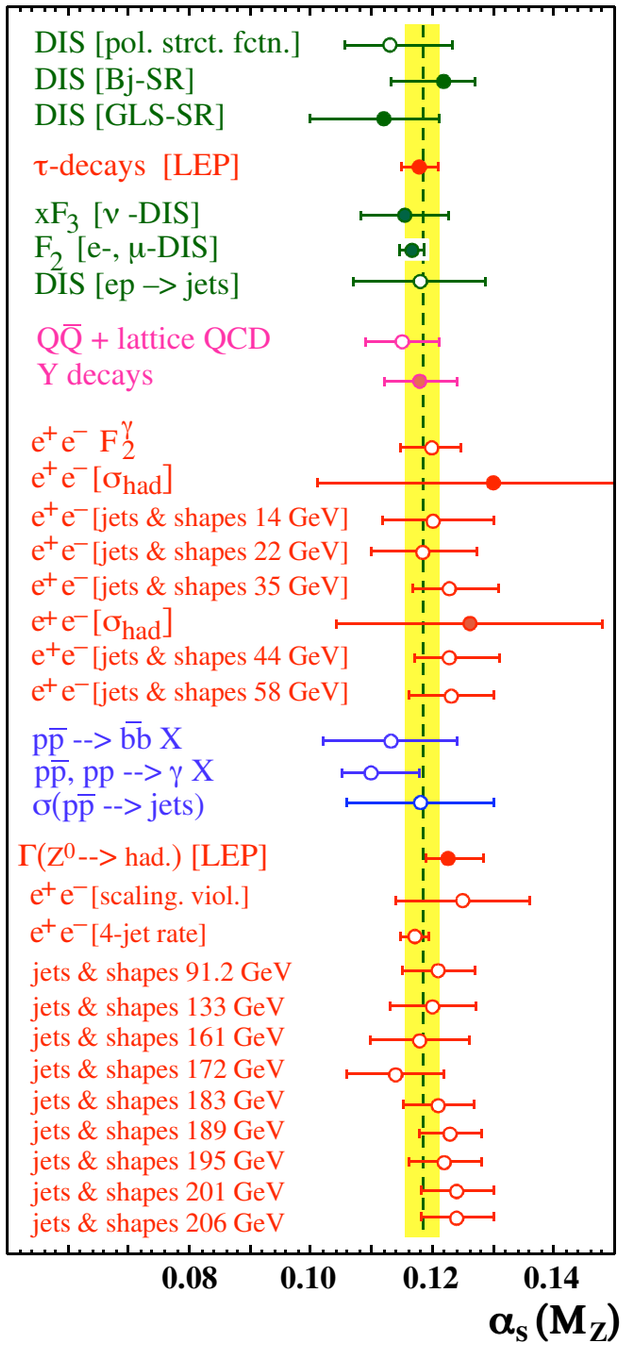
$$\alpha_s(M_{Z^0}) = 0.1189 \pm 0.0010$$



Measurements of the running coupling

<u>To remember:</u>

- overall consistent picture: α_s from very different observables compatible
- α_s is not so small at current scales
- α_s decreases very slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important



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Asymptotic freedom & confinement

Asymptotic freedom:

- coupling smaller at higher energies (smaller distances). Theory becomes effectively free
- a consequence of the sign of the beta function
- perturbation theory predicts asymptotic freedom

Confinement:

- related to the fact that the coupling increases at small energies
- however, the behavior is theoretically unknown because perturbation theory breaks down
- we do not have a rigorous explanation for confinement
- we just observe that all partons are confined into color singlet hadrons
- we assume that confinement always holds. Proof worth a Nobel Price.

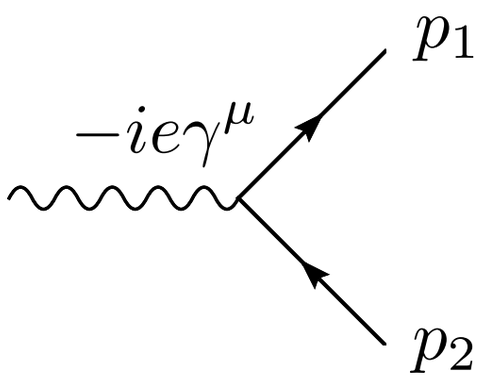
QCD often relies on unproven assumptions

The soft approximation

Let's consider again the R-ratio. This is determined by $\gamma^* \to q\bar{q}$

At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$



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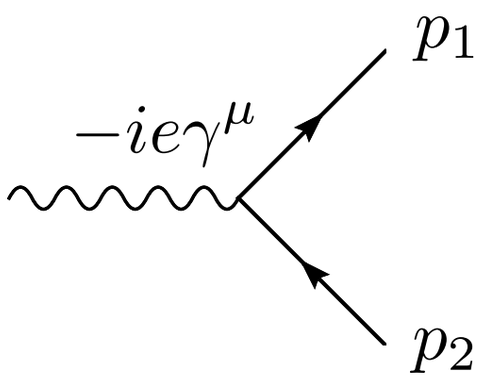
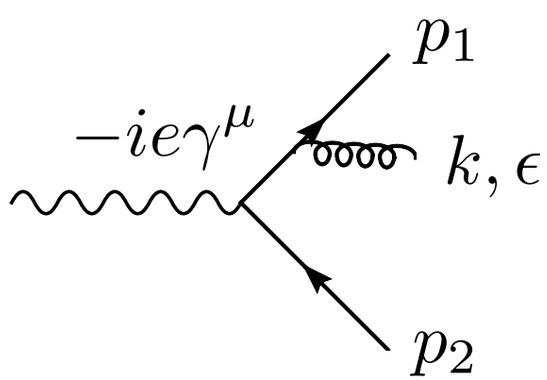


Emit one gluon:

$$\begin{aligned} M_{q\bar{q}g}^\mu &= \bar{u}(p_1)(-ig_s t^a \not{\epsilon})\frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &- \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_2 + \not{k})}{(p_2+k)^2}(-ig_s t^a \not{\epsilon})v(p_2) \end{aligned}$$



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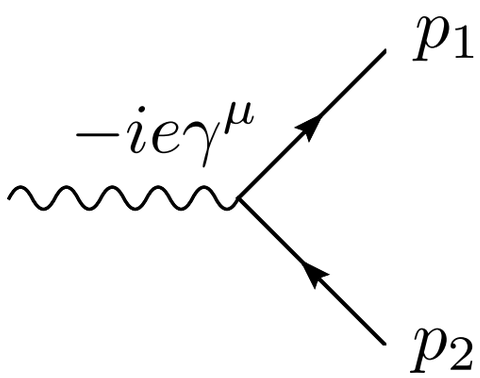

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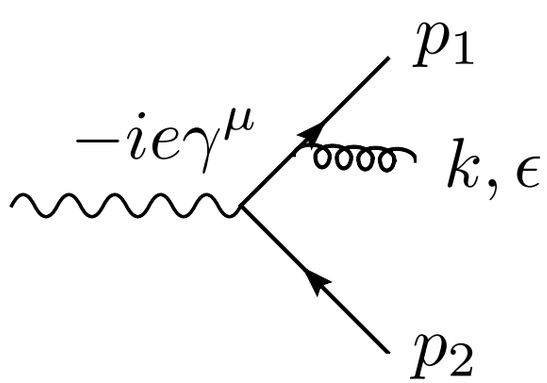

Consider the soft approximation: $k \ll p_1, p_2$

$$M_{q\bar{q}g}^\mu = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

$\Rightarrow$ factorization of soft part

Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1 p_2}{(p_1 k)(p_2 k)}$$

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Including phase space

$$d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 \quad = \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

$$= \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi} \frac{2\alpha_s C_F}{\pi} \frac{1}{\omega^2(1-\cos^2\theta)}$$

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The differential cross section is

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

Soft & collinear divergences

Cross section for producing a qq-pair and a gluon is ∞ (infrared divergent)!

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

<u>$\omega \to 0$</u>: soft divergence

<u>$\theta \to 0$</u>: collinear divergence

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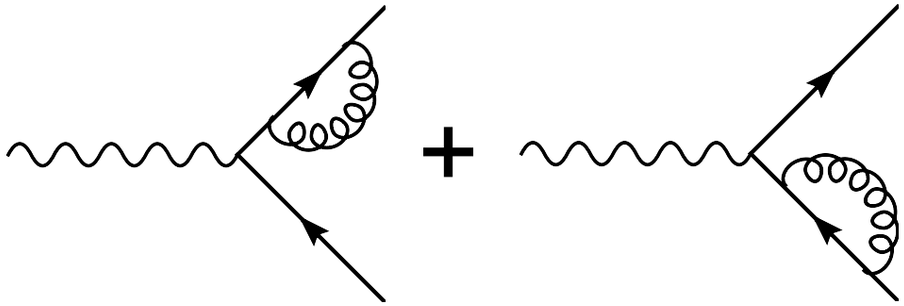
<u>$\omega \to 0$</u>: soft divergence

<u>$\theta \to 0$</u>: collinear divergence

But the full O(α_s) correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets the α_s/π relative correction as before

Soft & collinear divergences

$\omega \to 0$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\theta \to 0$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

Exercise: repeat the above calculation with massive fermions and show that the fermion mass regulates the collinear divergences and that only the infrared divergence survives

Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get ∞ , which can not be renormalized away (by the way: why not?)

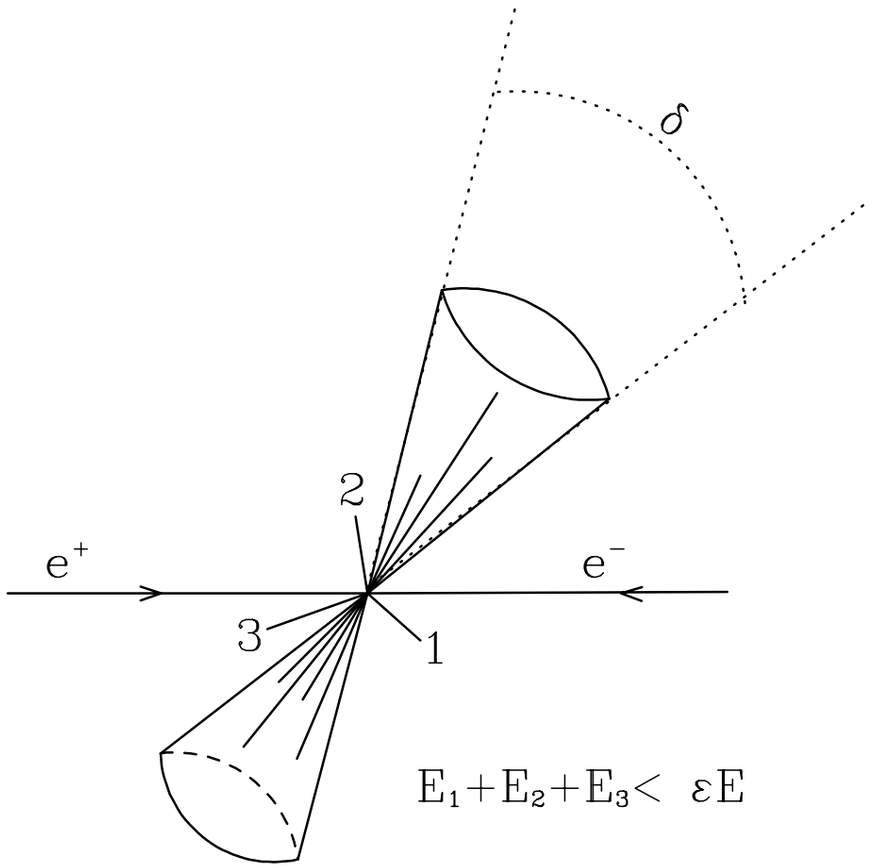
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters ε and δ: a pair of Sterman-Weinberg jets are two cones of opening angle δ that contain all the energy of the event excluding at most a fraction ε



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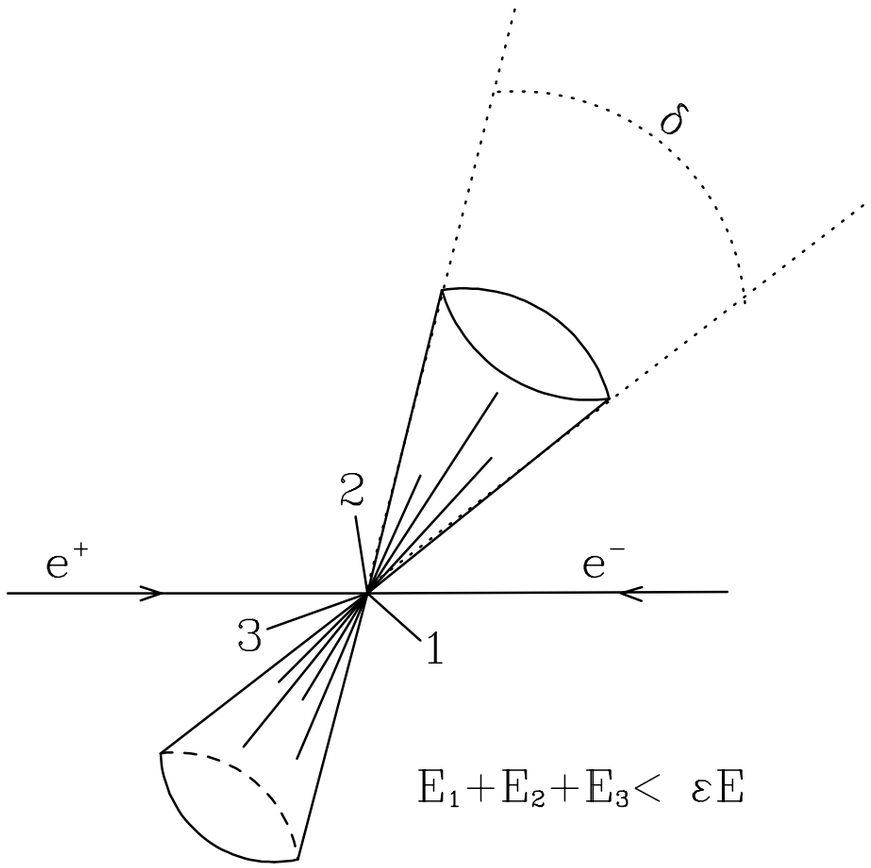

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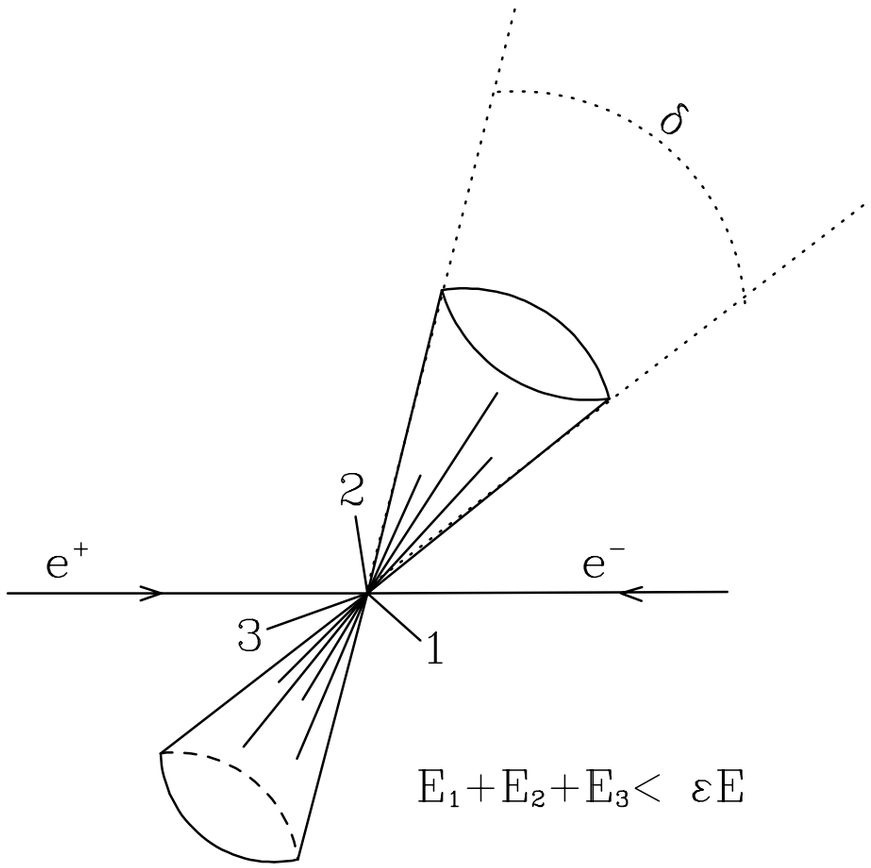

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Kinoshita-Lee-Nauenberg (KLN) theorem:
final-state infrared divergences cancel in measurable quantities (transition probabilities, cross-sections summed over indistinguishable states...)

Sterman-Weinberg jets

Exercise: show that the Sterman-Weinberg jet cross-section up to $O(\alpha_s)$ is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln \epsilon \ln \delta^2 \right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not α_s

α_s-expansion enhanced by a double log: left-over from real-virtual cancellation

NB:

- if ε and/or δ become too small the above result does not make sense
- if more gluons are allowed, one gets for each allowed gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm $\ln\varepsilon$
  - a collinear logarithm $\ln\delta$
- if the logs are large, fixer order meaningless, one needs to resum large infrared and collinear logarithms to all orders in the coupling constant

Infrared safety: definition

An observable $\mathcal{O}$ is infrared and collinear safe if

$$\mathcal{O}_{n+1}(k_1, k_2, \ldots, k_i, k_j, \ldots k_n) \to \mathcal{O}_n(k_1, k_2, \ldots k_i + k_j, \ldots k_n)$$

whenever one of the k_i/k_j becomes soft or k_i and k_j are collinear

i.e. the observable is insensitive to emission of soft particles or to collinear splittings

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IR-safe or not?

- multiplicity of the event
- cross section for producing one gluon with $E > E_{\rm min}$ and $\theta > \theta_{\rm min}$
- cross section for producing any number of gluons each with $E > E_{\rm min}$ and $\theta > \theta_{\rm min}$

Recap

- QCD is in principle a simple theory based on a simple Lagrangian with gauge group is SU(3)
- There are UV divergences but they are dealt with by renormalization (coupling + masses)
- The coupling runs $\Rightarrow$ beta-function. Because the coupling is small-ish at high energies we can use perturbation theory
- The theory is asymptotically free and consistent with confinement
- There are potential infrared and collinear divergences $\Rightarrow$ not all quantities can be computed in PT, but we saw what is the property of observables guaranties IR-finiteness