Effective potential at 3 loops

Stephen P. Martin

Northern Illinois University

LoopFest XVIII

Fermilab, August 14, 2019

Based on 1709.02397, and code written with Dave Robertson:

1907.02500 `SMDR`

1610.07720 `3VIL`

Motivations

The effective potential

$$V_{\text{eff}}(\phi_j) = V^{(0)} + \frac{1}{16\pi^2} V^{(1)} + \frac{1}{(16\pi^2)^2} V^{(2)} + \frac{1}{(16\pi^2)^3} V^{(3)} + \ldots$$

is useful for:

- Addressing (meta-)stability of the Standard Model electroweak vacuum
- Relating the Standard Model VEV to the Lagrangian parameters $m^2,\ \lambda,\ y_t,\ g_3,\ g,\ g',\ \ldots$
  - ⋆ Typically, eliminate m^2 in favor of VEV with high precision.
- Precise treatment of spontaneous symmetry breaking in your favorite New Physics model
  - ⋆ What symmetries are broken?
  - ⋆ What are the scales of VEVs?

I will report on the computation of V_{eff} through full 3-loop order in a general renormalizable theory, and specialization to the Standard Model.

In the Standard Model at tree level:

$$V^{(0)} = m^2|\Phi|^2 + \lambda|\Phi|^4$$

Easy recipe: the rest of the effective potential is computed as the sum of 1-particle-irreducible vacuum (no external legs) Feynman diagrams in Landau gauge, with masses and couplings derived with a constant scalar background field.

In electroweak perturbation theory, expand the Higgs field about the VEV:

$$\Phi = \frac{v}{\sqrt{2}} + H$$

where v is a constant background field of order 246 GeV.

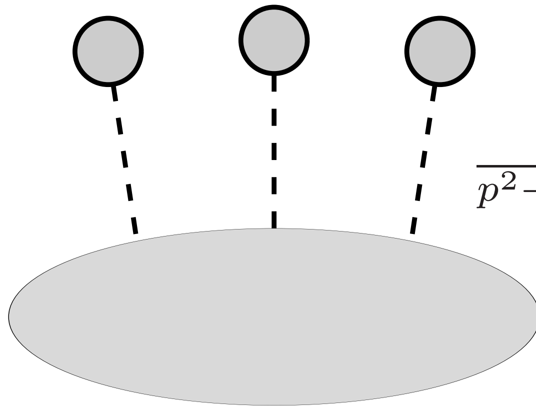
But there are at least two distinct ways that this is commonly done. . .

Two common definitions of the VEV:

- Tree-level VEV: $v_{\text{tree}} = \sqrt{-m^2/\lambda}$
  - Advantage: manifestly gauge-invariant
  - Disadvantage: must include tadpole graphs, perturbation theory includes factors $1/\lambda^n$ at loop order n
- Loop-corrected VEV: $v =$ minimum of the full effective potential
  - Advantage: tadpole graphs vanish, need not be included. Sum of all Higgs tadpoles $\propto \partial V_{\text{eff}}/\partial\phi = 0$.
  - Disadvantage: depends on gauge choice; at 3-loop order, only tractable in Landau gauge. (See SPM and Hiren Patel, 1808.07615, for 2-loop order V_{eff} with general gauge fixing.)

The first definition is often used, but I prefer the “tadpole-free” scheme following from expanding the Higgs field around the VEV v in the second definition.

The problem with tadpoles:



$$\frac{1}{p^2+m_h^2} = \frac{1}{2\lambda v^2}$$

Perturbation theory converges more slowly if one expands the Higgs field around the tree-level VEV. For observables (such as pole masses, G_F, etc.), the leading loop-expansion parameter is

$$\frac{N_c y_t^4}{16\pi^2\lambda} \qquad \text{(expand around } v_{\rm tree}, \text{ need tadpoles)}$$

$$\frac{N_c y_t^2}{16\pi^2} \qquad \text{(expand around } v, \text{ tadpoles vanish).}$$

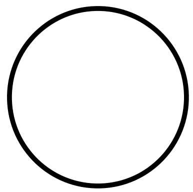
Coleman-Weinberg (1-loop) effective potential in $\overline{\text{MS}}$ scheme:

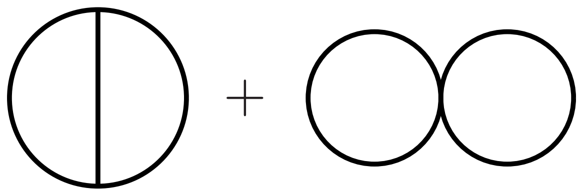
$$V^{(1)}(\phi) = \sum_{\text{real scalars}} \frac{(M^2)^2}{4}\left[\overline{\ln}(M^2) - 3/2\right]$$
$$- 2 \sum_{\text{Weyl fermions}} \frac{(M^2)^2}{4}\left[\overline{\ln}(M^2) - 3/2\right]$$
$$+ 3 \sum_{\text{real vectors}} \frac{(M^2)^2}{4}\left[\overline{\ln}(M^2) - 5/6\right]$$

where $\overline{\ln}(x) = \ln(x/Q^2)$, with $Q =$ renormalization scale, and $M^2 = \overline{\text{MS}}$ squared mass, dependent on background field ϕ.

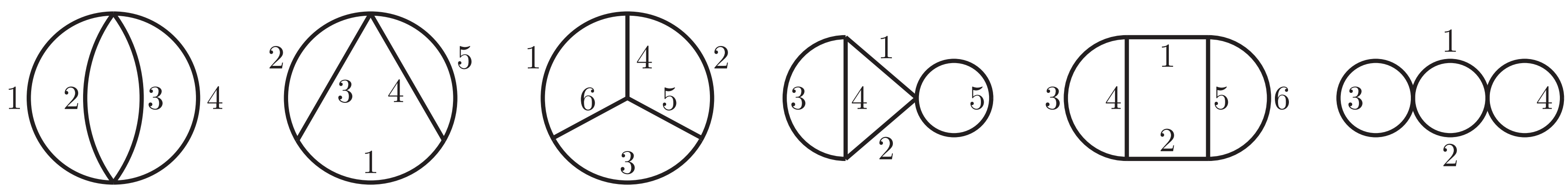
Beyond 1 loop, ϕ also enters through field-dependent couplings...

Topologies of loop corrections to the effective potential:

$V^{(1)} =$  Coleman and E. Weinberg

$V^{(2)} =$  Ford, Jack, Jones hep-ph/0111190

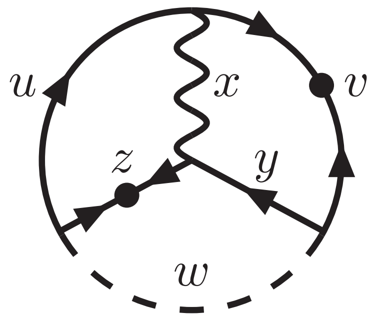
$V^{(3)} =$



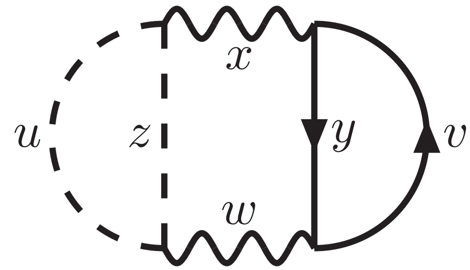
E_{1234} G_{12345} H_{123456} J_{12345} K_{123456} L_{1234}

After taking into account symmetries and gauge invariance, the $V^{(3)}$ for a general renormalizable field theory can be written in terms of 89 loop integrals.

Examples:



$H_{F\overline{F}SVF\overline{F}}(u, v, w, x, y, z)$



$K_{VVSSFF}(x, w, u, z, y, v)$

Propagator labels:

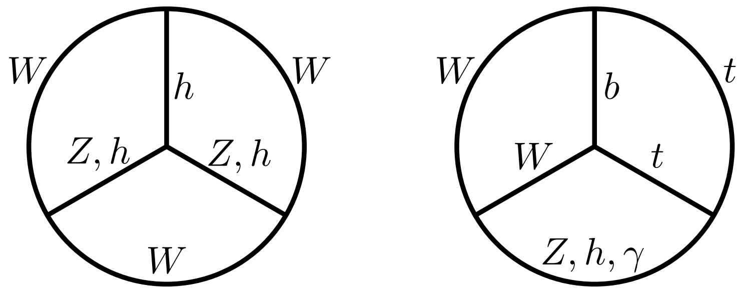
- $S =$ scalar
- $F =$ helicity-preserving fermion
- $\overline{F} =$ helicity-violating fermion (mass insertion)
- $V =$ vector

$$H_{SSSSSS},\ K_{SSSSSS},\ J_{SSSSS},\ G_{SSSSS},\ L_{SSSS},\ E_{SSSS},\ H_{FF\overline{F}SSS},\ H_{\overline{FFF}SSS},$$
$$H_{FFSSFF},\ H_{FFSS\overline{FF}},\ H_{F\overline{F}SSF\overline{F}},\ H_{\overline{FF}SS\overline{FF}},\ K_{SSSSFF},\ K_{SSSS\overline{FF}},\ K_{FFFSSF},$$
$$K_{FF\overline{F}SS\overline{F}},\ K_{\overline{FF}FSSF},\ K_{\overline{F}F\overline{F}SSF},\ K_{\overline{FFF}SS\overline{F}},\ K_{SSFFFF},\ K_{SSFF\overline{FF}},\ K_{SS\overline{FFFF}},$$
$$J_{SSFFS},\ J_{SS\overline{FF}S},\ H_{SSSSSV},\ H_{VVSSSS},\ H_{SSVVSS},\ H_{VVVSSS},\ H_{SSSVVV},$$
$$H_{VVSSVS},\ H_{SSVVVV},\ H_{SVVVSV},\ K_{SSSSSV},\ K_{SSSSVV},\ K_{SSSVVS},\ K_{VVSSSS},$$
$$K_{SSSVVV},\ K_{VVSSVS},\ K_{SSVVVV},\ K_{VVSVVS},\ J_{SSVSS},\ J_{SSVVS},\ G_{VSVVS},$$
$$H_{\text{gauge},\,S},\ K_{\text{gauge},\,S},\ K_{\text{gauge},\,SS},\ H_{FFVVFF},\ H_{FFVV\overline{FF}},\ H_{F\overline{F}VVF\overline{F}},\ H_{\overline{FF}VV\overline{FF}},$$
$$H_{FFFVVV},\ H_{F\overline{FF}VVV},\ K_{FFFVVF},\ K_{FF\overline{F}VV\overline{F}},\ K_{\overline{FF}FVVF},\ K_{\overline{F}F\overline{F}VVF},\ K_{\overline{FFF}VV\overline{F}},$$
$$K_{VVFFFF},\ K_{VVFF\overline{FF}},\ K_{VV\overline{FFFF}},\ K_{\text{gauge},\,FF},\ K_{\text{gauge},\,\overline{FF}},\ H_{FFSVFF},\ H_{FFSV\overline{FF}},$$
$$H_{F\overline{F}SV\overline{F}F},\ H_{F\overline{F}SVF\overline{F}},\ H_{\overline{FF}SV\overline{FF}},\ H_{FFFVSS},\ H_{F\overline{FF}VSS},\ H_{\overline{FF}FVSS},\ H_{F\overline{F}FSVV},$$
$$H_{FF\overline{F}SVV},\ H_{\overline{FFF}SVV},\ K_{FFFSVF},\ K_{FF\overline{F}SV\overline{F}},\ K_{\overline{FF}FSVF},\ K_{\overline{F}F\overline{F}SVF},\ K_{\overline{F}FFSV\overline{F}},$$
$$K_{\overline{FFF}SV\overline{F}},\ K_{SSSVFF},\ K_{SSSV\overline{FF}},\ K_{SSVVFF},\ K_{SSVV\overline{FF}},\ K_{VVSSFF},$$
$$K_{VVSS\overline{FF}},\ K_{VVSVFF},\ K_{VVSV\overline{FF}},\ H_{\text{gauge}},\ K_{\text{gauge}}.$$

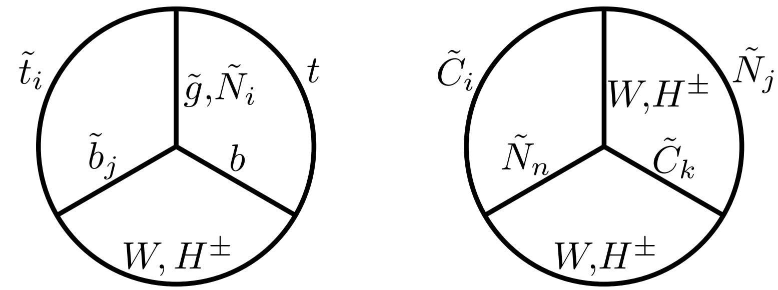
These 89 functions each depend on squared mass arguments $x, y, z, \ldots$ and the $\overline{\text{MS}}$ renormalization scale Q. Most are far too lengthy to be given in print, so are provided in an electronic file in terms of 3-loop basis vacuum integrals, which have to be computed numerically.

Need to be able to systematically compute hundreds of integrals, for example:

Standard Model



Supersymmetry

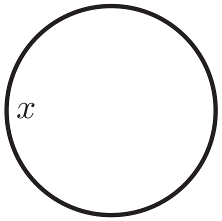


In Standard Model case, the mass hierarchies are not all large.

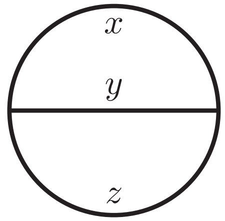
In SUSY cases, the mass hierarchies not known in advance.

Reduce to basis ("master") integrals, compute numerically using differential equations in the squared mass arguments.

The basis integrals for 3-loop vacuum diagrams are:



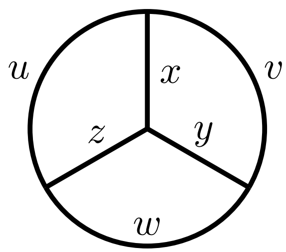
$\mathbf{A}(x)$



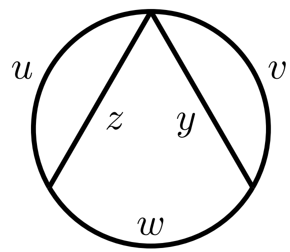
$\mathbf{I}(x, y, z)$

Known analytically, present no problems.

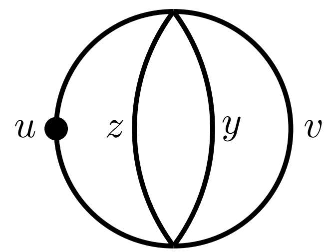
and genuinely three-loop scalar integrals:



$\mathbf{H}(u, v, w, x, y, z)$



$\mathbf{G}(w, u, z, v, y)$



$\mathbf{F}(u, v, y, z)$

which are known in 1-scale and some 2-scale special cases, but in general require numerical computation.

Any 3-loop vacuum integral can be written as linear combinations of these, with coefficients that are rational functions of the squared masses, obtained using integration-by-parts identities.

The generic case: consider the master tetrahedral topology, and all corresponding basis integrals obtained by removing propagator lines:

$$H(u, v, w, x, y, z),$$
$$G(w, u, z, v, y),\ \ G(x, u, v, y, z),\ \ G(u, v, x, w, z),$$
$$G(y, v, w, x, z),\ \ G(v, u, x, w, y),\ \ G(z, u, w, x, y),$$
$$F(w, u, x, y),\ \ F(w, v, x, z),\ \ F(x, u, w, y),\ \ F(x, v, w, z),$$
$$F(u, v, y, z),\ \ F(u, w, x, y),\ \ F(y, u, v, z),\ \ F(y, u, w, x),$$
$$F(v, u, y, z),\ \ F(v, w, x, z),\ \ F(z, u, v, y),\ \ F(z, v, w, x),$$

products of I and A functions

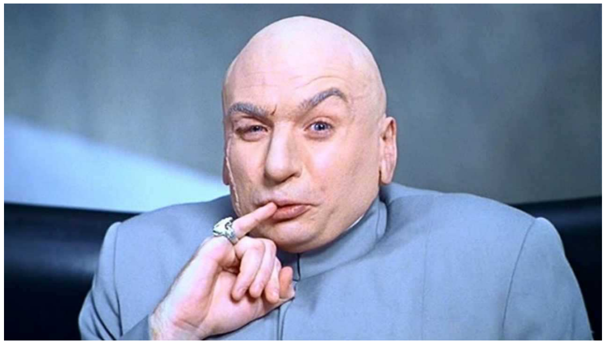
The derivatives of all of these with respect to any squared mass argument u, v, w, x, y, z are also 3-loop integrals, and so are linear combinations of the basis.

Solve differential equations in the masses to compute these using Runge-Kutta, starting from known analytical values at a fixed reference squared mass a as initial conditions:

$$H(a, a, a, a, a, a),\ \ G(a, a, a, a, a),\ \ F(a, a, a, a),\ \ I(a, a, a),\ \ A(a).$$

3VIL = 3-loop Vacuum Integral Library

SPM and Dave Robertson, 1610.07720



- Written in C, can be called from C, C++, Fortran, . . .
- Uses analytic results where available, otherwise differential equations method
- Evaluation for generic mass inputs:
  - Time < 1 second for generic cases on reasonably modern hardware
  - Relative accuracy $\lesssim 10^{-10}$
  - When computing a basis integral $H(u, v, w, x, y, z)$, simultaneously computes all subordinate basis integrals formed by removing propagators.
- See also TVID (S. Bauberger and A. Freitas), uses dispersion relations.

In the Standard Model, the field-dependent squared masses that enter into the computation of $V_{\rm eff}$ are:

$$\begin{aligned} t &= y_t^2 v^2/2, & W &= g^2 v^2/4, & Z &= (g^2 + g'^2) v^2/4, \\ H &= m^2 + 3\lambda v^2, & G &= m^2 + \lambda v^2. \end{aligned}$$

A problem: the Goldstone boson squared mass G can be very small, or negative.

1) If $G < 0$, then $V_{\rm eff}$ is complex even at 1-loop, due to terms with $\overline{\ln}(G)$.

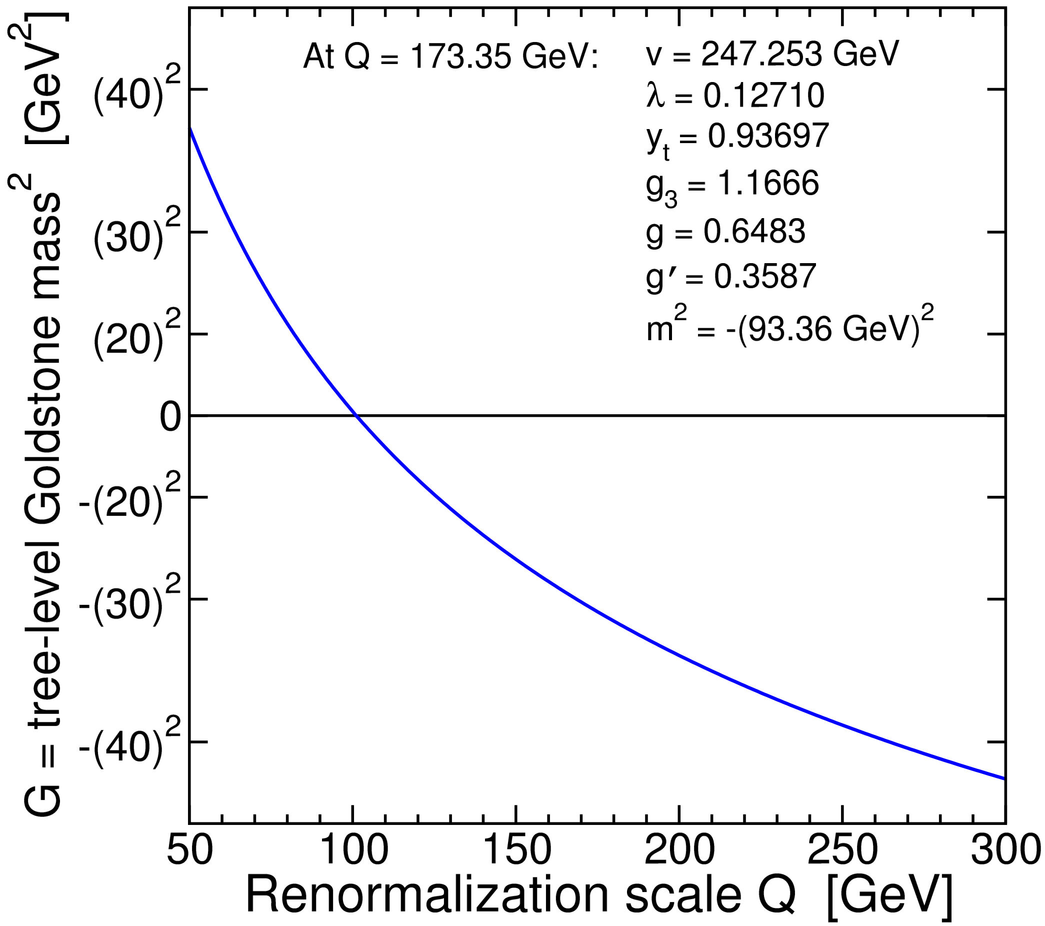
Usually, a complex $V_{\rm eff}$ means instability, but there is no physical instability here.

2) If $G \to 0$, then starting at 3-loop order, get infrared divergences in $V_{\rm eff}$.

Need to make sure IR divergences do not infect physical observables.

Goldstone boson tree-level $(\text{mass})^2$

G as a function of renormalization scale Q, at minimum of V_{eff}:



For $Q \gtrsim 100$ GeV, we really do have tachyonic Goldstones: $G < 0$.

The Goldstone Boson Catastrophe

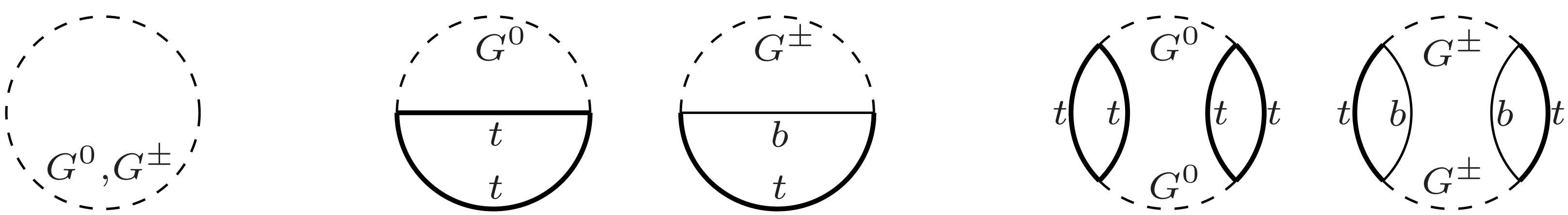
The leading behavior as $G \to 0$ is:

$$V^{(1)} \sim \frac{3}{4} G^2 \,\overline{\ln} G, \qquad \text{2nd derivative singular as } G \to 0$$

$$V^{(2)} \sim -3 N_c y_t^2 t \left(\overline{\ln} t - 1\right) G \,\overline{\ln} G, \qquad \text{1st derivative singular as } G \to 0$$

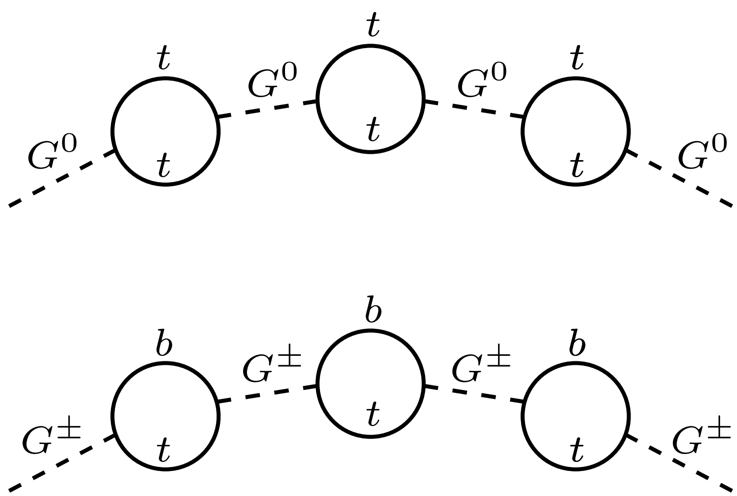
$$V^{(3)} \sim 3 \left[N_c y_t^2 t (\overline{\ln} t - 1)\right]^2 \,\overline{\ln} G. \qquad \text{singular as } G \to 0$$

with $t, G =$ squared masses of top, Goldstone. These come from diagrams:



At higher loop orders, the $G \to 0$ singularities get worse. . .

From ℓ-loop diagrams with $\ell - 1$ top or top/bottom one-loop subdiagrams:



For $\ell \geq 4$,

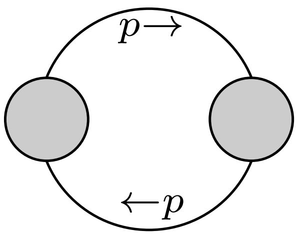
power-law singularity as $G \to 0$.

$$V^{(\ell)} \sim (N_c y_t^2)^{\ell-1} t^2 \left(\frac{t}{G}\right)^{\ell-3}$$

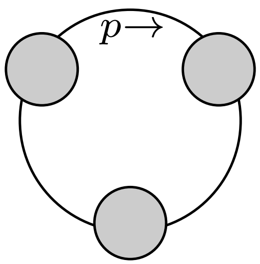
Since G is unphysical, one expects that these singularities will not affect physical observables. Indeed, resummation of the Goldstone propagators to all orders eliminates the problem. (SPM 1406.2355; Elias-Miro, Espinosa, Konstandin 1406.2652)

More generally, starting at 3-loop order, there is a problem to worry about in $V_{\rm eff}$ that doesn't occur at lower loop orders: IR divergences from doubled propagators of massless fields.

Regulating the IR divergences using an infinitesimal squared mass δ:



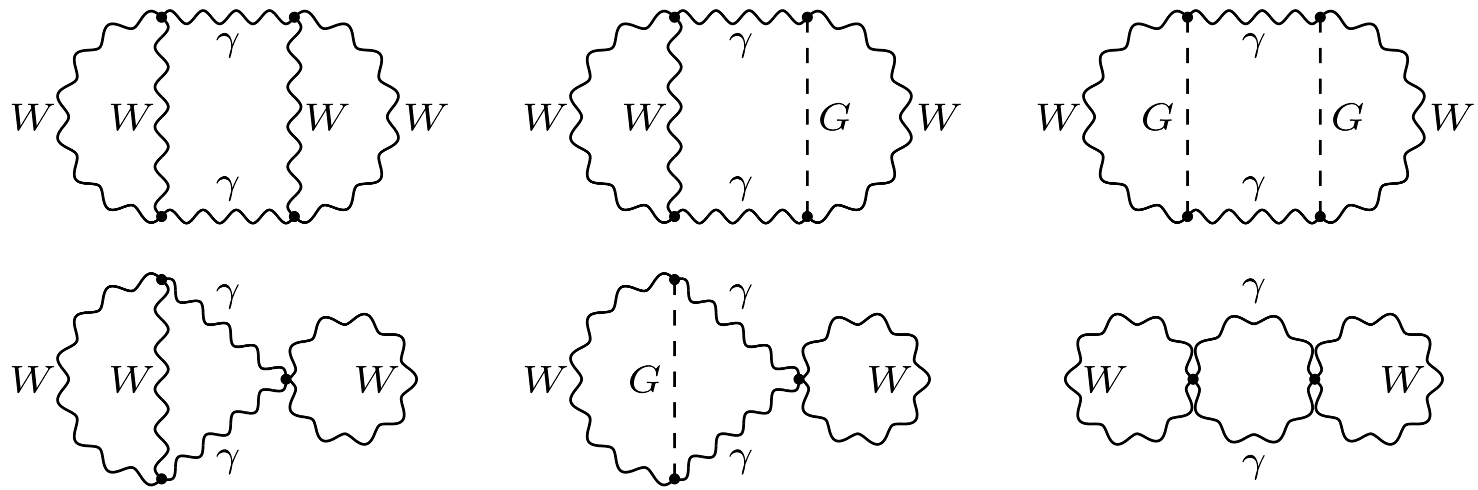
$$\sim \int \frac{d^D p}{(p^2+\delta)^2} \sim \overline{\ln}(\delta) = \text{log IR divergence}$$



$$\sim \int \frac{d^D p}{(p^2+\delta)^3} \sim \frac{1}{\delta} = \text{linear IR divergence}$$

I've checked that when the massless propagators are gluons, and the blobs are quark loops, the IR divergences cancel at each of 3-loop and 4-loop orders.

For doubled photons, the IR divergences again cancel, EXCEPT for those due to the following diagrams:



The total, with photon IR regulator mass δ, is:

$$V^{(3)} \quad \sim \quad \frac{27e^4}{16}\left(\frac{WG\ln(W/G)}{W-G}\right)^2 \overline{\ln}(\delta)$$

This is a genuine, uncancelled IR divergence in the 3-loop Standard Model effective potential! In retrospect, perhaps not so surprising; $V_{\rm eff}$ is not a physical observable. This IR divergence does not infect the VEV or its minimization condition, after Goldstone boson resummation, essentially because it's **quadratic** in G.

After resummation, the Goldstone boson propagators completely decouple! This means that the equation determining the VEV is:

$$v^2 = -\frac{1}{\lambda}\left(m^2 + \sum_{\ell=1}^{\infty} \frac{1}{(16\pi^2)^\ell} \Delta^{(\ell)}\right)$$

where the corrections $\Delta^{(\ell)}$ do not depend at all on the Goldstone boson squared mass G, or on the Lagrangian Higgs squared mass parameter m^2. Each $\Delta^{(\ell)}$ is found in terms of ℓ-loop order vacuum integrals for ℓ up to 3, provided in an electronic file.

In addition, Dave Robertson and I have incorporated these results into a public code:

SMDR = Standard Model in Dimensional Regularization

Written in C, includes 3VIL, can be called from C++, Fortran, etc.

But wait! There's more. For no extra charge, it also does. . .

SMDR provides a complete map between Standard Model $\overline{\text{MS}}$ inputs in the tadpole-free scheme and observable outputs, using all known loop effects.

Tadpole-free pure $\overline{\text{MS}}$ inputs:

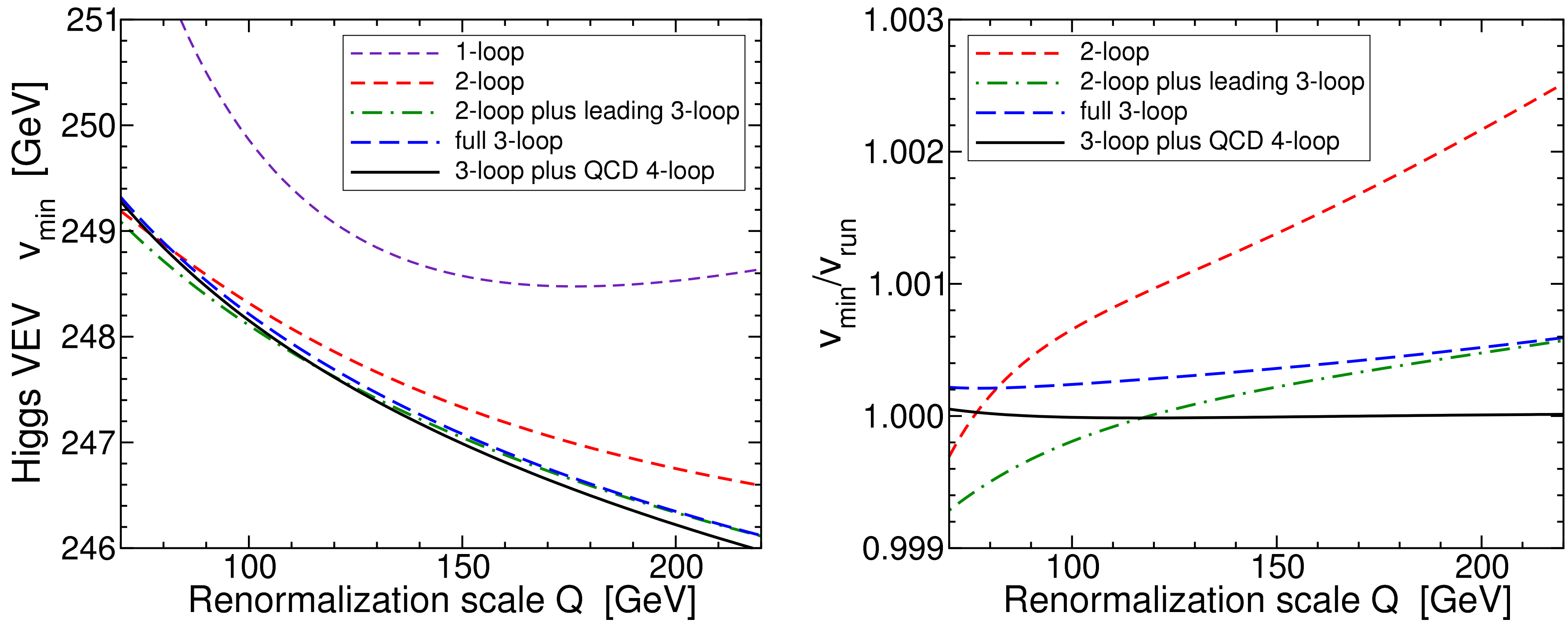
$$Q,\, v,\, \lambda,\, g_3,\, g,\, g',\, y_t,\, y_b,\, y_c,\, y_s,\, y_d,\, y_u,\, y_\tau,\, y_\mu,\, y_e,\, \Delta\alpha^{(5)}_{\text{had}}(M_Z)$$

$\updownarrow$ **SMDR**

On-shell observable outputs:

heavy particle pole masses:	$M_t,\, M_h,\, M_Z,\, M_W,$
running light quark masses:	$m_b(m_b),\, m_c(m_c),\, m_s(2\text{ GeV}),\, m_d(2\text{ GeV}),\, m_u(2\text{ GeV}),$
lepton pole masses:	$M_\tau,\, M_\mu,\, M_e,$
5-quark QCD coupling:	$\alpha_S^{(5)}(M_Z),$
Fermi constant:	$G_F = 1.1663787\ldots \times 10^{-5}\text{ GeV}^{-2},$
fine structure constant:	$\alpha = 1/137.035999139\ldots$ and $\Delta\alpha^{(5)}_{\text{had}}(M_Z)$

Scale-dependence of VEV v, for a typical set of input parameters specified at $Q = M_{\rm top}$:

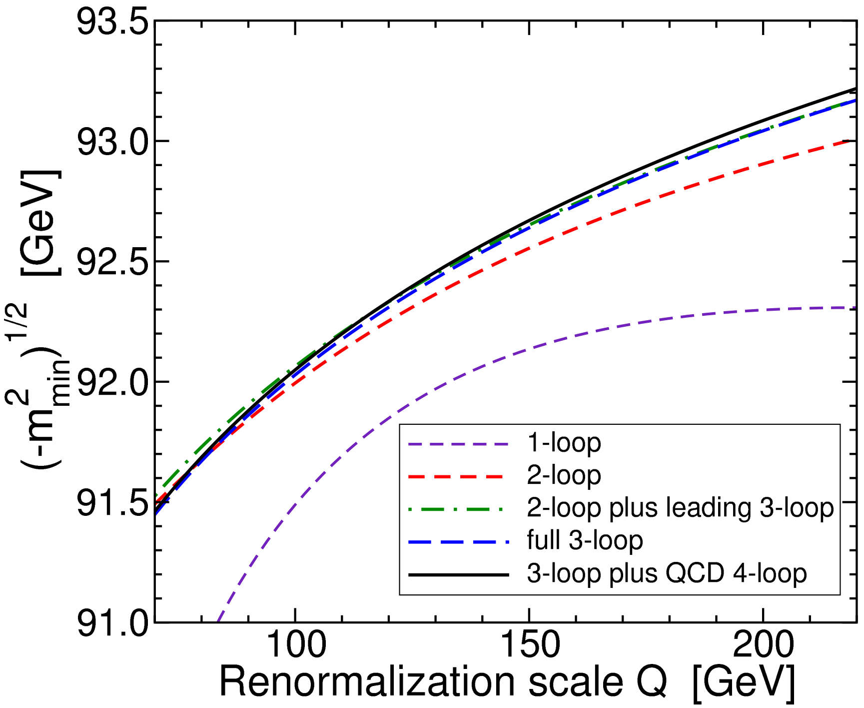


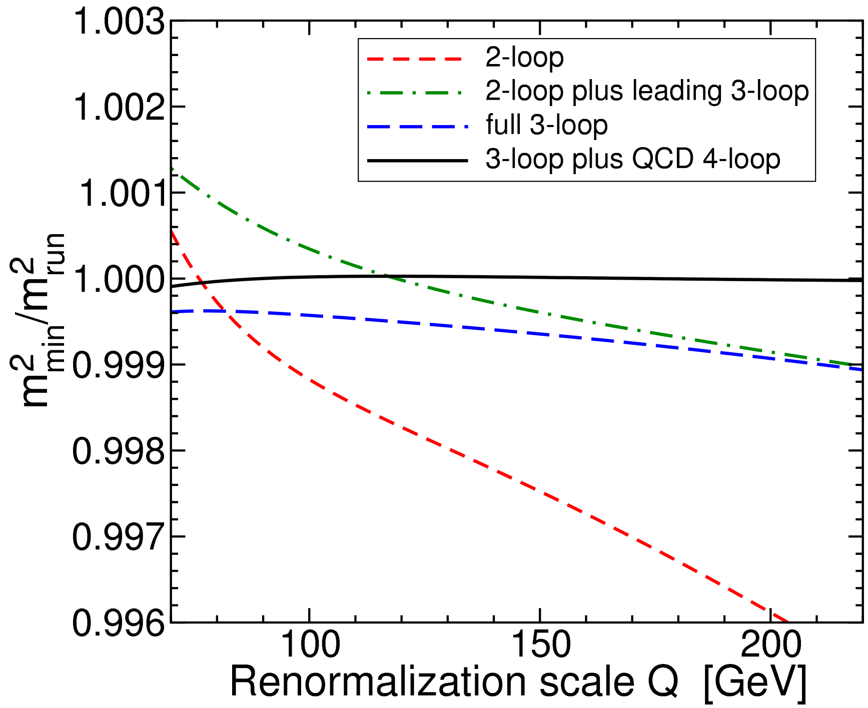
Right graph shows ratio of v to value $v_{\rm run}$ obtained by RG running from the input scale.

For 100 GeV $< Q <$ 200 GeV, RG scale dependence $< 1 \times 10^{-5}$, so about $\Delta v = \pm 2$ MeV.

Conservatively, purely theoretical error might be an order of magnitude larger, $\sim$20 MeV.

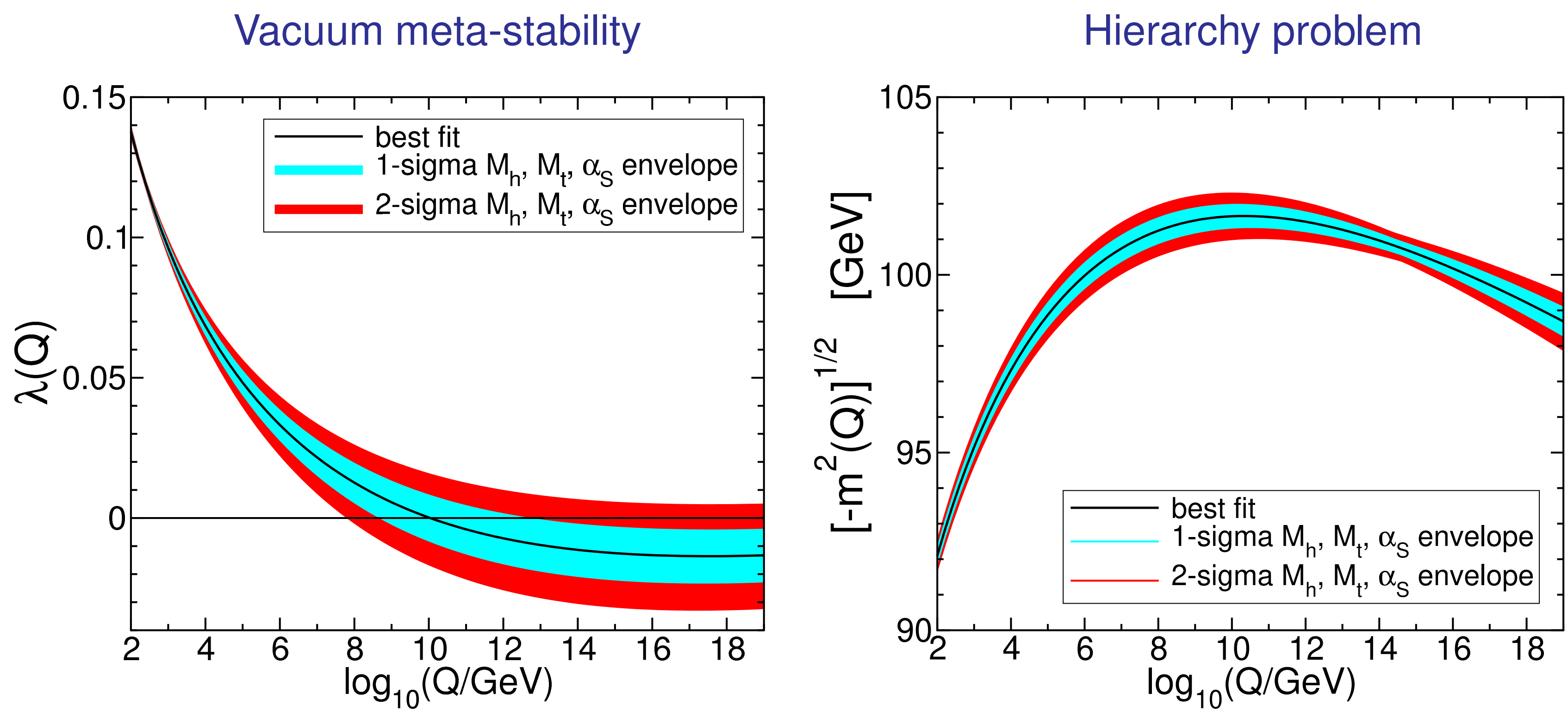
Converse of previous slide: take v as given at $Q = M_{\rm top}$, require $V_{\rm eff}$ to be minimized to obtain m^2:





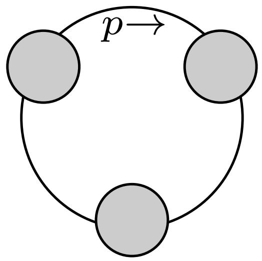
Note the small RG scale dependence shown above involves only purely theoretical sources of error.

Trade knowledge of the Higgs pole mass M_h and the Higgs VEV for the $\overline{\text{MS}}$ parameters m^2 and λ. Including all 3-loop V_{eff} and leading 3-loop M_h corrections, and parametric errors from M_t, α_S, and M_h:



Could there be a “Massless Fermion Catastrophe” in the effective potential?

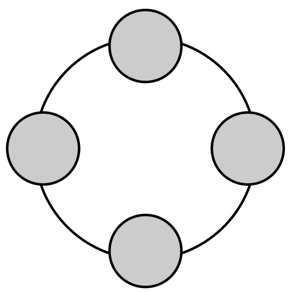
At three loops:



$$\sim \int \frac{d^D p}{\left(\not{p} - \sqrt{\delta}\right)^3} \sim \sqrt{\delta}.$$

No! Also checked explicitly.

At four loops:



$$\sim \int \frac{d^D p}{\left(\not{p} - \sqrt{\delta}\right)^4} \sim \overline{\ln}(\delta).$$

Maybe?

However, I believe there won’t be a “Goldstino Catastrophe” (IR divergence for massless fermions due to spontaneously broken global SUSY), at any loop order, because a Goldstino will have only derivative interactions.

In any case, on physical grounds I expect that, if present, this will not infect physical observables. Might require resummation?

Outlook

- Effective potential found in $\overline{\text{MS}}$ scheme to full 3-loop order in general renormalizable field theory, and specialized to Standard Model.
- 4-loop corrections at leading order in QCD also known (SPM, 1508.00912).
- Infrared divergence in V_{eff} at 3 loops from doubled photon propagators, but doesn't infect physical observables after Goldstone boson resummation of higher-order contributions.
- Not applicable to SUSY gauge theories yet, because $\overline{\text{MS}}$ explicitly violates SUSY when gauge vector supermultiplets are present.
  More work to be done; find $V^{(3)}$ in the SUSY-respecting $\overline{\text{DR}}$ scheme.
- See backup slides for more SMDR propaganda plots. All plots in this talk are made with the SMDR code, using example programs included with the distribution. Download it here:
  `http://www.niu.edu/spmartin/SMDR/`

BACKUP: SMDR Propaganda

A benchmark model point in parameter space, defined by a fit to central values from the Particle Data Group's 2018 Review of Particle Properties:

$$M_t = 173.1\text{ GeV}, \qquad M_h = 125.1\text{ GeV},$$

$$M_{Z,\,\text{Breit-Wigner}} = 91.1876\text{ GeV},$$

$$G_F = 1.1663787 \times 10^{-5}\ \mathrm{GeV}^2,$$

$$\alpha = 1/137.035999139,$$

$$\alpha_S^{(5)}(M_Z) = 0.1181,$$

$$m_b(m_b) = 4.18\text{ GeV}, \qquad m_c(m_c) = 1.27\text{ GeV},$$

$$m_s(2\text{ GeV}) = 0.093\text{ GeV} \quad m_d(2\text{ GeV}) = 0.00467\text{ GeV}, \quad m_u(2\text{ GeV}) = 0.00216\text{ GeV},$$

$$M_\tau = 1.77686\text{ GeV}, \qquad M_\mu = 0.1056583745\text{ GeV}, \qquad M_e = 0.000510998946\text{ GeV},$$

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z) = 0.02764$$

The corresponding benchmark values for the $\overline{\text{MS}}$ parameters are found to be:

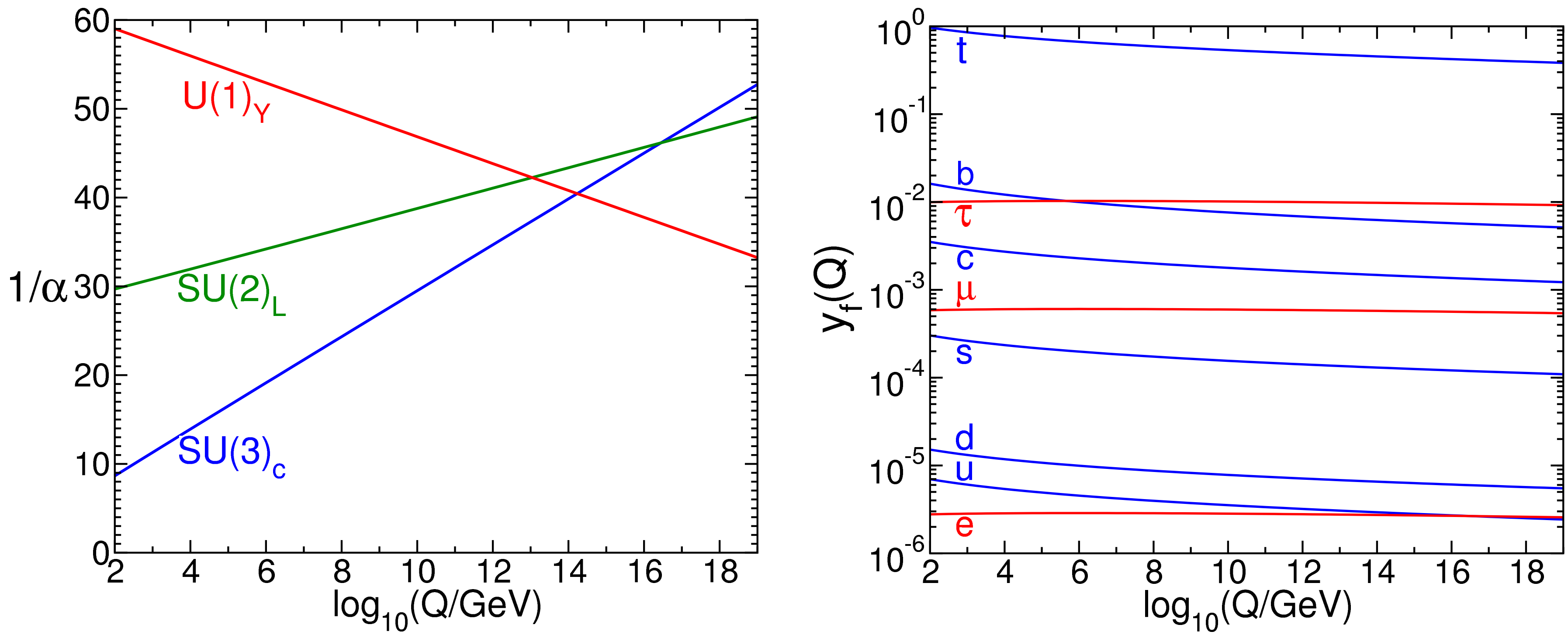
Benchmark input $\overline{\text{MS}}$ parameters:

$$
\begin{aligned}
Q_0 &= 173.1\,\text{GeV}, \\
v(Q_0) &= 246.601\,\text{GeV}, \qquad \lambda(Q_0) \;=\; 0.126038, \\
g_3(Q_0) &= 1.163624, \qquad g_2(Q_0) \;=\; 0.647660, \qquad g'(Q_0) \;=\; 0.358539, \\
y_t(Q_0) &= 0.934801, \qquad y_b(Q_0) \;=\; 0.0154801, \qquad y_\tau(Q_0) \;=\; 0.00999444, \\
y_c(Q_0) &= 0.0033820, \qquad y_s(Q_0) \;=\; 0.000290945, \qquad y_\mu(Q_0) \;=\; 0.000588380, \\
y_d(Q_0) &= 1.46098 \times 10^{-5}, \qquad y_u(Q_0) \;=\; 6.72278 \times 10^{-6}, \\
y_e(Q_0) &= 2.7929820 \times 10^{-6}.
\end{aligned}
$$

SMDR (incorporating state-of-the-art theory) allows you to:

- change the RPP observables, and find the corresponding best-fit $\overline{\text{MS}}$ parameters, or vice versa.
- compute individual observables in terms of the $\overline{\text{MS}}$ inputs
- do renormalization group running and decoupling of $\overline{\text{MS}}$ parameters

Run gauge couplings and Yukawa couplings from the input scale up to very high scales:



Uses 5-loop QCD and 3-loop for everything else.

Note: the Review of Particle Properties definitions of $\overline{\text{MS}}$ electroweak couplings $\widehat{\alpha}(M_Z)$ and $\widehat{s}^2_W(M_W)$ decouple the top quark, but not the W boson.

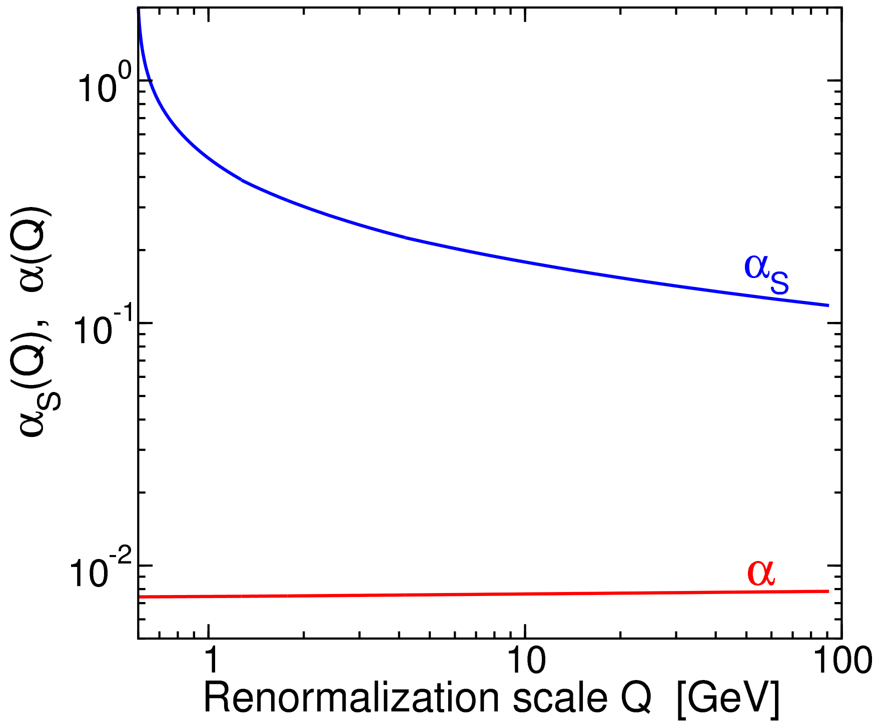
Not actually $\overline{\text{MS}}$ couplings in the usual sense, since the effective theory with a massive W boson is not renormalizable.

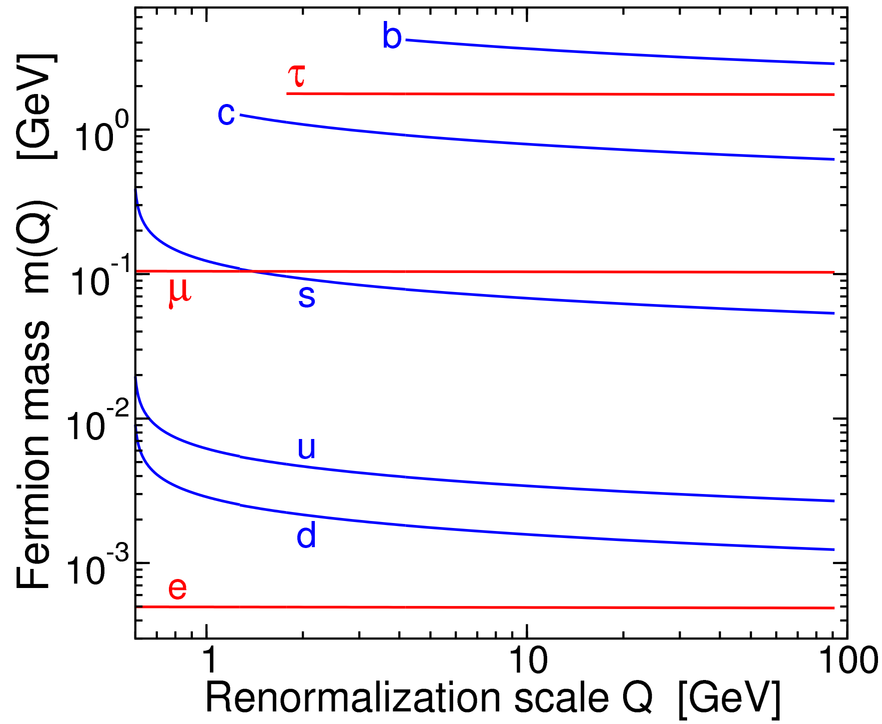
We prefer to decouple t, h, Z, W **simultaneously**, at a common matching scale.

The high-energy ("non-decoupled") theory has gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$.

The low-energy theory has gauge group $SU(3)_c \times U(1)_{\text{EM}}$, with 5 massive quarks and 3 massive leptons.

Decouple top, Higgs, Z, W simultaneously at a scale $Q_{\rm dec}$ of your choice, and run down to lower Q, decoupling bottom, tau, charm at scales of your choice:





5-loop RG running and 4-loop decoupling for QCD, 3-loop running and 2-loop decoupling for other parameters. Complete 2-loop decoupling from SPM 1812.04100.

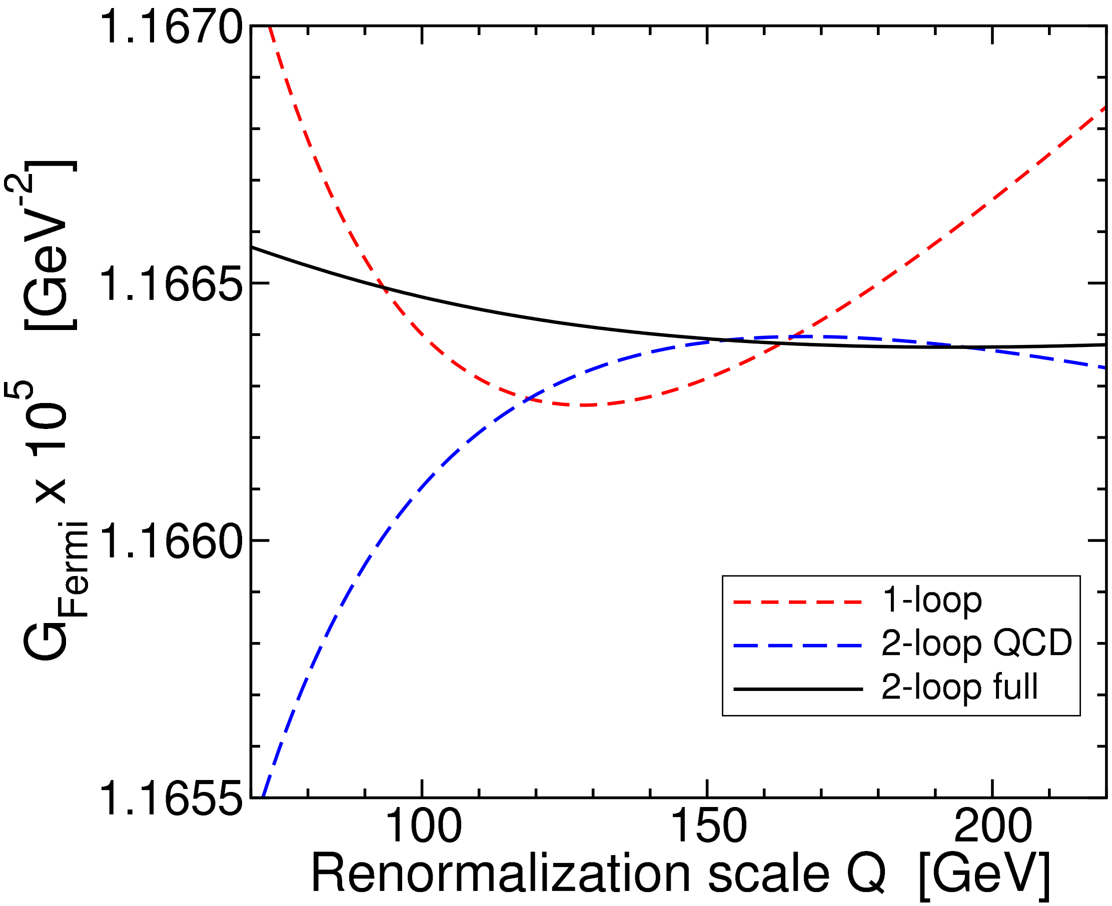
Determination of Fermi decay constant in terms of $\overline{\text{MS}}$ parameters:

$$G_F = \frac{1 + \Delta\overline{r}}{\sqrt{2} v_{\text{tree}}^2} = \frac{1 + \Delta\widetilde{r}}{\sqrt{2} v^2}.$$

In the tree-level VEV scheme, $\Delta\overline{r}$ has been at full 2-loop order given by Kniehl, Pikelner and Veretin 1401.1844, 1503.02138, 1601.08143 in their computer program `mr`.

We have obtained $\Delta\widetilde{r}$ at full 2-loop order in the tadpole-free VEV scheme, and checked that $1/\lambda$ and $1/\lambda^2$ terms are absent.

RG scale dependence of $G_{\rm Fermi}$, as a function of the Q where it is computed:



Scale dependence is less than 1 part in 10^{-4}, for 100 GeV $< Q <$ 220 GeV.

Compares well to an interpolating formula given by Degrassi, Gambino, Giardino 1411.7040 using another scheme; difference corresponds to $\Delta M_W \sim 5$ MeV.

Relate Sommerfeld fine structure constant $\alpha = 1/137.035999139\ldots$ to $\overline{\text{MS}}$ parameters in non-decoupled theory at $Q = M_Z$:

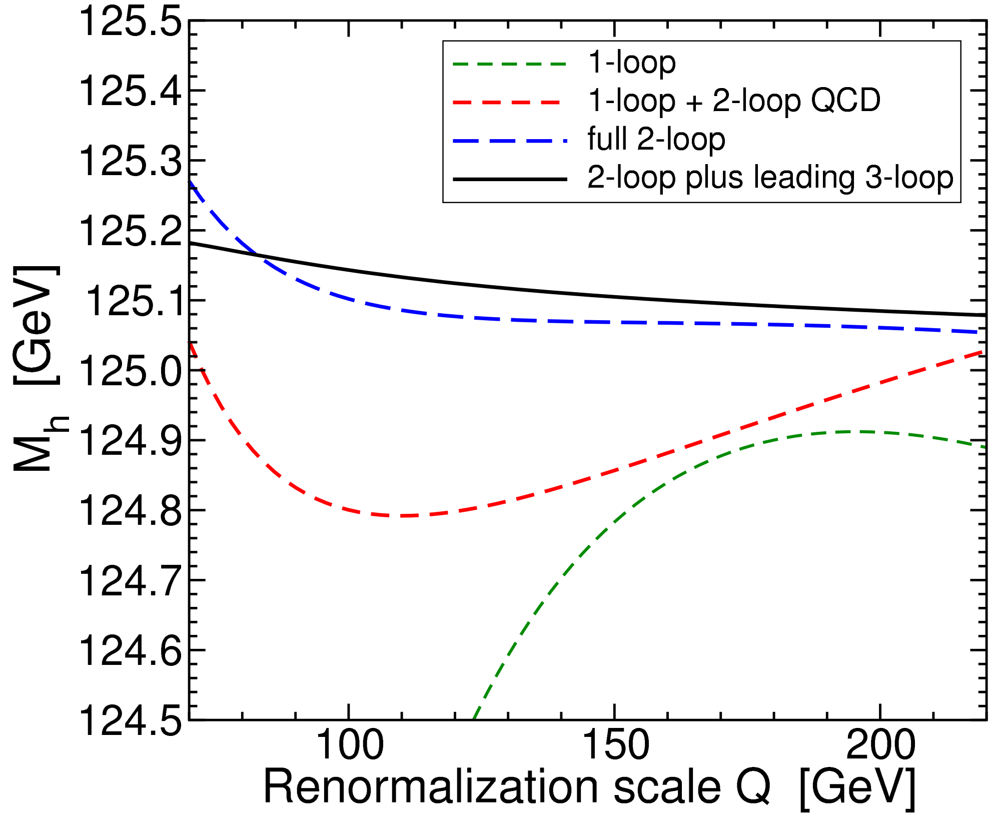
$$\alpha = \frac{g^2(M_Z) g'^2(M_Z)}{4\pi \left[g^2(M_Z) + g'^2(M_Z)\right]} \left[1 - \Delta\alpha^{(5)}_{\text{had}}(M_Z) - \Delta\alpha^{\text{LO}}_{\text{pert}} - \Delta\alpha^{\text{HO}}_{\text{pert}}\right],$$

where $\Delta\alpha^{(5)}_{\text{had}}(M_Z)$ contains contributions from b, c, s, d, u including non-perturbative effects, and the sum of 1-loop contributions from t, W, τ, μ, e are:

$$\begin{aligned}\Delta\alpha^{\text{LO}}_{\text{pert}} \quad = \quad & \frac{\alpha}{4\pi}\Bigg[\frac{202}{27} + 14\ln(M_W/M_Z) - \frac{32}{9}\ln(M_t/M_Z) - \frac{8}{3}\ln(M_\tau/M_Z) \\ & -\frac{8}{3}\ln(M_\mu/M_Z) - \frac{8}{3}\ln(M_e/M_Z)\Bigg]\end{aligned}$$

The higher-order contributions $\Delta\alpha^{\text{HO}}_{\text{pert}}$ were given in an interpolating formula by Degrassi Gambino Giardino 1411.7040.

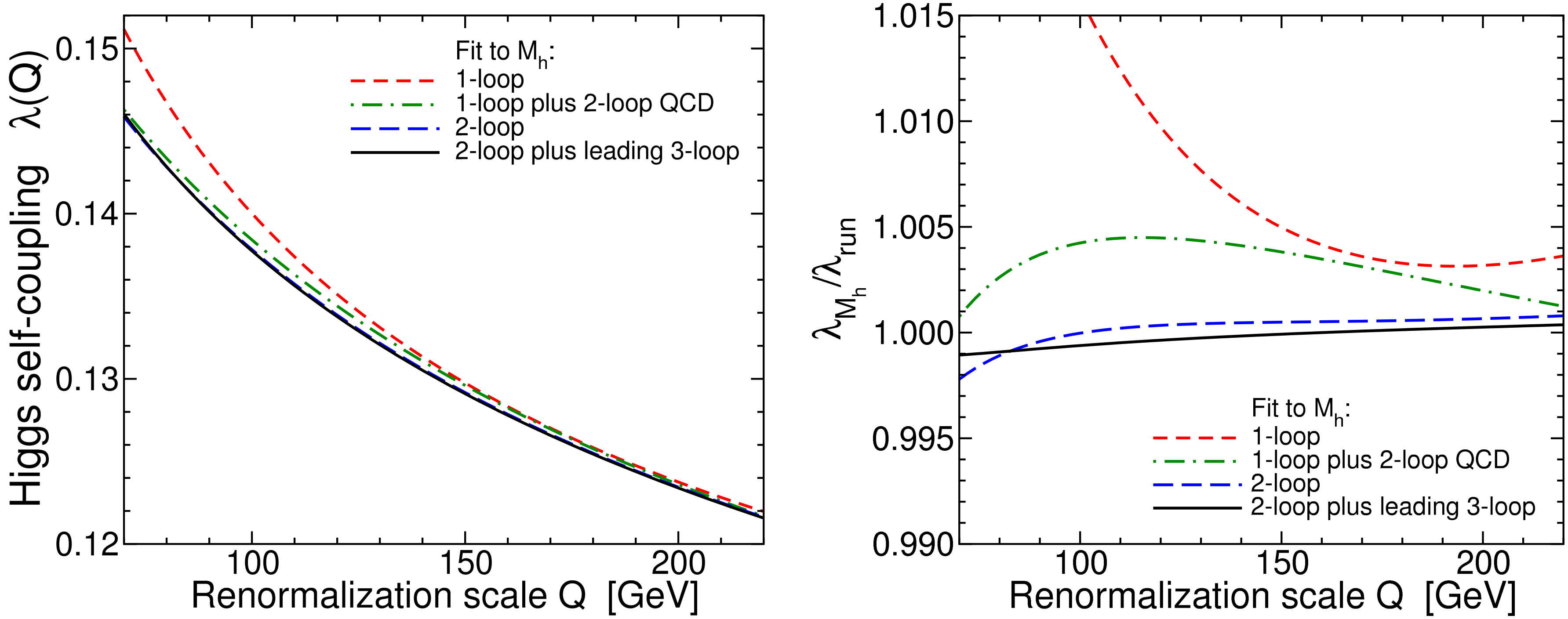
Higgs boson pole mass state-of-the-art: full 2-loop plus leading 3-loop approximation $g_3^2, y_t^2 \gg g^2, g'^2, \lambda$. (SPM and D.G. Robertson, 1407.4336)



SMDR subsumes and replaces our previous program SMH.

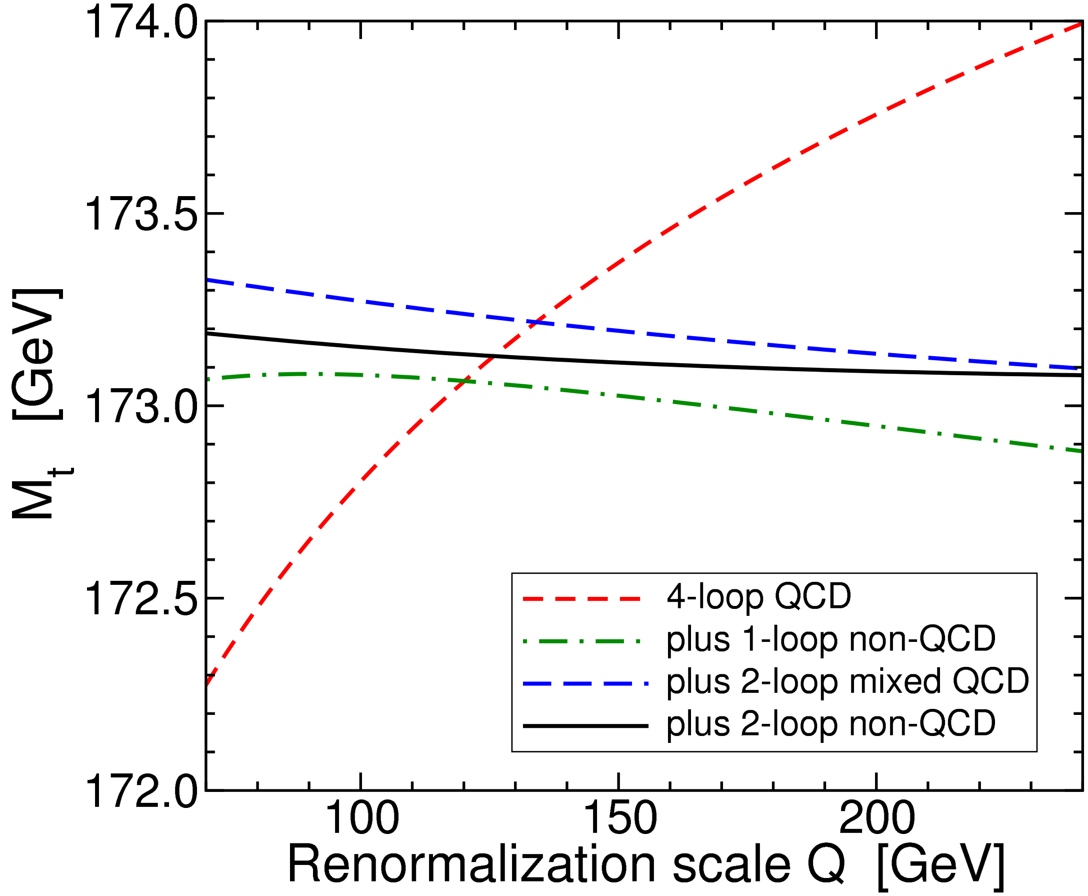
Scale dependence is tens of MeV, but we showed that $Q \sim 160$ GeV is preferred; higher-order effects are minimized with that choice.

Conversely, given the pole mass M_h as an input, can determine the Higgs self-coupling parameter λ as an output:



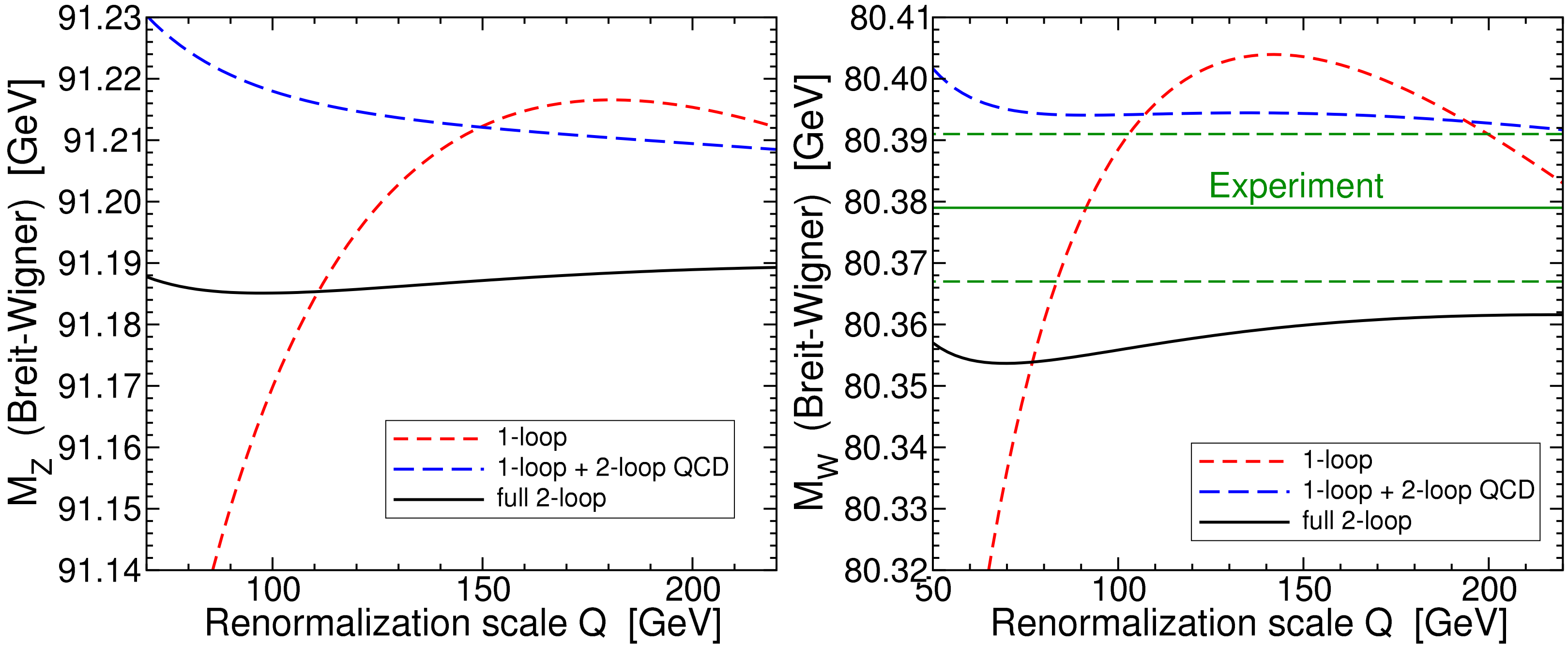
Again, the effects of higher-order contributions are minimized by choosing $Q \approx 160$ GeV, so that is the default used by SMDR when fitting λ.

Top-quark pole mass: state-of-the-art is 4-loop order in QCD plus full 2-loop order.



Note that neglecting electroweak and y_t effects is not justified. (Bad scale dependence is hidden if one also neglects electroweak and y_t contributions to beta functions.)

Z and W boson pole masses at full 2-loop order in tadpole-free pure $\overline{\text{MS}}$ scheme, from SPM 1505.04833 and 1503.03782 respectively:



Note: at this level of accuracy, need to convert pole masses (theoretical calculation, gauge invariant) to Breit-Wigner variable-width masses (used by experimentalists and PDG), using $M^2_{\rm BW} = M^2_{\rm pole} + \Gamma^2$.

$M_{Z,{\rm BW}} = M_{Z,{\rm pole}} +$ 34.1 MeV, $M_{W,{\rm BW}} = M_{W,{\rm pole}} +$ 27.1 MeV.

SMDR uses many multi-loop calculations by other authors, including:

- Standard Model 3-loop beta functions: Tarasov 1982, Mihaila, Salomon, Steinhauser 1201.5868, Chetyrkin, Zoller 1205.2892 and 1303.2890, Bednyakov, Pikelner, Velizhanin 1210.6873, 1212.6829, 1303.4364, 1310.3806, and 1406.7171

- QCD 5-loop beta functions van Ritbergen, Vermaseren, Larin hep-ph/9701390, Czakon hep-ph/0411261, Baikov, Chetyrkin, Kuhn 1606.08659, Herzog, Ruijl, Ueda, Vermaseren, Vogt 1701.01404, Chetyrkin hep-ph/9703278, Vermaseren, Larin, van Ritbergen hep-ph/9703284, Baikov, Chetyrkin, Kühn 1402.6611

- QCD contributions to quark pole masses at 4-loop order: Melnikov, van Ritbergen hep-ph/9912391, Marquard, A. Smirnov, V. Smirnov, M. Steinhauser 1502.01030

- Matching relations for decoupling in QCD at 4-loop order: Chetyrkin, Kniehl, Steinhauser 9708255, Grozin, Hoeschele, Hoff, Steinhauser 1107.5970, Schroder and Steinhauser 0512058, Bednyakov 1410.7603, Liu and Steinhauser 1502.04719

- Matching relations for α and y_f at full 2-loop order: SPM 1812.04100

Some other public code with overlapping aims:

- **RunDec, CRunDec** by Chetyrkin, Kuhn, Herren, Schmidt, Steinhauser hep-ph/0004189, 1201.6149, 1703.03751

  QCD 5-loop running, 4-loop decoupling and pole masses

- **mr** by Kniehl, Pikelner, Veretin 1601.08143

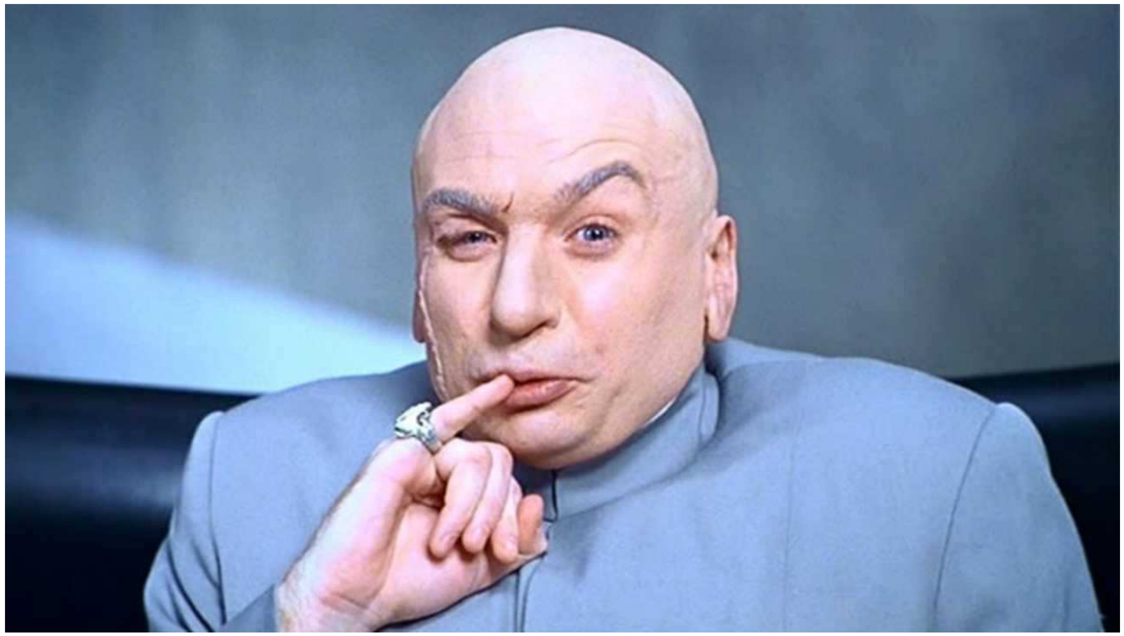
  Standard Model, uses tree-level VEV scheme

Sample run in calculator mode:

```
[smdr]# ./calc_all ReferenceModel.dat
INPUT PARAMETERS read from "ReferenceModel.dat":
Q = 173.100000;
Higgs vev = 246.600746;
Higgs mass^2 parameter = -8636.365174;
Higgs self-coupling lambda = 0.126203;
gauge  couplings: g3 = 1.163624;      g = 0.647659;      gp = 0.358539;
Yukawa couplings: yt = 0.934799;     yb = 0.015480;    ytau = 0.00999446;
                  yc = 0.0034009;    ys = 0.00029720;   ymu = 0.000588381;
                  yu = 0.0000068473; yd = 0.000014704;   ye = 0.00000279299;
Delta_hadronic^(5) alpha(MZ) = 0.027640

OUTPUT QUANTITIES:

Mt = 173.100000; Gammat = 1.372897;   (* complex pole *)

Mh = 125.180000; Gammah = 0.003409;   (* complex pole *)

MZ = 91.153552;  GammaZ = 2.491674;   (* complex pole *)
MZ = 91.187600;  GammaZ = 2.490744;   (* Breit-Wigner, compare to PDG *)

MW = 80.333307;  GammaW = 2.084131;   (* complex pole *)
MW = 80.360337;  GammaW = 2.083430;   (* Breit-Wigner, compare to PDG *)
```

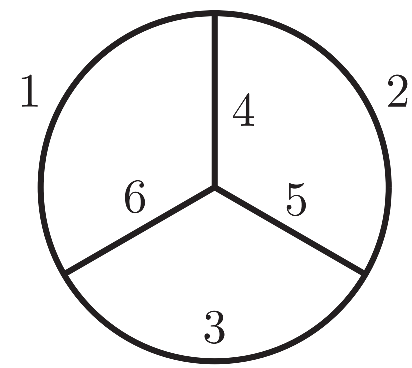
```
MSbar quantities at Q = MZ, full Standard Model, nothing decoupled:
  alphaS = 0.117053;   alpha = 1/128.114214;   sin^2_thetaW = 0.231417;

MSbar quantities at Q = MZ, only top quark decoupled (PDG convention):
  alphaS = 0.118100;   alpha = 1/127.945062;   sin^2_thetaW = 0.231228;

MSbar bottom and charm masses:
  mb(mb) = 4.180000;   (MSbar mass in 5-quark + 3-lepton QCD+QED theory)
  mc(mc) = 1.275000;   (MSbar mass in 4-quark + 2-lepton QCD+QED theory)

Light quark MSbar masses (at Q = 2 GeV, in 4-quark + 3-lepton QCD+QED theory):
  ms = 0.095000;     mu = 0.0022000;     md = 0.0047000;

Lepton pole masses:
  Mtau = 1.776860;      Mmuon = 0.105658375;      Melectron = 0.0005109989;

Sommerfeld fine structure and Fermi constants:
  alpha = 1/137.03599914;     GFermi = 1.16637870 10^-5;

Total calculation time: 2.68 seconds
```

BACKUP: 3VIL Propaganda



3VIL provides a fast, accurate, and flexible (valid for all masses, doesn’t rely on predetermined hierarchical expansions) numerical computation of vacuum integrals up to 3 loops.

Using partial fractions, any 3-loop vacuum integral can be reduced to this topology of scalar integral in $d = 4 - 2\epsilon$ Euclidean dimensions with $\int_p = \mu^{4-d} \int d^d p/(2\pi)^d$, where the $\overline{\text{MS}}$ renormalization scale is defined by $Q^2 = 4\pi e^{-\gamma_E} \mu^2$:



$$\mathbf{T}^{(n_1,n_2,n_3,n_4,n_5,n_6)}(x_1,x_2,x_3,x_4,x_5,x_6) \;=\; (16\pi^2)^3 \int_p \int_q \int_k$$

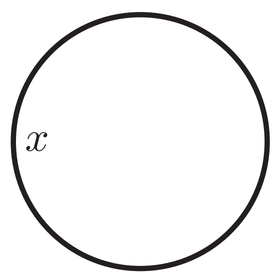
$$\frac{1}{[p^2+x_1]^{n_1}[q^2+x_2]^{n_2}[k^2+x_3]^{n_3}[(p-q)^2+x_4]^{n_4}[(q-k)^2+x_5]^{n_5}[(k-p)^2+x_6]^{n_6}}$$

The propagator powers n_i can be positive, negative, or zero. Using integration by parts, can always reduce all integrals of this type to a few basis integrals. . .

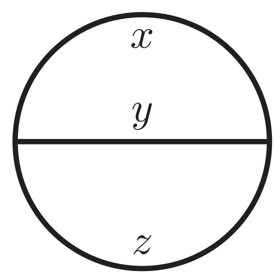
Basis integrals:

$$
\begin{aligned}
\mathbf{H}(u,v,w,x,y,z) &= \mathbf{T}^{(1,1,1,1,1,1)}(u,v,w,x,y,z),\\
\mathbf{G}(w,u,z,v,y) &= \mathbf{T}^{(1,1,1,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{F}(u,v,y,z) &= \mathbf{T}^{(2,1,0,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{I}(v,w,y) &= \mathbf{T}^{(1,1,1,0,1,0)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(w) &= \mathbf{T}^{(1,1,1,0,0,0)}(u,v,w,x,y,z),
\end{aligned}
$$

The last two are just products of 1-loop and 2-loop basis integrals:



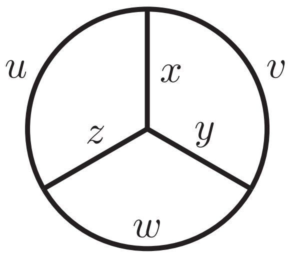
$\mathbf{A}(x)$



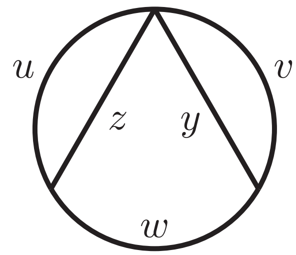
$\mathbf{I}(x,y,z)$

These are known analytically, and present no problems.

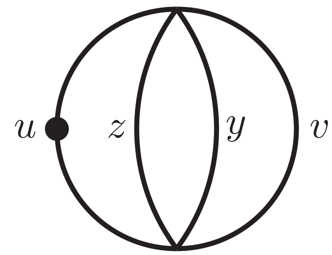
The genuinely 3-loop integrals in the basis are $\mathbf{H}$, $\mathbf{G}$, and $\mathbf{F}$:



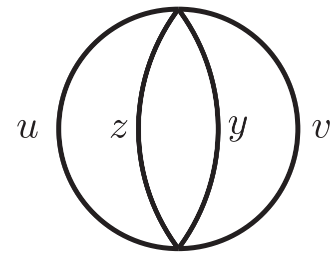
$\mathbf{H}(u, v, w, x, y, z)$



$\mathbf{G}(w, u, z, v, y)$



$\mathbf{F}(u, v, y, z)$



$\mathbf{E}(u, v, y, z)$

The dot on the $\mathbf{F}$ integral denotes a doubled propagator for the first squared mass argument; all other propagators are single.

The 4-propagator integral $\mathbf{E}$ is not part of the basis. By dimensional analysis:

$$\mathbf{E}(u, v, y, z) = \left[u\mathbf{F}(u, v, y, z) + v\mathbf{F}(v, u, y, z) + y\mathbf{F}(y, u, v, z) + z\mathbf{F}(z, u, v, y)\right]/(-2 + 3\epsilon),$$

so it is redundant. However, it is still useful. Note:

$$\mathbf{F}(u, v, y, z) = -\frac{\partial}{\partial u}\mathbf{E}(u, v, y, z).$$

Renormalized quantities are much more succinctly written in terms of modified basis integrals in which UV sub-divergences have been subtracted.

For example, at 2-loop order, define:

$$I(x, y, z) \quad = \quad \lim_{\epsilon \to 0} \left[\mathbf{I}(x, y, z) - I^{(1)}_{\rm div}(x, y, z) - I^{(2)}_{\rm div}(x, y, z) \right],$$

where

$$\begin{aligned} I^{(1)}_{\rm div}(x, y, z) &= \frac{1}{\epsilon}[\mathbf{A}(x) + \mathbf{A}(y) + \mathbf{A}(z)], \\ I^{(2)}_{\rm div}(x, y, z) &= \frac{1}{2}(x + y + z)\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right). \end{aligned}$$

The modified basis integral $I(x, y, z)$ is finite, by construction. It is known in terms of dilogarithms. Note it is **not** just the same thing as the ϵ^0 term in the ϵ expansion!

For the 3-loop, 4-propagator integrals, define:

$$E(u,v,y,z) = \lim_{\epsilon\to 0}\left[\mathbf{E}(u,z,y,v) - E^{(1)}_{\rm div}(u,v,y,z) - E^{(2)}_{\rm div}(u,v,y,z) - E^{(3)}_{\rm div}(u,v,y,z)\right],$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are, respectively,

$$\begin{aligned}
E^{(1)}_{\rm div}(u,v,y,z) &= \frac{1}{\epsilon}\mathbf{A}(u)\mathbf{A}(v) + \text{(5 permutations)},\\
E^{(2)}_{\rm div}(u,v,y,z) &= \left[\frac{1}{2\epsilon^2}(v+y+z) + \frac{1}{2\epsilon}\left(\frac{u}{2} - v - y - z\right)\right]\mathbf{A}(u) + \text{(3 permutations)},\\
E^{(3)}_{\rm div}(u,v,y,z) &= \left[\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon}\right](uv+uy+uz+vy+vz+yz)\\
&\quad + \left[\frac{1}{6\epsilon^2} - \frac{3}{8\epsilon}\right](u^2+v^2+y^2+z^2).
\end{aligned}$$

Renormalized quantities are written in terms of the ϵ-independent modified basis functions:

$$F(u,v,y,z) = -\frac{\partial}{\partial u}E(u,v,y,z).$$

Similarly, define the modified basis function:

$$G(w,u,z,v,y) \;=\; \lim_{\epsilon\to 0}\Big[\mathbf{G}(w,u,z,v,y) - G^{(1)}_{\rm div}(w,u,z,v,y) - G^{(2)}_{\rm div}(w,u,z,v,y) - G^{(3)}_{\rm div}(w,u,z,v,y)\Big],$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are:

$$G^{(1)}_{\rm div}(w,u,z,v,y) \;=\; \frac{1}{\epsilon}\left[\mathbf{I}(w,u,z) + \mathbf{I}(w,v,y)\right],$$

$$G^{(2)}_{\rm div}(w,u,z,v,y) \;=\; \left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\right)\left[\mathbf{A}(u) + \mathbf{A}(v) + \mathbf{A}(y) + \mathbf{A}(z)\right] - \frac{1}{\epsilon^2}\mathbf{A}(w),$$

$$G^{(3)}_{\rm div}(w,u,z,v,y) \;=\; \left(-\frac{1}{6\epsilon^3} + \frac{1}{2\epsilon^2} - \frac{2}{3\epsilon}\right)(u+v+y+z) + \left(-\frac{1}{3\epsilon^3} + \frac{1}{3\epsilon^2} + \frac{1}{3\epsilon}\right) w.$$

$\mathbf{H}$ has no 1-loop and 2-loop sub-divergences, but does have a 3-loop UV divergence. So, define:

$$H(u,v,w,x,y,z) \;=\; \lim_{\epsilon\to 0}\left[\mathbf{H}(u,v,w,x,y,z) - H^{(3)}_{\rm div}(u,v,w,x,y,z)\right]$$

where

$$H^{(3)}_{\rm div}(u,v,w,x,y,z) \;=\; 2\zeta(3)/\epsilon.$$

The function $F(u, v, y, z)$ has an IR log divergence as $u \to 0$. Therefore, further define:

$$\overline{F}(u, v, y, z) \equiv F(u, v, y, z) + \overline{\ln}(u) F(u, v, y, z)$$

where

$$\overline{\ln}(u) = \ln(u/Q^2)$$

with $Q = \overline{\text{MS}}$ renormalization scale. The function $\overline{F}$ is well-defined for all values of its squared mass arguments, including $u = 0$.

For convenience, our program `3VIL` outputs all E, F, and $\overline{F}$ functions, for given input arguments.

(Also can output the ϵ expansions of the original bold-faced integrals $\mathbf{I}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$.)

The following are known analytically:

- All 1-scale integrals E, F, $\overline{F}$, G, H, with squared masses all equal to 0 or a single non-zero value x. Broadhurst 1992, 1999; Avdeev+Fleischer+Mikhailov+Tarasov, 1994; Fleischer+Tarasov, 1994; Avdeev 1995; Fleischer+Kalmykov 1999; Schröder+Vuorinen 2005.

- The following 2-scale integral cases, and integrals E, F related to them, and permutations implied by symmetries of the graphs: $\overline{F}(x,0,0,y)$, $\overline{F}(0,0,x,y)$, $\overline{F}(x,x,y,y)$, $\overline{F}(x,0,y,y)$, $\overline{F}(y,0,y,x)$, $G(0,0,0,x,y)$, $G(0,0,x,0,y)$, $G(x,0,0,0,y)$, $G(x,0,x,0,y)$, $G(0,x,x,y,y)$, $G(x,0,0,y,y)$, $G(y,x,x,x,x)$, $H(0,0,x,y,x,x)$. Davydychev+Kalmykov 2003, Kalmykov 2005, Bytev+Kalmykov+Kniehl 2009, our paper.

Our program `3VIL` knows about these cases and uses them whenever possible. Computation time $\approx$ 0.

The generic case: consider the master tetrahedral topology, and all corresponding basis integrals obtained by removing propagator lines:

$$H(u,v,w,x,y,z),$$
$$G(w,u,z,v,y),\;\; G(x,u,v,y,z),\;\; G(u,v,x,w,z),$$
$$G(y,v,w,x,z),\;\; G(v,u,x,w,y),\;\; G(z,u,w,x,y),$$
$$\overline{F}(w,u,x,y),\;\; \overline{F}(w,v,x,z),\;\; \overline{F}(x,u,w,y),\;\; \overline{F}(x,v,w,z),$$
$$\overline{F}(u,v,y,z),\;\; \overline{F}(u,w,x,y),\;\; \overline{F}(y,u,v,z),\;\; \overline{F}(y,u,w,x),$$
$$\overline{F}(v,u,y,z),\;\; \overline{F}(v,w,x,z),\;\; \overline{F}(z,u,v,y),\;\; \overline{F}(z,v,w,x),$$

products of I and A functions

The derivatives of all of these with respect to any squared mass argument u,v,w,x,y,z are also 3-loop integrals, and so are linear combinations of the basis.

Solve differential equations in the masses to compute these, starting from known analytical values at a fixed but arbitrary reference squared mass a as initial conditions:

$$H(a,a,a,a,a,a),\;\; G(a,a,a,a,a),\;\; \overline{F}(a,a,a,a),\;\; I(a,a,a),\;\; A(a).$$

Define an integration variable t, and:

$$\begin{aligned} U &= a + t(u-a), & V &= a + t(v-a), & W &= a + t(w-a), \\ X &= a + t(x-a), & Y &= a + t(y-a), & Z &= a + t(z-a). \end{aligned}$$

and consider basis integrals as functions of U, V, W, X, Y, Z.

- At $t = 0$, have $U = V = W = X = Y = Z = a$, so all integrals are known.
- At $t = 1$, have desired values of squared mass arguments: $(U, V, W, X, Y, Z) = (u, v, w, x, y, z)$.

Denoting the basis integrals generically by Φ_i, have first-order coupled linear differential equations in t:

$$\frac{d}{dt}\Phi_j = \sum_k c_{jk}\Phi_k + c_j$$

where the coefficients c_{jk} and c_j are ratios of polynomials in t and fixed values a, u, v, w, x, y, z.

Integrate differential equations numerically from $t = 0$ to $t = 1$.

Differential equations method for evaluation of loop integrals

Kotikov 1991, Remiddi 1997, Caffo+Czyz+Laporta+Remiddi 1998, Caffo+Czyz+Remiddi 2002, SPM 2003, SPM+Robertson 2005, . . .

Allows analytic evaluation in favorable cases; otherwise Runge-Kutta numerical integration.

When computing tetrahedral integral $H(u, v, w, x, y, z)$, we simultaneously get all subordinate basis integrals G, F, $\overline{F}$, E.

However, there are complications. . .

$$\frac{d}{dt}\Phi_j = \sum_k c_{jk}\Phi_k + c_j$$

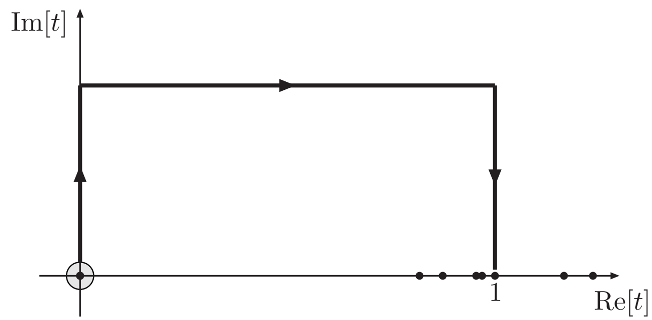
A complication: the coefficients c_{jk} and c_j have poles in t.

- All poles can be made simple by use of partial fractions on the coefficients.
- There are always poles at $t = 0$.

  Use a power series expansion around $t = 0$, up to order t^8.

  Start integration at $t = 0.01$
- All poles are on the real t axis. Sometimes poles exist for $0 < t < 1$.

  In that case, integrate on a contour in the complex plane to avoid them:



Otherwise, integrate straight along $\mathrm{Re}[t]$ axis.

Recall $U = a + t(u - a)$, etc.

The fixed reference squared mass a is arbitrary. In principle, results should not depend on it. Can be changed as a check. By default `3VIL` uses:

$$a = 2\mathrm{Max}(u, v, w, x, y, z).$$

Avoids numerical problems that can arise in certain special cases.

Other checks:

- analytical special cases compared to Runge-Kutta evaluation
- vanishing of imaginary parts of basis integrals when squared mass inputs are positive
- change shape of contour in complex plane, including height in the $\mathrm{Im}[t]$ direction

Initialization at $t = 0.01$:

$$H(U,V,W,X,Y,Z) = H(a,a,a,a,a,a) + \sum_{n\geq 1} t^n H^{(n)}(u,v,w,x,y,z;a),$$

$$G(W,U,Z,V,Y) = G(a,a,a,a,a) + \sum_{n\geq 1} t^n G^{(n)}(w,u,z,v,y;a),$$

$$\overline{F}(U,V,Y,Z) = \overline{F}(a,a,a,a) + \sum_{n\geq 1} t^n \overline{F}^{(n)}(u,v,y,z;a),$$

with:

$$\overline{F}(a,a,a,a) = a\left[53/12 + (3\sqrt{3}\mathrm{Ls}_2 - 3/2)\overline{\ln}(a) + \frac{3}{2}\overline{\ln}^2(a) - \frac{1}{2}\overline{\ln}^3(a)\right]$$

$$G(a,a,a,a,a) = a\left[-97/3 + 12\sqrt{3}\mathrm{Ls}_2 + 6\zeta_3 + (26 - 6\sqrt{3}\mathrm{Ls}_2)\overline{\ln}(a) - 8\overline{\ln}^2(a) + \overline{\ln}^3(a)\right]$$

$$H(a,a,a,a,a,a) = 16\mathrm{Li}_4(1/2) - \frac{17\pi^4}{90} + \frac{2}{3}\ln^2(2)[\ln^2(2) - \pi^2] - 9(\mathrm{Ls}_2{}^2) + 6\zeta_3[1 - \overline{\ln}(a)]$$

and

$$H^{(1)}(u,v,w,x,y,z;a) = \zeta_3(6a - u - v - w - x - y - z)/a,$$

etc. All expansion coefficients through $n = 7$ included, so that at $t = 0.01$ the relative error from truncation is same order as that of long double arithmetic, 10^{-16}.

For most of the integration, 3VIL uses a 6-stage, 5th order Runge-Kutta algorithm with automatic step-size adjustment.

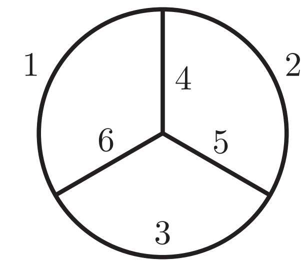
However, can have poles in the coefficients at the integration endpoint $t = 1$. Usual Runge-Kutta routines fail!

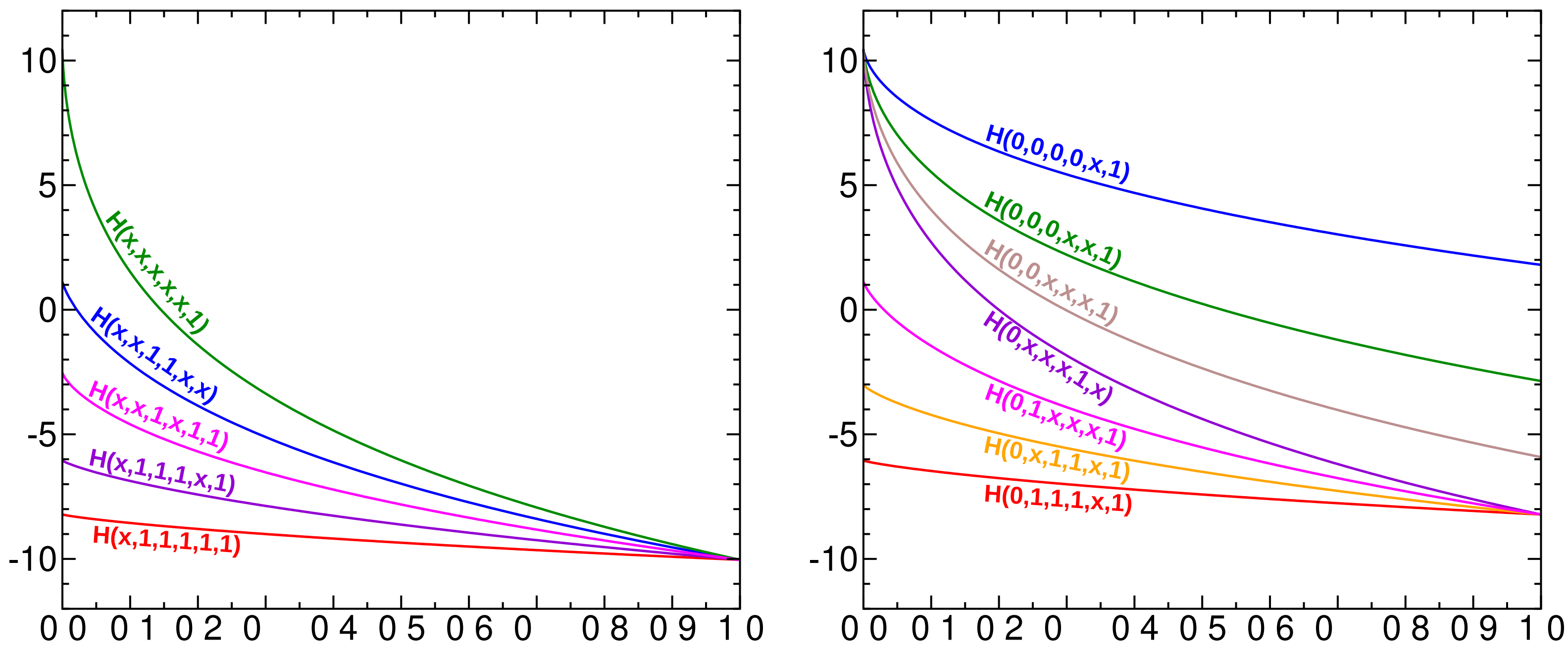
Key property needed: no evaluations of derivatives at the endpoint of the integration step.

No 4-stage Runge-Kutta algorithms with this property exist, but we found a 5-stage, 4th order algorithm. (Invented for a very similar situation for our program TSIL = Two-loop Self-energy Integration Library, hep-ph/0501132.)

Note: although the coefficients in the differential equations have poles, the basis functions themselves are completely finite and smooth! Only pseudo-thresholds, no thresholds.

Some examples of the basis integral H, as a function of a squared mass argument x, with other squared mass arguments fixed to 0 or 1.

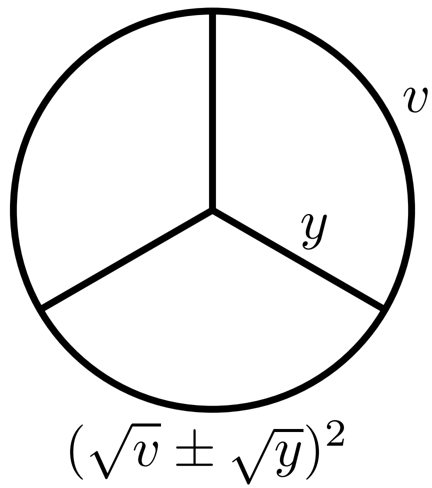




The endpoints at $x = 0$ and $x = 1$ are known analytically in terms of logs.

For all other $0 < x < 1$, computed analytically with 3VIL.

Pseudo-thresholds = numerically difficult cases:



with $v \neq 0$ and $y \neq 0$.

Can take longer (5 seconds), with some loss of accuracy.

Note that these cases are “unnatural”; not consequences of any possible symmetry in a quantum field theory. Don’t arise in Standard Model, but may occur in continuous parameter scans in Beyond Standard Model theories.