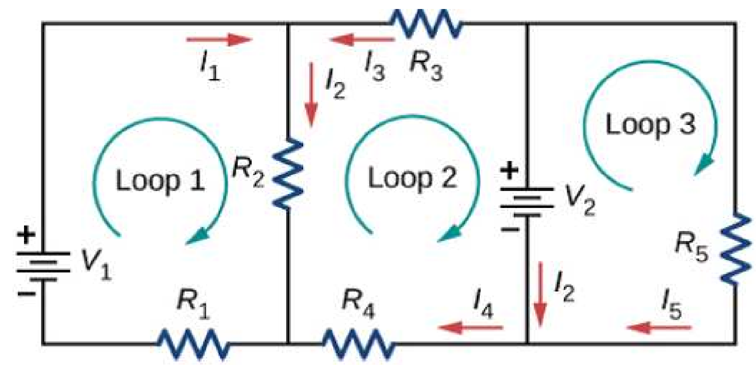
On the numerical evaluation of 3-loop self-energy integrals

A. Freitas

University of Pittsburgh



1. Introduction

2. Review: General 3-loop vacuum integrals

3. Planar-type 3-loop self-energy integrals

4. Public program TVID 2

Need for 3-loop corrections:

- Electroweak precision tests:

	Current exp.	Current theory*	CEPC	FCC-ee
M_W [MeV]	15	4	1	0.5–1
Γ_Z [MeV]	2.3	0.4	0.5	0.1
$R_b = \Gamma_\mathrm{Z}^b/\Gamma_\mathrm{Z}^\mathrm{had}$ [10^{-5}]	66	10	4.3	6
$\sin^2\theta^\ell_\mathrm{eff}$ [10^{-5}]	16	4.5	$<$1	0.5

* Full 2-loop and leading 3-/4-loop corrections

- Higgs mass calculation in SUSY — Harlander, Kant, Mihaila, Steinhauser '08,10; Reyes, Fazio '19

- Mixed EW-QCD corrections to Higgs prod. at LHC — Bonetti, Melnikov, Tancredi '17; Anastasiou et al. '18

- ...

Two-loop electroweak corrections

- Analytical evaluation of master integrals with diff. eq. or Mellin-Barnes rep.

  Kotikov '91; Remiddi '97; Smirnov '00,01; Henn '13; ...

$\longrightarrow$ Result in terms of Goncharov polylogs / multiple polylogs

Goncharov '98
Gehrmann, Remiddi '00,01

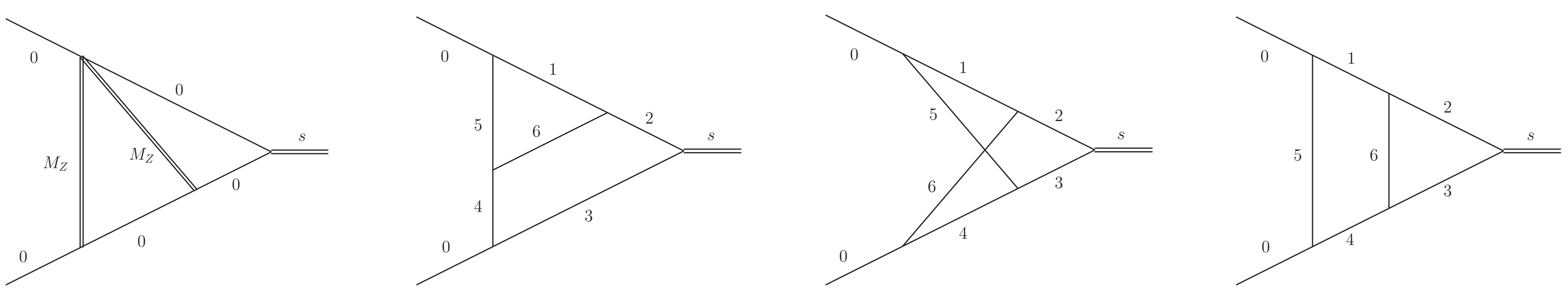
$\longrightarrow$ Some problems need iterated elliptic integrals / elliptic multiple polylogs

Levin, Racinet '07; Bloch, Vanhove '07
Adams, Bogner, Weinzierl '14; ...

$\longrightarrow$ **Full set of functions for all 2-loop diagrams not known**

Two-loop electroweak corrections

- Problem has multiple scales: M_Z, M_W, M_H, m_t $(m_f \to 0,\ f \neq t)$
- Numerical techniques needed
- Self-energies (incl. from renormlization) and vertices with sub-loop bubbles using dispersion relation technique
  S. Bauberger et al. '95
  Awramik, Czakon, Freitas '06
- Non-trivial vertex diagrams: Dubovyk, Freitas, Gluza, Riemann, Usovtisch '16,18
  - Sector decomposition
  - Mellin-Barnes representations (MB / AMBRE 3 / MBnumerics)
  - No tensor reduction (besides trivial cancellations)
    $\longrightarrow$ $>$ 1000 different two-loop vertex integrals



Direct numerical integration

Two general (automizable) approches:

- **Sector decomposition:** Binoth, Heinrich ’00,03

  Advantageous for diagrams with many massive propagators

  Public programs: `SecDec` Carter, Heinrich ’10; Borowka et al. ’12,15,17
  `FIESTA` Smirnov, Tentyukov ’08; Smirnov ’13,15

- **Mellin-Barnes representations:** Smirnov ’99; Tausk ’99; Czakon ’06; Anastasiou, Daleo ’06

  ... with fewer independent parameters

  Public programs: `MB/MBresolve` Czakon ’06; Smirnov, Smirnov ’09
  `AMBRE/MBnumerics` Gluza, Kajda, Riemann ’07; Dubovyk, Gluza, Riemann ’15; Usovitsch, Dubovyk, Riemann ’18

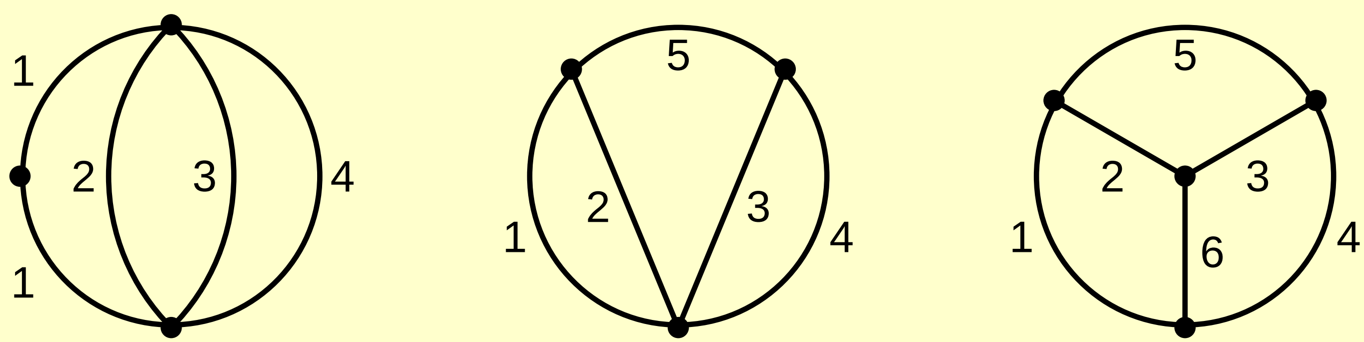
- Can be applied to any number of scales and loop order
- Automated extraction of UV and IR divergencies
- Requires sizeable computing resources
- Diagrams with internal thresholds can cause numerical instabilities

General 3-loop vacuum integrals

- Relevant for low-energy precision observables ($p^2 \ll M_Z$)
- Coefficients of low-momentum expansions
- Building block for more general 3-loop calculations

Master integrals:

$$M(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6;\, m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2)$$
$$= i\frac{e^{3\gamma_E\epsilon}}{\pi^{3D/2}}\int d^Dq_1\, d^Dq_2\, d^Dq_3\, [q_1^2-m_1^2]^{-\nu_1}[(q_1-q_2)^2-m_2^2]^{-\nu_2}$$
$$\times [(q_2-q_3)^2-m_3^2]^{-\nu_3}[q_3^2-m_4^2]^{-\nu_4}[q_2^2-m_5^2]^{-\nu_5}[(q_1-q_3)^2-m_6^2]^{-\nu_6}$$



$U_4 = M(2,1,1,1,0,0)$ $\quad U_5 = M(1,1,1,1,1,0)$ $\quad U_6 = M(1,1,1,1,1,1)$

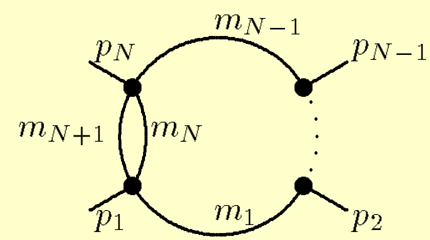
Sub-loop dispersion relations

Topologies with **self-energy sub-loop** can easily be integrated by using dispsersion relation for B_0 function: S. Bauberger et al. '95

$$B_0(p^2, m_1^2, m_2^2) = -\int_{(m_1+m_2)^2}^{\infty} \mathrm{d}s \, \frac{\Delta B_0(s, m_1^2, m_2^2)}{s - p^2}$$

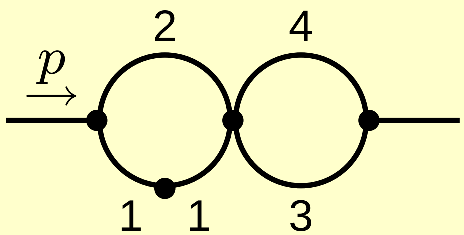
with $$\Delta B_0(s, m_1^2, m_2^2) = (4\pi\mu^2)^{4-D} \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \frac{\lambda^{(D-3)/2}(s, m_1^2, m_2^2)}{s^{D/2-1}},$$

$\lambda(a, b, c) = (a - b - c)^2 - 4bc$



$$T_{N+1}(p_i; m_i^2) = -\int_{s_0}^{\infty} \mathrm{d}s \, \Delta B_0(s, m_N^2, m_{N+1}^2)$$

$$\times \int \mathrm{d}^4 q \, \frac{1}{q^2 - s} \, \frac{1}{(q+p_1)^2 - m_1^2} \cdots \frac{1}{(q+p_1+\cdots+p_{N-1})^2 - m_{N-1}^2}$$

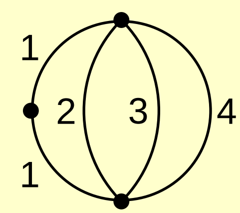


$$= B_{0,m_1}(p^2, m_1^2, m_2^2) B_0(p^2, m_3^2, m_4^2)$$
$$= \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{s - p^2 - i\varepsilon}$$

$$\Delta I_{\mathrm{db}}(s, m_1^2, m_2^2, m_3^2, m_4^2) = \Delta B_{0,m_1}(s, m_1^2, m_2^2)\, B_0(s, m_3^2, m_4^2) + B_{0,m_1}(s, m_1^2, m_2^2)\, \Delta B_0(s, m_3^2, m_4^2),$$

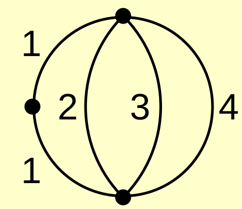
$$\Delta B_0(s, m_a^2, m_b^2) = \frac{1}{s} \lambda(s, m_a^2, m_b^2)\, \Theta\Big(s - (m_a + m_b)^2\Big)$$
$$\Delta B_{0,m_1}(s, m_a^2, m_b^2) = \frac{m_a^2 - m_b^2 - s}{s\, \lambda(s, m_a^2, m_b^2)}\, \Theta\Big(s - (m_a + m_b)^2\Big)$$

$$U_4(m_1^2, m_2^2, m_3^2, m_4^2) = -\frac{e^{\gamma_{\mathrm{E}}\epsilon}}{i\pi^{D/2}} \int d^D q_3 \int_0^\infty ds \, \frac{\Delta I_{\mathrm{db}}(s)}{q_3^2 - s + i\varepsilon}$$
$$= -\int_0^\infty ds\, A_0(s)\, \Delta I_{\mathrm{db}}(s)$$



U_4

Problem: U_4 is divergent



Solution:

$$U_4(m_1^2, m_2^2, m_3^2, m_4^2) = U_4(m_1^2, m_2^2, 0, 0) + U_4(m_1^2, 0, m_3^2, 0) + U_4(m_1^2, 0, 0, m_4^2) - 2\,U_4(m_1^2, 0, 0, 0) + U_{4,\text{sub}}(m_1^2, m_2^2, m_3^2, m_4^2)$$

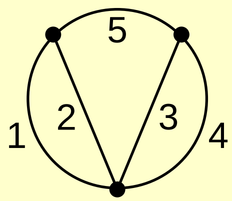
$\longrightarrow$ $U_4(m_X^2, m_Y^2, 0, 0)$ can be computed analytically

$\longrightarrow$ $U_{4,\text{sub}}$ is finite

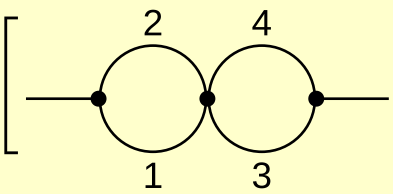
$$U_{4,\text{sub}}(m_1^2, m_2^2, m_3^2, m_4^2) = -\int_0^\infty ds\, A_{0,\text{fin}}(s)\, \Delta I_{\text{db,sub}}(s)$$

$$\begin{aligned} I_{\text{db,sub}}(s, m_1^2, m_2^2, m_3^2, m_4^2) = &\\ &\Delta B_{0,m_1}(s, m_1^2, m_2^2)\ \text{Re}\Big\{B_0(s, m_3^2, m_4^2) - B_0(s,0,0)\Big\}\\ &- \Delta B_{0,m_1}(s, m_1^2, 0)\ \text{Re}\Big\{B_0(s, 0, m_3^2) + B_0(s, 0, m_4^2) - 2B_0(s,0,0)\Big\}\\ &+ \text{Re}\Big\{B_{0,m_1}(s, m_1^2, m_2^2)\Big\}\Big[\Delta B_0(s, m_3^2, m_4^2) - \Delta B_0(s,0,0)\Big]\\ &- \text{Re}\Big\{B_{0,m_1}(s, m_1^2, 0)\Big\}\Big[\Delta B_0(s, 0, m_3^2) + \Delta B_0(s, 0, m_4^2) - 2\,\Delta B_0(s,0,0)\Big] \end{aligned}$$

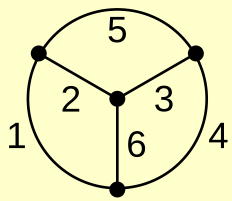
$U_5,\ U_6$



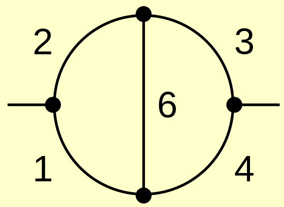
$$= -\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int_0^\infty ds \int \frac{d^D q_3}{[q_3^2 - s][q_3^2 - m_5^2]} \times \text{Disc}\Big[\quad\Big]_s$$



$$= \int_0^\infty ds\, B_0(0, s, m_5^2)\, \text{Disc}[\ldots]_s$$



$$= -\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int_0^\infty ds \int \frac{d^D q_3}{[q_3^2 - s][q_3^2 - m_5^2]} \times \text{Disc}\Big[\quad\Big]_s$$



$$= \int_0^\infty ds\, B_0(0, s, m_5^2)\, \text{Disc}[\ldots]_s$$

2-loop self-energy known in terms of 1-dimensional numerical integral

Bauberger, Böhm ’95

- U_5, U_6 made UV-finite by subtracting terms that can be computed analytically

Results

- U_4, U_5 given in terms of one-dimensional numerical integrals
- U_6 given in terms of two-dimensional numerical integral
- Special cases (e.g. $m_1 = 0$) can also be handled

Public code: **TVID 1** Freitas '16; Bauberger, Freitas '17

- Algebraic part (`Mathematica`) performs subtraction of UV-divergencies
- Numerical part (`C++`) performs numerical integrals

  Timing (single core Xeon 3.7 GHz):
  $\lesssim$ 0.1 s for U_4, U_5
  $\lesssim$ 30 s for U_6

- At least ten digit agreement with literature (for one/two-scale cases)

  Broadhurst '98; Chetyrkin, Steinhauser '99
  Grigo, Hoff, Marquard, Steinhauser '12

- Available at `www.pitt.edu/˜afreitas/`

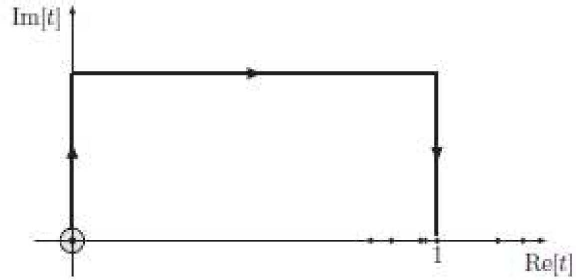
Alternative approach

Differential equations: (with respect to mass parameters) Martin, Robertson '16

$$\frac{d}{dt}\Phi_j = \sum_k c_{jk}\Phi_k + c_j$$

A complication: the coefficients c_{jk} and c_j have poles in t.

- All poles can be made simple by use of partial fractions on the coefficients.
- There are always poles at $t = 0$.
  Use a power series expansion around $t = 0$, up to order t^8.
  Start integration at $t = 0.01$
- All poles are on the real t axis. Sometimes poles exist for $0 < t < 1$.
  In that case, integrate on a contour in the complex plane to avoid them:

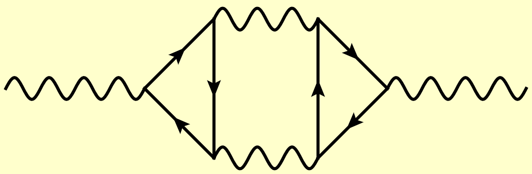


[slide from S. Martin]

Public code: **3VIL** (timing below 1 sec. and similar precision as TVID)

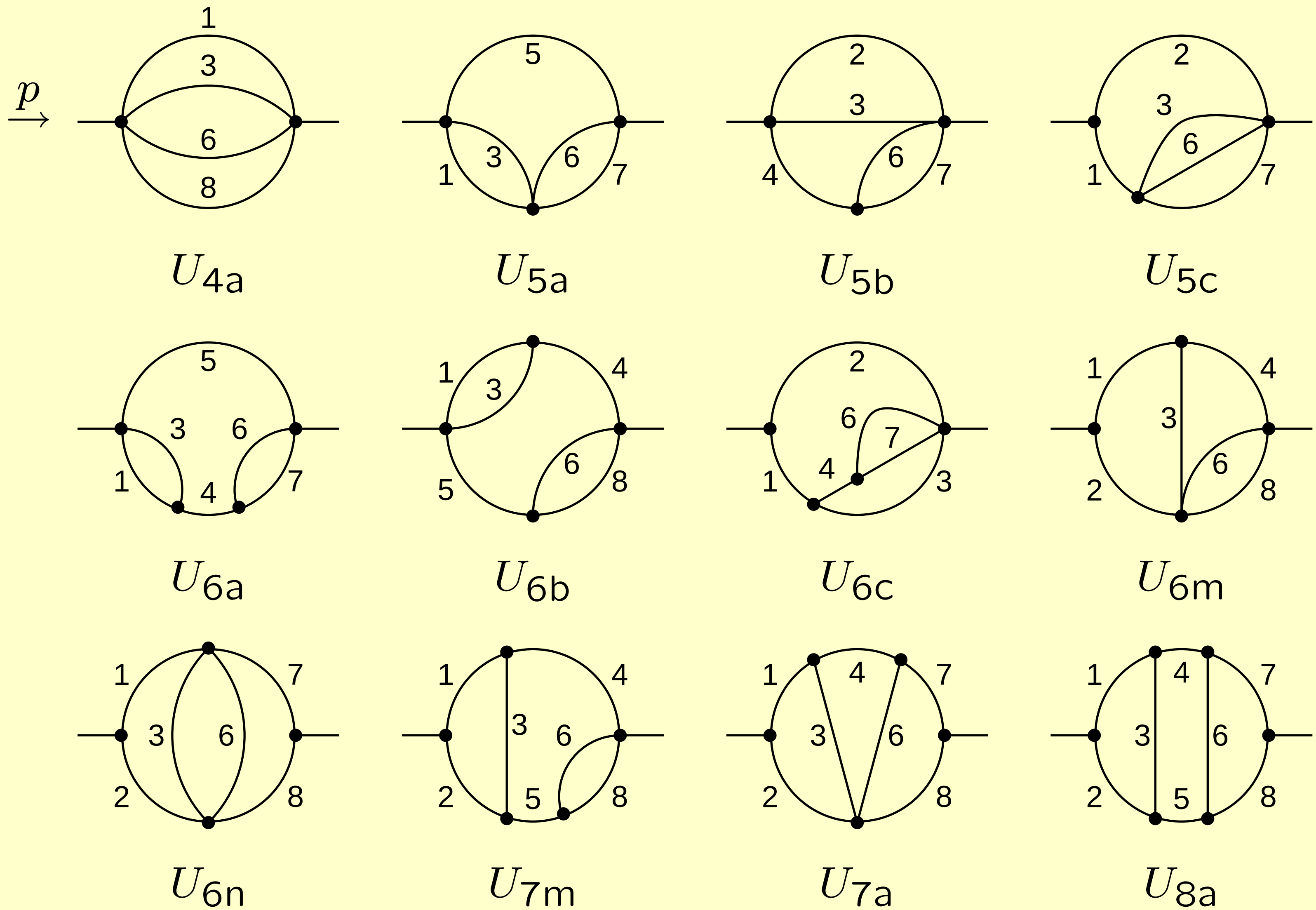
Planar-type 3-loop self-energy integrals

- 3-loop self-energy diagrams
  with ladder-type topology

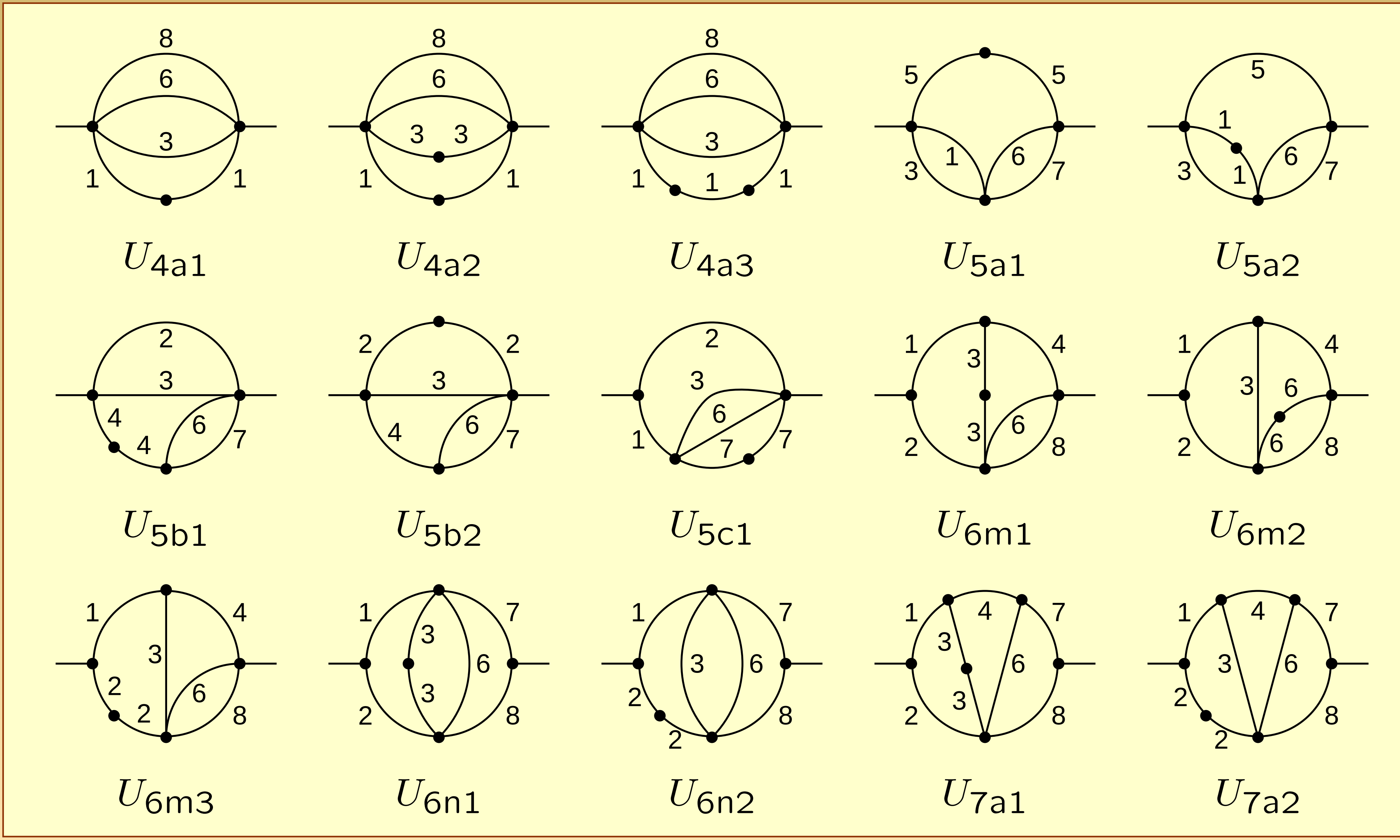


- Useful for:
  - On-shell renormalization in full SM
  - Higgs mass corrections in SUSY
  - ...
- Find set of master integrals:
  - Generate diagrams with FEYNARTS 3 [Hahn '01]
  - Reduce using IBP relations with FIRE 5 [Smirnov '14]
- Set of masters with **only denominators**, no numerators (may not be *optimal*)
- All masters can be evaluated in terms of 2-dim. numerical integrals
  $\longrightarrow$ Fast and high-precision results

Master integrals

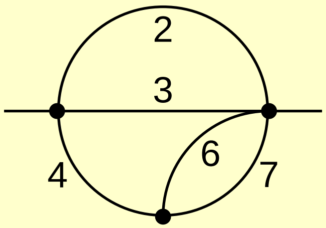


Master integrals with doubled propagators



Double-bubble integrals

Example: U_{5b}



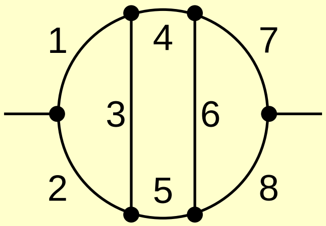
$$\int_0^\infty ds_1 \int_0^\infty ds_2 \, \Delta B_0(s_1, m_6^2, m_7^2) \, \Delta B_0(s_2, m_1^2, m_3^2) \times \underbrace{\frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \int d^D q_2 \frac{1}{[q_2^2 - m_4^2][q_2^2 - s_1][(q_2+p)^2 - s_2]}}_{\text{one-loop integral, known analytically}}$$

Subtraction of divergencies:

$$\int_0^\infty ds_1 \int_0^\infty ds_2 \, \frac{\Delta B_0(s_1, m_6^2, m_7^2) \, \Delta B_0(s_2, m_1^2, m_3^2)}{s_1 - m_4^2 - i\varepsilon} \, B_0(p^2, s_1, s_2)$$

$$= \int_0^\infty ds_1 \int_0^\infty ds_2 \, \frac{\Delta B_0(s_1, m_6^2, m_7^2) \, \Delta B_0(s_2, m_1^2, m_3^2)}{s_1 - m_4^2 - i\varepsilon} \times \Big[\underbrace{B_0(0, s_1, s_2) + p^2 B_0'(0, s_1, s_2)}_{\text{vacuum integrals}} + \underbrace{[B_0(p^2, s_1, s_2) - B_0(0, s_1, s_2) - p^2 B_0'(0, s_1, s_2)]}_{\text{finite}} \Big]$$

Planar master topology: $U_{8\mathrm{a}}$



$$\int \frac{d^4q}{i\pi^2} \frac{C_0(p^2, y, x, m_1^2, m_2^2, m_3^2) C_0(x, p^2, y, m_6^2, m_7^2, m_8^2)}{[x - m_4^2 + i\varepsilon][y - m_5^2 + i\varepsilon]}$$

$$x = q^2, \qquad y = (q+p)^2$$

- In cms frame, $p = (p_0, \vec{0})$: $\quad x = q_0^2 - |\vec{q}|^2, \quad y = q_0^2 - |\vec{q}|^2 + p^2 + 2q_0 p_0$
- Integrate angles of $\vec{q}$; variable transformation $(q_0, |\vec{q}|) \to (x, y)$:

$$U_{8\mathrm{a}} = \frac{1}{2i\pi p^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \sqrt{\lambda(x, y, p^2)} \, \Theta(\lambda(x, y, p^2))$$

Ghinculov '96

$$\times \frac{C_0(p^2, y, x, m_1^2, m_2^2, m_3^2) C_0(x, p^2, y, m_6^2, m_7^2, m_8^2)}{[x - m_4^2 + i\varepsilon][y - m_5^2 + i\varepsilon]} .$$

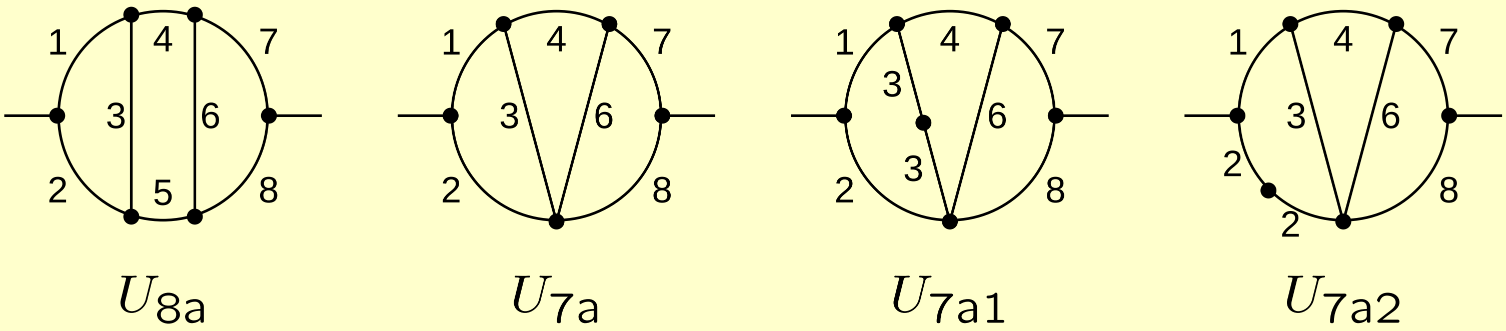
$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + xz)$$

- Integration over poles:

$$\int_{-\infty}^{\infty} dx \, \frac{f(s)}{x - \xi \pm i\varepsilon} = \mp i\pi f(\xi) + \int_0^{\infty} dx' \, \frac{f(\xi + x') - f(\xi - x')}{x'} .$$

Planar master topologies

- Similar approach for



⟶ UV-finite, no subtractions necessary

⟶ Two-dimensional numerical integrals, suitable for high-precision evaluation

- Implementation in TVID 2

- Adaptive Gauss-Kronrod integration
  [QUADPACK Piessens, de Doncker-Kapenga, Überhuber, Kahanger '83],
  quad precision floating points

- **Note:** Evaluation of C_0 in quad precision is rather slow
  ⟶ Use double precision C_0 from FF/LoopTools Hahn, Perez-Victoria '99
  ⟶ Final precision reduced to 6–8 digits

Implementation notes

- Currently no treatment for IR divergencies or threshold singularities

- Numerical instabilites in tail of $\int_{s_0}^{\infty} ds \ldots$
  $\longrightarrow$ Use asymptotic formulas for integrand above $s > s_{\text{cut}}$

- Some double-propagator integrals

$$U_{\text{6m1}}(\ldots) = \frac{\partial}{\partial m_3^2} U_{\text{6m}}(\ldots)$$
$$U_{\text{6m3}}(\ldots) = \frac{\partial}{\partial m_2^2} U_{\text{6m}}(\ldots)$$
$$U_{\text{6n2}}(\ldots) = \frac{\partial}{\partial m_2^2} U_{\text{6n}}(\ldots)$$

  produce big rational expressions in integrand,
  leading to 0/0 instabilities in integration region

  $\longrightarrow$ Use numerical differentiation of final function, using 5-point stencil

$$U_{\text{6m1}}(\ldots) \approx \frac{-U_{\text{6m}}(m_3^2 + 2\delta) + 8U_{\text{6m}}(m_3^2 + \delta) - 8U_{\text{6m}}(m_3^2 - \delta) + U_{\text{6m}}(m_3^2 - 2\delta)}{12\delta}$$

Checks and performance

$p^2 = 1.0$, $m_1^2 = 1.1$, $m_2^2 = 1.2$, $m_3^2 = 1.3$, $m_4^2 = 1.4$, $m_5^2 = 1.5$, $m_6^2 = 1.6$, $m_7^2 = 1.7$, $m_8^2 = 1.8$

	TVID 2.0		FIESTA 4.1 [Smirnov '16]	
	Result	Time* [s]	Result	Time* [s]
U_{4a}	38.7964435845(4)	6.6	38.80(1)	283
$\widetilde{U}_{5a}$	9.828362321(2)	0.5	9.830(2)	283
$\widetilde{U}_{6a}$	1.196967810(2)	0.5	1.1970(1)	315
$\widetilde{U}_{6m}$	−9.64795183(6)	160	−9.6480(1)	336
U_{8a}	0.1224166(1)	502	0.122418(1)	542
U_{4a1}	−1.4651121210(1)	1.5	1.465(3)	163
U_{7a2}	0.2200785(2)	559	0.220080(3)	269

* only numerical integration time, for $\mathcal{O}(\epsilon^0)$ parts

- $\widetilde{U}_{5a}$, $\widetilde{U}_{6a}$, $\widetilde{U}_{6m}$ are linear combinations that avoid $O(\epsilon)$ terms of 2-loop self-energy integrals
- FIESTA parameters:

```
CurrentIntegratorSettings = {{"epsrel","1e-05"},{"maxeval","5e6"}};
ComplexMode = False;
```

Checks and performance

$p^2 = 40$, $m_1^2 = 1.1$, $m_2^2 = 1.2$, $m_3^2 = 1.3$, $m_4^2 = 1.4$, $m_5^2 = 1.5$, $m_6^2 = 1.6$, $m_7^2 = 1.7$, $m_8^2 = 1.8$

	TVID 2.0		FIESTA 4.1 [Smirnov '16]	
	Result	Time* [s]	Result	Time* [s]
U_{4a}	$-149.6944621(5)$ $+9.6099138(5)\,i$	17.6	$-149.7(1)$ $+9.6(1)\,i$	3052
$\widetilde{U}_{5a}$	$53.705925142(1)$ $-20.874552008(1)\,i$	0.5	$53.71(8)$ $-20.88(8)\,i$	2865
$\widetilde{U}_{6m}$	$-11.094545131(6)$ $+4.390391111(6)\,i$	989	$-11.094(7)$ $+4.391(7)\,i$	5585
U_{8a}	$0.01238717(2)$ $-0.16344185(2)\,i$	253	$0.012353(3)$ $-0.016361(3)\,i$	11407

* only numerical integration time, for $\mathcal{O}(\epsilon^0)$ parts

- $\widetilde{U}_{5a}$, $\widetilde{U}_{6a}$, $\widetilde{U}_{6m}$ are linear combinations that avoid $O(\epsilon)$ terms of 2-loop self-energy integrals

- FIESTA parameters:

```
CurrentIntegratorSettings = {{"epsrel","1e-05"},{"maxeval","5e6"}};
ComplexMode = True;
```

Summary

Public code: **TVID 2**

Bauberger, Freitas, Wiegand '19

- Algebraic part (`Mathematica`) performs subtraction of UV-divergencies
  $\longrightarrow$ Symbolic expressions can get large!
- Numerical part (`C++`) performs numerical integrals
  $\longrightarrow$ Uses LoopTools for some integrals

	precision [digits]	timing [single core]
3-loop vacuum integrals	≥ 10	0.1s–30s
3-loop self-energy integrals	9–10 some cases 6–8	0.5s–16m

- Tested on Linux systems
- Soon available at `www.pitt.edu/~afreitas/`