The $O(\alpha^2)$ Initial State Radiation to e^+e^- Annihilation into a Neutral Vector Boson Revisited

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based on:

J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B791 (2019)
and further work in preparation

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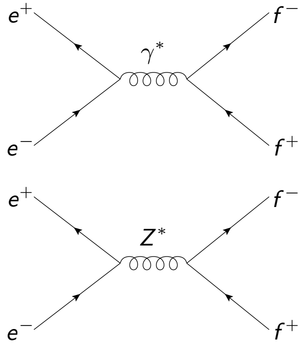

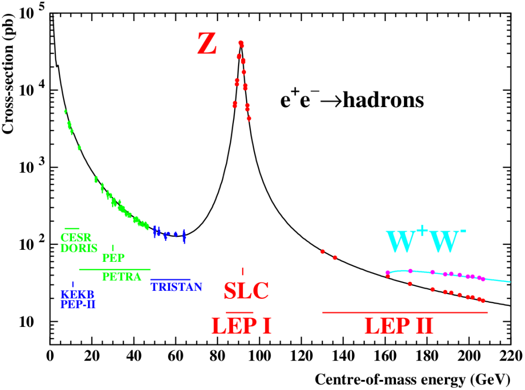

- We revisit the initial state corrections to $e^+ e^-$ annihilation to a neutral vector boson.
- This corrections are important for the prediction of the Z-boson peak and for $t\,\bar{t}$ production at LEP, ILC and FCC-ee, and at Higgs factories through $e^+ e^- \to Z^* H^0$.

We look at the process:

$$e^- + e^+ \rightarrow \gamma^*/Z^* \rightarrow f^- + f^+$$

with the invariants

$$(p_- + p_+)^2 = s, \qquad p_-^2 = p_+^2 = m_e^2, \qquad q^2 = s'$$

The initial state radiation (ISR) of n particles can be described according to the Drell-Yan mechanism

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s'} = \frac{\sigma^0(s')}{4s} \int \mathrm{d}^4 q \delta^+(q^2 - s') \frac{1}{(2\pi)^{3n}} \prod_{i=1}^{n} \int \mathrm{d}^4 k_i \delta^+(k_i^2 - m_i^2) \delta^{(4)}(p_- + p_+ - q - K) \, |T^{(n)}|^2$$

where $\sigma^0(s')$ describes the leading order process and $T^{(n)}$ the matrix element of the ISR process.

The $\mathcal{O}(\alpha)$ Corrections

- The first radiative corrections come from the process

$$e^+ + e^- \to \gamma^*/Z^* + \gamma.$$

- To stay in $d = 4$, we can split the contributions into hard, soft and virtual photons.
- The hard part is characterized by demanding $k^0 > \frac{\sqrt{s}\Delta}{2}$.
- The soft and virtual parts of the cross section have to be made infrared finite by introducing a small photon mass λ.
- The cross section is then given by

$$\frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s}\left(\frac{\alpha}{\pi}\right)\left[\delta\left(1-z\right)\left(\delta_1^{S_1}(\lambda,\Delta) + \delta_1^{V_1}(\lambda)\right) + \theta(1-z-\Delta)\delta_1^{H_1}(z)\right].$$

- The result is given by

$$\frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s}\frac{\alpha}{\pi}\left[\delta\left(1-z\right)\left(-2+\frac{3}{2}L+2\zeta_2+2(L-1)\ln(\Delta)\right)\right.$$
$$\left. + \theta(1-z-\Delta)\frac{1+z^2}{1-z}(L-1)\right] + \mathcal{O}\left(\frac{m^2}{s}\right)$$

with $L = \ln(s/m_e^2)$.

Why is it a 'revisit'?

ISR corrections have been calculated up to $O(\alpha^2)$ in the asymptotic limit $m_e^2/s \ll 1$ with two different techniques:

1. Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))
    - Full calculation with massive electrons in the limit $m_e^2 \ll s$ calculation in $d = 4$ with soft-hard separation, including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.
    - Expansion in $m_e^2 \ll s$ on integrand level (no details given).

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   - Expansion in $m_e^2 \ll s$ on integrand level (no details given).
2. Blümlein, De Freitas, van Neerven (Nucl. Phys. B855 (2012))
   - Direct calculation of the asymptotic limit $m_e^2 \ll s$ using massive light-cone operator matrix elements.
   - The technique is based on asymptotic factorization.

     Buza, Matiounine, Smith, Migneron, van Neerven (Nucl.Phys. B472 (1996))
   - It was already used in Berends et al, but only for the logarithmically enhanced terms, claiming it works only at that level.

Factorization in the Asymptotic Region

In the asymptotic region $m_e^2 \ll s$ the cross section factorizes

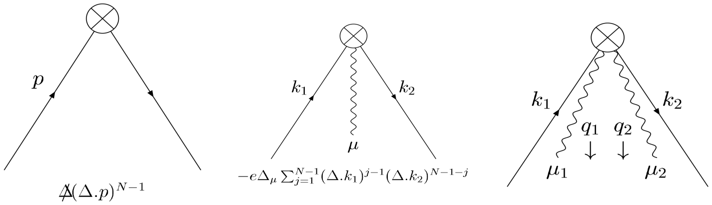
$$\frac{d\sigma_{ij}(s')}{ds'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{l,i}\left(z, \frac{\mu^2}{m_e^2}\right) \otimes \tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right) \otimes \Gamma_{k,j}\left(z, \frac{\mu^2}{m_e^2}\right)$$

into

- massless cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$
  Hamberg, van Neerven, Matsuura (Nucl. Phys. B359 (1991))
  Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))
- massive operator matrix elements $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence
  Blümlein, De Freitas, van Neerven (Nucl.Phys. B855 (2012))

$\sigma^{(0)}(s')$ is the Born cross section and the Mellin convolution $\otimes$ is given by

$$f(z) \otimes g(z) = \int_0^1 dz_1 \int_0^1 dz_2 \, f(z_1) g(z_2) \delta(z - z_1 z_2).$$



$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e| O_F^{\text{NS,S}} |e\rangle ,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma| O_F^{\text{S}} |\gamma\rangle ,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e| O_V^{\text{S}} |e\rangle ,$$

$$O_{F;\mu_1,\dots,\mu_N}^{\text{NS,S}} = i^{N-1}\text{S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\dots D_{\mu_N}\psi\right] - \text{traces},$$
$$O_{V;\mu_1,\dots,\mu_N}^{\text{S}} = 2i^{N-2}\text{S}\left[F_{\mu_1\alpha}D_{\mu_2}\dots D_{\mu_{N-1}}F_{\mu_N}^{\alpha}\right] - \text{traces}$$

- The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$.
- In the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process
    Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))

  through analytic calculations.

Factorization in the Asymptotic Region

The comparison between both calculations shows:

- the one-loop, i.e. $O(\alpha)$, results agree between both calculations
- the logarithmically enhanced terms at two-loops ($O(\alpha^2)$) agree between both calculations
- the constant terms do not agree

$\Rightarrow$ breakdown of asymptotic factorization or calculation errors?

The $\mathcal{O}(\alpha^2)$ Corrections

- In Berends et al the $\mathcal{O}(\alpha^2)$ corrections have been split up into four distinct processes:
  - Process I, photon radiation
  - Process II, non-singlet fermion pair production
  - Process III, pure-singlet fermion pair production
  - Process IV, interference between non-singlet and pure-singlet fermion pair production
- In the calculation of Blümlein et al (Nucl. Phys. B855 (2012)) process I and IV are combined owed to the nature of the OMEs.
- In the following we will present the calculation of fermion pair production, i.e. processes II-IV .

Our Approach to the recalculation:

- Full integration over the phase space in $d = 4$, i.e. no a-priori expansion in the electron mass.
- The phase space can be parametrized as

$$\int dPS_3 = \frac{1}{(2\pi)^6} \int d^4 q \int d^4 k_- \int d^4 k_+ \Big\{ \delta\left(q^2 - s'\right) \delta\left(k_-^2 - m^2\right) \delta\left(k_+^2 - m^2\right) \times \delta^{(4)}\left(p_- + p_+ - q - k_- - k_+\right) |T^{(2)}|^2 \Big\}$$

$$= \frac{1}{(4\pi)^4} \frac{1}{2\pi s} \int ds'' \int_{s_3^-}^{s_3^+} ds_3 \int_{-1}^{1} d\cos(\theta) \int_0^{\pi} d\phi |T^{(2)}|^2,$$

with the phase space boundaries

$$4m^2 \leq s'' \leq (\sqrt{s} - \sqrt{s'})^2,$$

$$s_3^{\pm} = \frac{1}{2}\left(s + s' - s'' + 2m^2 \pm \sqrt{1 - \frac{4m^2}{s''}} \lambda^{1/2}(s, s', s'') \right)$$

and $\lambda(s, s', s'') = s^2 + s'^{\,2} + s''^{\,2} - 2ss' - 2ss'' - 2s's''$.

Recalculation

Our Approach to the recalculation:

- Full integration over the phase space in $d = 4$, i.e. no a-priori expansion in the electron mass.

- Through partial fractioning, the angular integrals can be mapped to the form

$$I_{l,k} = \int_0^\pi d\theta \int_0^\pi d\phi \frac{\sin(\theta)}{[a + b\cos(\theta)]^l} \frac{1}{[A + B\cos(\theta) + C\sin(\theta)\cos(\phi)]^k}.$$

- All relevant integrals can be found in Beenakker et al., Phys. Rev. D40 (1989) , but have been recalculated for the current calculation.

- For example one finds:

$$I_{1,2} = \frac{2\pi(a(B^2 + C^2) - bAB)}{(A^2 - B^2 - C^2)X} + \frac{b(bA - aB)\pi}{X^{3/2}} \ln\left(\frac{aA - bB + \sqrt{X}}{aA - bB - \sqrt{X}}\right)$$

with $X = (aA - bB)^2 - (a^2 - b^2)(A^2 - B^2 - C^2)$

- After rationalizing the appearing square root, one can integrate the first invariant with standard techniques.

Recalculation

Our Approach to the recalculation:

- Full integration over the phase space in $d = 4$, i.e. no a-priori expansion in the electron mass.
- Three out of four integrations can be performed using standard techniques.
- The integrand of the last integral contains rational, logarithmic and polylogarithmic expressions with involved argument structures.
- The last integration is performed in terms of iterated integrals after determining the minimal set of contributing letters.

Iterated Integrals

- Iterated integrals can be recursively defined according to

$$\mathrm{H}_{w_1,\ldots,w_n}(x) = \int_0^x dt\, f_{w_n}(t)\, \mathrm{H}_{w_1,\ldots,w_{n-1}}(x).$$

- The letters w_i can in general be any function of t so that the integral on the right hand side is defined.
- The Kummer-Poincare (Goncharov) polylogarithms are defined by linear letters

$$f_{w_a}(t) = \frac{1}{t-a}, \quad a \in \mathbb{C}$$

  a special case are the harmonic polylogarithms
  Remiddi, Vermaseren (Int.J.Mod.Phys A15 (2000))

$$f_0(t) = \frac{1}{t}, \qquad f_1(t) = \frac{1}{1-t}, \qquad f_{-1}(t) = \frac{1}{1+t}.$$

- The letters can also contain square roots and dependence on external kinematic variables.
  Ablinger, Blümlein, Raab, Schneider (J.Math.Phys. 55 (2014))
- Iterated integrals are solutions to differential equation which factorize into first order terms.

Recalculation

- For the current calculation we also have to introduce the modified iterated integral

$$\tilde{\mathrm{H}}_{w_1,\dots,w_n}(x) = \int_x^1 dt\, f_{w_n}(t)\, \mathrm{H}_{w_1,\dots,w_{n-1}}(x).$$

- We want to use iterated integrals so we can work in a differential field.
- The steps to transform the last integrand to iterated integrals include:
  - Express all logarithms and polylogarithms in terms of iterated integrals evaluated at the last integration variable through linear differential equations.
  - Find relations between the occurring letters and square roots to get rid of redundancies.
  - Compactify the integrand expressed in terms of iterated integrals as far as possible.
    $\rightarrow$ Since we express everything in linearly independent quantities, the complexity of the last integral can be drastically reduced in this step.
- Some integrands took up $\mathcal{O}(1\,\mathrm{Mb})$ of disk space and the integration into iterated integrals needed $\mathcal{O}(1\,\mathrm{month})$.
- In total we need 37 letters to express the contributions due to fermion pair production.

$$v_1 = \frac{1}{\sqrt{1-4t}\sqrt{16t^2-8(1+z)t+(1-z)^2}}$$
$$v_2 = \frac{1}{t\sqrt{1-4t}\sqrt{16t^2-8(1+z)t+(1-z)^2}}$$
$$v_3 = \frac{1}{\sqrt{1-4t}\big(4t-(1+x)\big)\sqrt{16t^2-8(1+z)t+(1-z)^2}},$$
$$v_4 = \frac{1}{t\sqrt{1-t}},$$
$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_4 = \frac{1}{\big(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2\big)\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_5 = \frac{t}{\big(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2\big)\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_6 = \frac{1}{\big(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2\big)\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_7 = \frac{t}{\big(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2\big)\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_8 = \frac{1-z}{\big(4\rho-(1-z)t\big)\sqrt{1-t}},$$
$$d_9 = \frac{1}{\big(16\rho^2+4\big(z-2\rho(1+z)\big)t+(1-z)^2t^2\big)\sqrt{1-t}},$$
$$d_{10} = \frac{t}{\big(16\rho^2+4\big(z-2\rho(1+z)\big)t+(1-z)^2t^2\big)\sqrt{1-t}},$$
$$d_{11} = \frac{1}{t\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}},$$
$$d_{12} = \frac{1}{16\rho^2+4\big(z-2\rho(1+z)\big)t+(1-z)^2t^2},$$
$$d_{13} = \frac{t}{16\rho^2+4\big(z-2\rho(1+z)\big)t+(1-z)^2t^2},$$
$$d_{14} = \frac{1}{t(1-z)-4\rho},$$
$$d_{15} = \frac{1}{\sqrt{1-t}(t(1-z)-4\rho)},$$
$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2-16\rho^2}},$$
$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z)-4\rho)\sqrt{t(1-z)^2-16\rho^2}},$$
$$d_{18} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}},$$
$$d_{19} = \frac{1}{\sqrt{t}(t(1-z)-4\rho)\sqrt{t(1-z)^2-16\rho^2}},$$
$$d_{20} = \frac{1}{\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2-16\rho^2}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{23} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2-16\rho^2}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)},$$
$$d_{24} = \frac{1}{\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{25} = \frac{t}{\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{26} = \frac{1}{\sqrt{1-t}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{27} = \frac{t}{\sqrt{1-t}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2-16\rho^2}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{29} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)},$$
$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho(1+z)t+16\rho^2}},$$
$$d_{31} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2-16\rho^2}\big(t^2(1-z)^2-8\rho(1+z)t+4tz+16\rho^2\big)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}},$$
$$d_{33} = \frac{t}{\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}.$$

$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_4 = \frac{1}{\left(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\right)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_5 = \frac{t}{\left(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\right)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2 - 16\rho^2}},$$
$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z) - 4\rho)\sqrt{t(1-z)^2 - 16\rho^2}},$$
$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{26} = \frac{1}{\sqrt{1-t}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
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$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho(1+z)t + 16\rho^2}},$$
$$d_{31} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{33} = \frac{t}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}.$$

- 16 of these letters introduce elliptic structures, since multiple square roots cannot be rationalized at once.

- In total we need 37 letters to express the contributions due to fermion pair production.

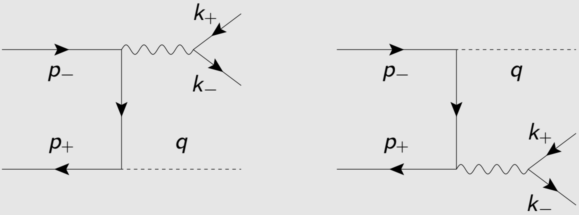
- The analytic results can be expanded in the electron mass.

- For the expansion one has to be careful not to simply expand the integrand of the iterated integral, since the integration boundary also depends on the mass.

- For the expansion we have a two stage procedure:
  1. Expand the integrand in m^2/s. This term serves as a subtraction term.
  2. Map the intgeration boundaries of the difference between the original integrand and the subtraction term to $(0, 1)$ and again expand in m^2/s, this will lead to a non-vanishing contribution.

- The result is validated through numerical integration to a high accuracy.

The Non-Singlet Case



- The two diagrams given above are contributing to the non-singlet process.
- The electron pair in the final state completely factorizes from the phase space integration.
- This property allows to derive a compact one dimensional integral representation.

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} &= \frac{\sigma^{(0)}(s')}{s}\, a^2 \Biggl\{ \frac{64}{3} z(1-z)(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_7} + \frac{256}{3} z\rho(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_6} \\
&+ \frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2}\tilde{\mathrm{H}}_{d_8,d_7} \\
&+ \frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3}\tilde{\mathrm{H}}_{d_8,d_6} \\
&+ \frac{16}{9(1-z)^2}\Bigl[(1+z)^2\bigl(4-9z+4z^2\bigr) + 2\bigl(9-16z+13z^2-2z^3\bigr)\rho + 32\rho^2\Bigr]\tilde{\mathrm{H}}_{d_2} \\
&+ \frac{512z\rho}{9(1-z)^4}\Bigl[3(1-z)^4 z - (1-z)^3\bigl(4+z^2\bigr)\rho - 2\bigl(9-29z+38z^2-17z^3+3z^4\bigr)\rho^2 \\
&- 4(2-z)\bigl(3+6z-5z^2\bigr)\rho^3 + 16\bigl(7-8z+9z^2\bigr)\rho^4 + 128(3-z)\rho^5\Bigr]\tilde{\mathrm{H}}_{d_4} \\
&- \frac{16}{9(1-z)^4}\Bigl[3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8\bigl(2-16z+9z^2 \\
&+ 4z^3-5z^4+2z^5\bigr)\rho + 16z\bigl(12-13z+18z^2-z^3\bigr)\rho^2 + 32\bigl(1+22z-7z^2\bigr)\rho^3\Bigr]\tilde{\mathrm{H}}_{d_1} \\
&- \frac{128z}{9(1-z)^4}\Bigl[1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2\bigl(7-55z+54z^2 \\
&+ 16z^3-17z^4+3z^5\bigr)\rho - 4\bigl(39-16z+16z^2+4z^3+5z^4\bigr)\rho^2 \\
&+ 16\bigl(8-23z+22z^2+9z^3\bigr)\rho^3 + 128\bigl(7+2z-z^2\bigr)\rho^4\Bigr]\tilde{\mathrm{H}}_{d_5} - \frac{64}{3}(2z+(1-z)\rho)\tilde{\mathrm{H}}_{d_3} \\
&+ \left[\frac{16}{3\sqrt{1-4\rho}}(1+z-4\rho)\tilde{\mathrm{H}}_{v_4} + \frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}}\tilde{\mathrm{H}}_{d_8}\right] \\
&\times \ln\left(\frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}\right)\Biggr\}
\end{aligned}
$$

The Non-Singlet Case

- The explicit expansion of the analytical result in the limit $m^2 \ll s$ gives

$$\frac{d\sigma^{(2),\mathrm{II}}(z)}{ds'} = \frac{\sigma^{(0)}(s')}{s} \left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ \frac{8}{3}\frac{1+z^2}{1-z}L^2 - \Bigg[\frac{16}{9}\frac{11-12z+11z^2}{1-z} - \frac{16}{3}\frac{1+z^2}{1-z}\mathrm{H}_0 - \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1\Bigg]L + \frac{32}{9(1-z)^3}\left(7-13z+8z^2-13z^3+7z^4\right) - \frac{16z}{9(1-z)^4}\left(3-36z+94z^2-72z^3+19z^4\right)\mathrm{H}_0 - \frac{8z^2}{3(1-z)}\mathrm{H}_0^2 + \left(\frac{32}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{2+z^2}{1-z}\mathrm{H}_0\right)\mathrm{H}_1 + \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1^2 + \frac{16z^2}{3(1-z)}\mathrm{H}_{0,1} - \frac{16\left(2+3z^2\right)}{3(1-z)}\zeta_2 \Bigg\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right),$$

with H the harmonic polylogarithms evaluated at argument z.

- The result contains higher denominator powers which have not been obtained by Berends et al.
- The result is in disagreement with Berends et al but agrees with the result from Bümlein et al using massive OMEs.
- Where does this disagreement come from?

- The disagreement can be traced back to the neglection of initial state masses.
- The non-singlet phase space for particles with different masses in the initial and final state is given by

$$\frac{d\sigma^{(2),\mathrm{II}}}{ds'} = \frac{\sigma^0(s')}{s}\, a^2 \int_{4m_f{}^2}^{s(1-\sqrt{z})^2} ds''\, \frac{16}{3s\; s''^{\,2}} \sqrt{1 - \frac{4m_f{}^2}{s''}}\,(2m_f{}^2 + s'')\Bigg\{$$

$$- \frac{\lambda^{1/2}(s,s',s'')\left[2s\,s'\,s'' + m_i{}^2\left(s^2 + (s'-s'')^2\right) + 4s\,m_i{}^4\right]}{s\,s'\,s'' + m_i^2\left(s^2 + (s'-s'')^2 - 2s\,(s'+s'')\right)}$$

$$+ \frac{(s'+s'')^2 + 4m_i{}^2(s-s'-s'') + s^2 - 8m_i{}^4}{\beta(s-s'-s'')} \ln\left(\frac{s-s'-s''+\beta\lambda^{1/2}(s,s',s'')}{s-s'-s''-\beta\lambda^{1/2}(s,s',s'')}\right)\Bigg\},$$

with $\beta = \sqrt{1 - 4m_i{}^2/s}$.

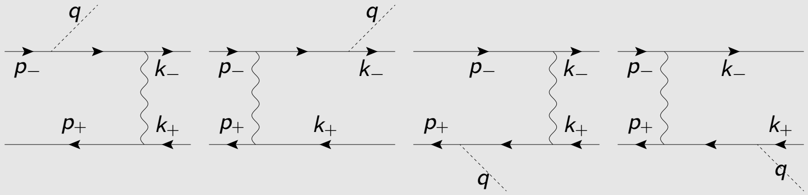
The Non-Singlet Case

- The disagreement can be traced back to the neglection of initial state masses.
- The non-singlet phase space for particles with different masses in the initial and final state is given by
- Neglecting initial state masses this expression reduces to

$$\frac{d\sigma^{(2),\mathrm{II}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\, a^2 \int_{4m^2}^{s(1-\sqrt{z})^2} ds'' \frac{16}{3s\; s''^{\,2}} \sqrt{1-\frac{4m^2}{s''}}(2m^2+s'')\Bigg\{ - 2\lambda^{1/2}(s,s',s'') + \frac{s^2+(s'+s'')^2}{s-s'-s''} \ln\left(\frac{s-s'-s''+\lambda^{1/2}(s,s',s'')}{s-s'-s''-\lambda^{1/2}(s,s',s'')}\right)\Bigg\}.$$

- This formula is the starting point of Berends et al (it was derived in Kniehl et al, Phys. Lett. B209 (1988))
- It is valid for the production of heavy particles, like muons, but not for electron pair production.

The Pure-Singlet Case



- In the pure-singlet case four diagrams contribute.
- The contributions of the left and right two diagrams contain mass divergences, their interference does not.
- The interference contribution in the limit $m^2 \ll s$ can simply be obtained by expanding in m^2/s and integrating the phase space.
- The squared contributions have to be treated as before, because of the more complicated topologies more involved letters are needed.

The Pure-Singlet Case

- The result expanded in $m^2 \ll s$ is given by

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{III}}}{ds'} &= \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ \left[\frac{4(1-z)\left(4+7z+4z^2\right)}{3z} + 8(1+z)\mathrm{H}_0\right] L^2 + \Bigg[-\frac{128(1-z)\left(1+4z+z^2\right)}{9z} \\
&- \frac{8\left(4+6z-3z^2-8z^3\right)}{3z}\mathrm{H}_0 - 16(1+z)\mathrm{H}_0^2 - \frac{16(1-z)\left(4+7z+4z^2\right)}{3z}\mathrm{H}_1 - 32(1+z)\mathrm{H}_{0,1} + 32(1+z)\zeta_2\Bigg] L \\
&- \frac{2(1-z)}{27z(1+z)^2}\left(80-303z-721z^2-789z^3-163z^4\right) - \Bigg(\frac{4}{9z(1+z)^3}\big(40+183z+339z^2+527z^3+825z^4 \\
&+ 462z^5+64z^6\big) - \frac{16\left(4+27z+3z^2-4z^3\right)}{3z}\mathrm{H}_{-1} + \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}^2\Bigg)\mathrm{H}_0 + \Bigg(\frac{4\left(-12-21z-12z^2+4z^3\right)}{3z} \\
&+ \frac{40\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\Bigg)\mathrm{H}_0^2 - \frac{8}{3}(3+5z)\mathrm{H}_0^3 + \Bigg(\frac{256(1-z)\left(1+4z+z^2\right)}{9z} - \frac{8\left(-4-18z+15z^2+4z^3\right)}{3z}\mathrm{H}_0 \\
&- \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_0^2\Bigg)\mathrm{H}_1 + \Bigg(\frac{16(1-z)\left(4+7z+4z^2\right)}{3z} - \frac{4\left(4-6z+3z^2\right)}{z}\mathrm{H}_0\Bigg)\mathrm{H}_1^2 \\
&+ \Bigg(\frac{8\left(4-6z+9z^2-12z^3\right)}{3z} + \frac{8\left(8+7z^2\right)}{z}\mathrm{H}_0 + \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1\Bigg)\mathrm{H}_{0,1} + \Bigg(\frac{16\left(-4-27z-3z^2+4z^3\right)}{3z} \\
&- \frac{32(5-2z)}{z}\mathrm{H}_0 + \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,-1} - \frac{32(2+z)}{z}\mathrm{H}_{0,0,1} + \frac{16\left(10-18z-5z^2\right)}{z}\mathrm{H}_{0,0,-1} \\
&- \frac{8\left(4-14z-5z^2\right)}{z}\mathrm{H}_{0,1,1} - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,-1,-1} + \Bigg(\frac{8\left(-8+27z+16z^3\right)}{3z} + 16(7-3z)\mathrm{H}_0 \\
&- \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1 - \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\Bigg)\zeta_2 + 32(5+z)\zeta_3\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right).
\end{aligned}
$$

- This result is again in disagreement with Berends et al but in agreement with the calculation by Blümlein et al using massive OMEs.

The Pure-Singlet Case

- Where does the disagreement come from?

- In the pure-singlet case a calculation done for massless partons was reused
  Schellekens, van Neerven (Phys.Rev. D21 (1980))
  - We agree with the interference term, which does not contain any mass singularity, although it was used with the wrong sign in Berends et al.

  - We disagree with the squared terms, which can be attributed to the neglection of mass effects going beyond the regularization of the integrals.

The Pure-Singlet–Non-Singlet Interference

- The last contribution of fermion pair production considered in Berends et al is the interference between the pure-singlet and non-singlet contributions.
- The expanded result is given by

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{IV}}}{ds'} &= \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2\Biggl\{-\Biggl[8(8-7z)+\frac{8\left(5-2z^2\right)}{1-z}\mathrm{H}_0+\frac{8\left(1+z^2\right)}{1-z}\left(\mathrm{H}_0^2+2\mathrm{H}_0\mathrm{H}_1\right.\\
&\quad\left.-2\mathrm{H}_{0,1}+2\zeta_2\right)\Biggr]L+\frac{8\left(27-42z+23z^2\right)}{1-z}+\Biggl[\frac{8}{(1-z)^2(1+z)}\left(3+10z-11z^2+22z^3-8z^4\right)\\
&\quad+\frac{64(1+z)}{1-z}\mathrm{H}_{-1}\Biggr]\mathrm{H}_0-\frac{8(1+z)^2}{1-z}\mathrm{H}_0^2-\frac{8\left(1+2z^2\right)}{3(1-z)}\mathrm{H}_0^3+\Biggl[16(8-7z)-\frac{8\left(3-2z-2z^2\right)}{1-z}\mathrm{H}_0\\
&\quad+\frac{16\left(2+z^2\right)}{1-z}\mathrm{H}_0^2\Biggr]\mathrm{H}_1+\frac{16}{1-z}\mathrm{H}_0\mathrm{H}_1^2+\Biggl[\frac{8\left(13-2z-6z^2\right)}{1-z}-\frac{16\left(5+4z^2\right)}{1-z}\mathrm{H}_0+\frac{32z^2}{1-z}\mathrm{H}_1\Biggr]\mathrm{H}_{0,1}\\
&\quad-\Biggl[\frac{64(1+z)}{1-z}-\frac{32\left(1+z^2\right)}{1-z}\mathrm{H}_0\Biggr]\mathrm{H}_{0,-1}+\frac{128\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,1}-\frac{64\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,-1}\\
&\quad-\frac{32\left(1+2z^2\right)}{1-z}\mathrm{H}_{0,1,1}-\Biggl[\frac{24\left(3-2z-2z^2\right)}{1-z}+\frac{16\left(2+3z^2\right)}{1-z}\mathrm{H}_0+\frac{32z^2}{1-z}\mathrm{H}_1\Biggr]\zeta_2\\
&\quad-\frac{16\left(3+z^2\right)}{1-z}\zeta_3\Biggr\}+\mathcal{O}\left(\frac{m^2}{s}L\right).
\end{aligned}
$$

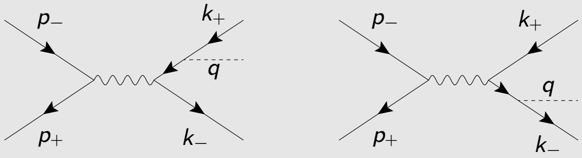
- This term shows less of the higher powers in the denominator than found in Berends et al.

The Pure-Singlet–Non-Singlet Interference

- Where do these discrepancies come from?
- We do not have an OME associated with the pure-singlet non-singlet interference. It has to be combined with the photon emissions.
- Berends et al do not provide more details on the calculation of this particular contribution.

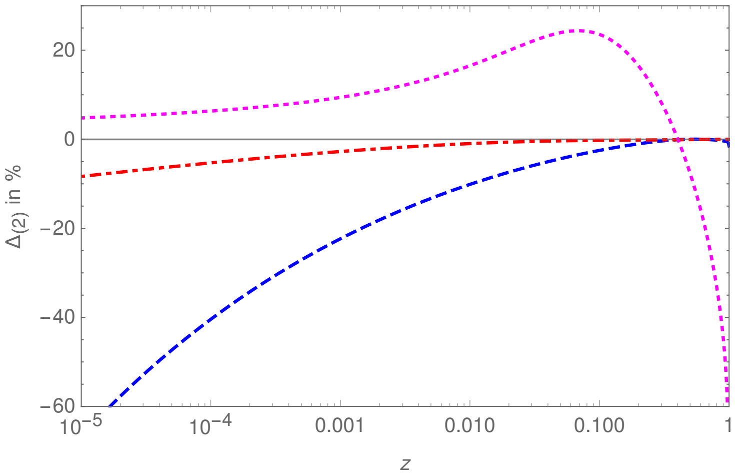
→ Probably the treatment of the mass expansion also lead to problems here.

Contributions not considered in Berends et al



- There are also contributions coming from the diagrams shown above, which have not been considered in Berends et al.
- We have the contributions from
  - only these diagrams,
  - their interference with the non-singlet (do only contribute for axial-vector couplings),
  - their interference with the pure-singlet.
- These contributions do not contain mass singularities and can simply be expanded in the limit $m^2 \ll s$ and have to agree with the massless calculation.
- We reproduce the results obtained in Hamberg et al, Nucl. Phys. B359 (1991).

Numerical Illustration



- Relative deviation of the non-singlet (red), pure-singlet (blue) and interference (magenta) contribution in %.

Corrections due to Photon Emissiom

- The corrections due to photon emission can be decomposed into six parts:
  - $\delta_2^{S_2}$, both photons are soft; ✓
  - $\delta_2^{V_2}$, both photons are virtual; ✓
  - $\delta_2^{S_1 V_1}$, one photon is soft, one virtual; ✓
  - $\delta_2^{S_1 H_1}$, one photon is soft, one hard; ✓
  - $\delta_2^{V_1 H_1}$, one photon is virtual, one hard;
  - $\delta_2^{H_2}$, both photons are hard. ✓

- The complete cross section can be expressed as

$$\frac{d\sigma}{ds'} = \frac{\sigma^{(0)}}{s}\left(\frac{\alpha}{\pi}\right)^2 \Bigg\{ \delta(1-z)\Big[\delta_2^{S_2}(\Delta,\lambda) + \delta_2^{V_2}(\lambda) + \delta_2^{S_1 V_1}(\Delta,\lambda)\Big]$$

$$+ \theta(1-z-\Delta)\Big[\delta_2^{S_1 H_1}(\Delta,\lambda,z) + \delta_2^{V_1 H_1}(\lambda,z) + \delta_2^{H_2}(\Delta,z)\Big]\Bigg\}.$$

Conclusions and Outlook

- ▶ All contributions due to fermion pair production have been recalculated.
- → In the non-singlet and pure-singlet processes agreement with the method based on asymptotic factorization has been found.
- → Numerically the differences at $\mathcal{O}(\alpha^2)$ are not negligible even though the logarithmically enhanced terms are unaffected.
- ⇒ Factorization in the asymptotic region works in the fermion-pair production channel also with massive external particles.
- ▶ The contributions due to axial couplings are work in progress. Since we work in $d = 4$ no problems with γ_5 arise.
- ▶ The last contribution due to photon production is work in progress.
- → All other terms have already been checked.