

# On The Four-Loop QCD Form Factors

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer, based on: Phys. Lett. **B744** (2015) 101,  
JHEP **1502** (2015) 120, Phys. Rev. **D93** (2016) 125014, Phys. Rev. **D95** (2017) 034030,  
Phys. Rev. **D99** (2019) 105010, Phys. Rev. **D99** (2019) 094014,  
JHEP **1905** (2019) 073, and work in progress

Michigan State University

# Outline

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  - The Dipole Conjecture
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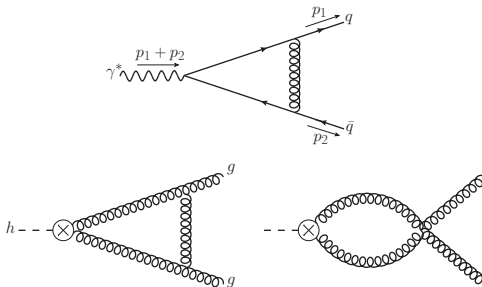
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$$q^2 \frac{\partial}{\partial q^2} \ln(\mathcal{F}(q^2/\mu^2, \alpha_s, \epsilon)) = 1/2\mathcal{K}(\alpha_s) + 1/2\mathcal{G}(q^2/\mu^2, \alpha_s, \epsilon)$$
$$\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{G}(q^2/\mu^2, \alpha_s, \epsilon) = \Gamma(\alpha_s)$$
$$\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{K}(\alpha_s) = -\Gamma(\alpha_s)$$



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At  $L$  loops,  $\Gamma_L$  characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

$\implies \Gamma_4$  is the last unknown ingredient needed for  $N^3LL$  resummation!

# A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, JHEP **0906** (2009) 081; E. Gardi and L. Magnea, JHEP **0903** (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar  $SU(N_c)$   $\mathcal{N} = 4$  super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1, \dots, p_n) = \exp \left\{ -\frac{1}{2} \sum_{L=1}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^L \mu_\epsilon^{2L\epsilon} \int_0^{\mu_\epsilon^2} d\mu^2 (\mu^2)^{-1-L\epsilon} \right. \\ \left. \sum_{\substack{i,j=1 \\ i < j}}^n \left( \Gamma_{1;L}^{\mathcal{N}=4} \ln \left( \frac{\mu^2}{-s_{ij}} \right) + \mathcal{G}_{1;L}^{\mathcal{N}=4} \right) \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{N_c} \right\} \sum_{L=0}^{\infty} \mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon; p_1, \dots, p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, JHEP **0706** (2007) 064). In a nutshell, the dipole conjecture is the suggestion that, with minor modifications, the above structure could hold for more general gauge theories like QCD.

# When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the  $n_f$  terms, it is now clear that the dipole conjecture fails for QCD due to three-loop calculations which probe the structure of the soft anomalous dimension matrix.

S. Caron-Huot, JHEP **1505** (2015) 093; Ø. Almelid *et. al.*, Phys. Rev. Lett. **117** (2016) 172002;

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In light of the above, one might expect Casimir scaling:

$$\Gamma_L^g \stackrel{?}{=} C_A/C_F \Gamma_L^q$$

to break down as well for some  $L$ .

R. Boels *et. al.*, Phys. Rev. Lett. **119** (2017) 201601; JHEP **1801** (2018) 153

S.-O. Moch *et. al.*, JHEP **1710** (2017) 041; Phys. Lett. **B782** (2018) 627

R. N. Lee *et. al.*, JHEP **1902** (2019) 172; J. M. Henn *et. al.*, Phys. Rev. Lett. **122** (2019) 201602

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⇒ To go beyond low-precision numerics, new multi-loop methods!

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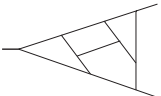
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Even for the cusp anomalous dimensions, many new developments!

# Complete Results For The Planar Master Integrals

A. von Manteuffel and RMS, JHEP **1905** (2019) 073

Complete **weight 8** results obtained for all **99** master integrals, *e.g.*

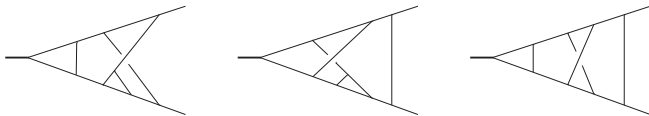


$$\begin{aligned}
 &= \frac{1}{\epsilon^7} \left( -\frac{1}{288} \right) + \frac{1}{\epsilon^6} \left( \frac{13}{576} \right) + \frac{1}{\epsilon^5} \left( -\frac{19}{144} \zeta_2 - \frac{101}{576} \right) + \dots - \frac{44097}{64} \zeta_7 \\
 &- \frac{577}{6} \zeta_5 \zeta_2 - \frac{1993}{16} \zeta_3 \zeta_2^2 + \frac{9101}{72} \zeta_3^2 + \frac{240637}{1344} \zeta_2^3 - \frac{24965}{32} \zeta_5 - \frac{61}{144} \zeta_3 \zeta_2 + \frac{99523}{480} \zeta_2^2 \\
 &- \frac{88745}{144} \zeta_3 + \frac{13319}{72} \zeta_2 + \frac{382375}{144} + \epsilon \left( \frac{4931}{30} \zeta_{5,3} - \frac{60385}{24} \zeta_5 \zeta_3 + \frac{1321}{36} \zeta_3^2 \zeta_2 \right. \\
 &- \frac{702196781}{1008000} \zeta_2^4 + \frac{21687}{8} \zeta_7 + \frac{9989}{24} \zeta_5 \zeta_2 + \frac{339821}{1440} \zeta_3 \zeta_2^2 - \frac{40519}{288} \zeta_3^2 - \frac{3681977}{5040} \zeta_3^3 \\
 &\left. + \frac{155087}{48} \zeta_5 - \frac{1457}{3} \zeta_3 \zeta_2 - \frac{179053}{288} \zeta_2^2 + \frac{381811}{144} \zeta_3 - \frac{10777}{24} \zeta_2 - \frac{2005247}{144} \right) + \mathcal{O}(\epsilon^2)
 \end{aligned}$$

# The $\mathcal{O}(N_{q\gamma}N_f)$ Quark Form Factor Calculation

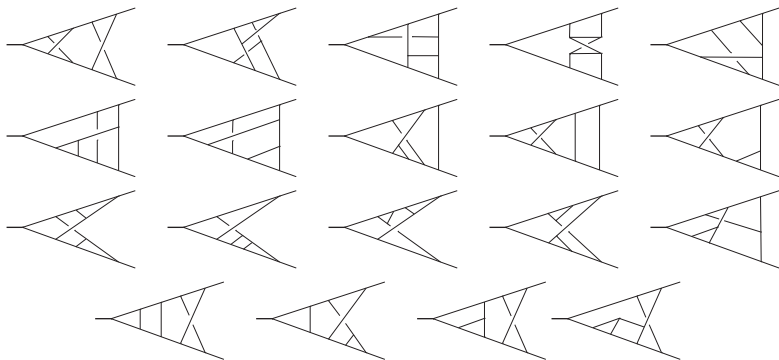
$$N_{q\gamma} = \frac{1}{e_q} \sum_{q'} e_{q'}$$

- **923** topologies to be reduced with up to **5** numerator insertions.
- The **3** non-planar top-level integral topologies are all **reducible**.
- **1** color structure and **45** master integrals (**42** topologies).
- **42** master integrals for  $\mathcal{O}(\epsilon^0)$  in the finite integral method.



# The $\mathcal{O}(N_f^2)$ Gluon Form Factor Calculation

- 1781 topologies to be reduced with up to 6 numerator insertions.
- 4 of 19 non-planar top-level integral topologies are **irreducible**.
- 4 color structures and 163 master integrals (151 topologies).
- 130 master integrals for  $\mathcal{O}(\epsilon^0)$  in the finite integral approach.





# The $\mathcal{O}(N_{q\gamma}N_f)$ Bare Four-Loop Quark Form Factor

In the  $\overline{\text{MS}}$  scheme, we find

$$\begin{aligned} \mathcal{F}_4^q(\epsilon) \Big|_{\frac{d_F^{abc} d_F^{abc}}{N_F}} &= \frac{1}{\epsilon} \left( \frac{1280}{3} \zeta_5 + \frac{32}{5} \zeta_2^2 - \frac{224}{3} \zeta_3 - 160 \zeta_2 - 64 \right) \\ &+ 2304 \zeta_3^2 + \frac{545792}{945} \zeta_3^3 - \frac{8512}{9} \zeta_5 - \frac{1888}{3} \zeta_2 \zeta_3 + \frac{12448}{45} \zeta_2^2 - \frac{3520}{9} \zeta_3 \\ &- \frac{16928}{9} \zeta_2 - \frac{11296}{9} + \mathcal{O}(\epsilon) \end{aligned}$$

in  $R_\xi$  gauge, keeping terms of  $\mathcal{O}(1 - \xi)$ , where

$$T_F = 1/2 \quad \text{and} \quad \frac{d_F^{abc} d_F^{abc}}{N_F} = \frac{(N_c^2 - 1)(N_c^2 - 4)}{16N_c^2}$$

# The $\mathcal{O}(N_f^2)$ Bare Four-Loop Gluon Form Factor

$$\begin{aligned}
 \mathcal{F}_4^g(\epsilon) \Big|_{C_A^2} &= \frac{1}{\epsilon^6} \left( \frac{41}{162} \right) + \frac{1}{\epsilon^5} \left( \frac{113}{486} \right) + \frac{1}{\epsilon^4} \left( -\frac{128}{81} \zeta_2 - \frac{151}{27} \right) + \frac{1}{\epsilon^3} \left( -\frac{5167}{243} \zeta_3 \right. \\
 &+ \left. \frac{799}{243} \zeta_2 - \frac{1085641}{34992} \right) + \frac{1}{\epsilon^2} \left( -\frac{7087}{405} \zeta_2^2 - \frac{35848}{729} \zeta_3 + \frac{425}{9} \zeta_2 - \frac{1144163}{209952} \right) \\
 &+ \frac{1}{\epsilon} \left( -\frac{73282}{405} \zeta_5 + \frac{26396}{243} \zeta_2 \zeta_3 - \frac{37894}{1215} \zeta_2^2 + \frac{28705}{324} \zeta_3 + \frac{2504629}{17496} \zeta_2 + \frac{541388327}{419904} \right) \\
 &+ \frac{717266}{729} \zeta_3^2 + \frac{58858}{945} \zeta_3^3 - \frac{62671}{1215} \zeta_5 - \frac{87329}{1458} \zeta_2 \zeta_3 + \frac{380701}{3240} \zeta_2^2 + \frac{27036815}{52488} \zeta_3 \\
 &- \frac{53253361}{104976} \zeta_2 + \frac{110016540845}{7558272} + \mathcal{O}(\epsilon) \\
 \mathcal{F}_4^g(\epsilon) \Big|_{C_F^2} &= \frac{1}{\epsilon^2} \left( \frac{1}{2} \right) + \frac{1}{\epsilon} \left( 320 \zeta_5 - 236 \zeta_3 - \frac{74}{3} \right) + \frac{2432}{3} \zeta_3^2 + \frac{32512}{135} \zeta_2^3 + 3360 \zeta_5 \\
 &+ \frac{128}{3} \zeta_3 \zeta_2 - \frac{484}{3} \zeta_2^2 - \frac{36260}{9} \zeta_3 + \frac{439}{9} \zeta_2 - \frac{80281}{216} + \mathcal{O}(\epsilon)
 \end{aligned}$$

# The $\mathcal{O}(N_f^2)$ Bare Four-Loop Gluon Form Factor

$$\begin{aligned}
 \mathcal{F}_4^g(\epsilon) \Big|_{C_A C_F} &= \frac{1}{\epsilon^4} \left( -\frac{7}{3} \right) + \frac{1}{\epsilon^3} \left( 26\zeta_3 - \frac{1115}{36} \right) \\
 &+ \frac{1}{\epsilon^2} \left( \frac{806}{45} \zeta_2^2 + \frac{580}{9} \zeta_3 + \frac{148}{9} \zeta_2 - \frac{12749}{108} \right) + \frac{1}{\epsilon} \left( \frac{442}{9} \zeta_5 - \frac{1132}{9} \zeta_2 \zeta_3 + \frac{1784}{45} \zeta_2^2 \right. \\
 &- \frac{5797}{27} \zeta_3 + \frac{1475}{18} \zeta_2 + \frac{642623}{1296} \Big) - \frac{4354}{3} \zeta_3^2 - \frac{143524}{945} \zeta_2^2 - \frac{47948}{27} \zeta_5 - \frac{3688}{9} \zeta_2 \zeta_3 \\
 &- \frac{13820}{81} \zeta_2^2 - \frac{538853}{162} \zeta_3 - \frac{27629}{81} \zeta_2 + \frac{3697777}{324} + \mathcal{O}(\epsilon) \\
 \mathcal{F}_4^g(\epsilon) \Big|_{\frac{d_F^{abcd} d_F^{abcd}}{N_A}} &= \frac{1}{\epsilon} \left( 128\zeta_3 - \frac{176}{3} \right) + 512\zeta_3^2 - 960\zeta_5 + \frac{384}{5} \zeta_2^2 + 1520\zeta_3 - \frac{9008}{9} \\
 &+ \mathcal{O}(\epsilon), \quad \text{where} \quad \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{(N_c^2 - 1)(N_c^4 - 6N_c^2 + 18)}{96N_c^2}
 \end{aligned}$$

## $\mathcal{O}(N_f^2)$ Casimir Scaling Works As Expected

$$\Gamma_4^q \Big|_{N_f^2} = C_A C_F \left[ -\frac{224}{15} \zeta_2^2 + \frac{2240}{27} \zeta_3 - \frac{608}{81} \zeta_2 + \frac{923}{81} \right] \\ + C_F^2 \left[ \frac{64}{5} \zeta_2^2 - \frac{640}{9} \zeta_3 + \frac{2392}{81} \right]$$

$$\Gamma_4^q \Big|_{N_{q\gamma} N_f} = 0$$

$$\Gamma_4^g \Big|_{N_f^2} = C_A^2 \left[ -\frac{224}{15} \zeta_2^2 + \frac{2240}{27} \zeta_3 - \frac{608}{81} \zeta_2 + \frac{923}{81} \right] \\ + C_A C_F \left[ \frac{64}{5} \zeta_2^2 - \frac{640}{9} \zeta_3 + \frac{2392}{81} \right]$$

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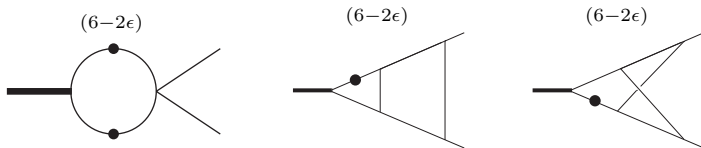
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- We could numerically check the expansion coefficients of the other masters to about part per mille precision using FIESTA 4.

# The Finite Integral Method

Andreas von Manteuffel, Erik Panzer, and RMS,  
JHEP **1502** (2015) 120, Phys. Rev. **D93** (2016) 125014

Very simple in principle, "just" choose a finite basis, *e.g.*



However...

A  $L$ -loop finite integral basis requires reducing integrals with  $L$  additional dots (*i.e.* propagators of higher multiplicity)

O. Tarasov, Phys. Rev. **D54** (1996) 6479

# Syzygy-Based IBPs Without Numerators

J. Gluza *et. al.*, Phys. Rev. **D83** (2011) 045012; RMS, JHEP **1201** (2012) 077;

R. N. Lee, Moriond QCD 2014 proceedings; T. Bitoun *et. al.*, Lett. Math. Phys. **109** (2019) 497;

Lee-Pomeransky representation:

$$I(\nu_1, \dots, \nu_N) = \mathcal{N} \left[ \prod_{i=1}^N \int_0^\infty dx_i x_i^{\nu_i-1} \right] G^{-d/2} \quad \text{with } G = \mathcal{U} + \mathcal{F}$$

linear relations from annihilators, of arbitrary order in principle:

$$\left[ c_0 + \sum_{i=1}^N c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^N c_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \dots \right] G^{-d/2} = 0$$

determine  $c_0(x_1, \dots, x_N), \dots$  via syzygy equations:

$$c_0 \left[ -\frac{2}{d} G^2 \right] + \sum_{i=1}^N c_i \left[ G \frac{\partial G}{\partial x_i} \right] + \sum_{i,j=1}^N c_{ij} \left[ G \frac{\partial^2 G}{\partial x_i \partial x_j} + \left( -\frac{d}{2} - 1 \right) \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_j} \right] + \dots = 0$$

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- It makes a **HUGE** difference whether you insert zeta values for the master integrals in a random basis or a “good” finite one.
- Here, good choices are those which decouple a maximal number of  $\epsilon$  expansion coefficients from the physical form factors.

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$$\begin{array}{c} (6-2\epsilon) \\ \text{Diagram} \end{array} = -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)$$

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In the conventional integral basis selected by **Reduze 2**, one would require the **weight 9**  $\epsilon$  expansion coefficients to recover **weight 6** information at the level of the gluon form factor!

# Finite + Uniform Weight Integrals

RMS, Phys. Rev. **D99** (2019) 105010; A. von Manteuffel and RMS, JHEP **1905** (2019) 073

It seems that the best you can do is to take a basis which is both **finite** AND **uniform weight**, such that non-factorizable topologies have as high a maximal weight at leading order in  $\epsilon$  as possible.



# Finite + Uniform Weight Integrals

RMS, Phys. Rev. **D99** (2019) 105010; A. von Manteuffel and RMS, JHEP **1905** (2019) 073

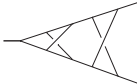
It seems that the best you can do is to take a basis which is both **finite** AND **uniform weight**, such that non-factorizable topologies have as high a maximal weight at leading order in  $\epsilon$  as possible.

$$\begin{aligned}
 & (6-2\epsilon) \text{ [Diagram: A triangle with a dot on a horizontal line inside] } \\
 &= \frac{35}{2}\zeta_7 - \frac{15}{4}\zeta_5\zeta_2 + \frac{18}{5}\zeta_3\zeta_2^2 - \frac{17}{4}\zeta_3^2 - \frac{131}{140}\zeta_2^3 - \frac{75}{4}\zeta_5 + \frac{1}{2}\zeta_3\zeta_2 \\
 &+ \frac{1}{10}\zeta_2^2 + 3\zeta_3 + \epsilon \left( -\frac{383}{20}\zeta_{5,3} - \frac{29}{2}\zeta_5\zeta_3 - \frac{9}{2}\zeta_3^2\zeta_2 + \frac{83581}{5250}\zeta_2^4 + \frac{2677}{16}\zeta_7 + 43\zeta_5\zeta_2 \right. \\
 &\left. + \frac{261}{20}\zeta_3\zeta_2^2 - 35\zeta_3^2 - \frac{1943}{105}\zeta_2^3 - 322\zeta_5 - 6\zeta_3\zeta_2 + \frac{18}{5}\zeta_2^2 + 60\zeta_3 \right) + \mathcal{O}(\epsilon^2)
 \end{aligned}$$

In practice, **compatibility** with uniform weight is all that matters.

# Outlook

Many obvious directions for future research!



$$\begin{aligned}
 &= \frac{1}{\epsilon^8} \left( \frac{1}{72} \right) + \dots - \frac{86152}{15} \zeta_{5,3} + \frac{47869}{12} \zeta_3 \zeta_5 + \frac{31999}{36} \zeta_2 \zeta_3^2 \\
 &+ \frac{33270103}{21000} \zeta_2^4 + \frac{134135}{2} \zeta_7 - 8110 \zeta_2 \zeta_5 + \frac{23202}{5} \zeta_2^2 \zeta_3 + \frac{948274}{315} \zeta_2^3 + \frac{8717}{3} \zeta_3^2 \\
 &- \frac{43085}{2} \zeta_5 + \frac{5230}{3} \zeta_2 \zeta_3 + \frac{117067}{30} \zeta_2^2 - \frac{76915}{6} \zeta_3 + \frac{28828}{3} \zeta_2 - \frac{183475}{24} + \mathcal{O}(\epsilon)
 \end{aligned}$$

- The full quark and gluon cusp anomalous dimensions.
- Further improvements to our no-numerator syzygy generator.
- Improved understanding of finite + uniform weight bases.
- Full results for the four-loop form factors.