Master Integrals for the two-loop, non-planar QCD corrections to top-quark pair production in the quark-annihilation channel

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Outline

- Analytic two-loop corrections to top pair hadroproduction
- Calculation of the last missing two-loop Master Integrals for the two-loop amplitude in the quark-annihilation channel

Talk based on **M. Becchetti, R. Bonciani, V. Casconi, AF, S. Lavacca, A. von Manteuffel arXiv:1904.10834**;
(see also **S. Di Vita, T. Gehrmann, S. Laporta, P. Mastrolia, A. Primo, and U. Schubert arXiv:1904.10964**)

Top-pair production at NNLO

Top-pair hadroproduction total and differential cross sections are known to NNLO in QCD

Barnreuter, Czakon, Fiedler, Heymes, Mitov (2012-2016)

→ Crucial result for LHC phenomenology but also a landmark calculation in perturbative QCD

Further improvements:

- Matching of NNLO QCD with NLO EW corrections

  Czakon, Heymes, Mitov, Pagani, Tsinikis, Zaro (2017)

- Resummation of threshold and small mass logarithms for the top pair invariant mass and top quark transverse momentum distributions to NNLO+NNLL' in QCD

  Czakon, AF, Heymes, Mitov, Pecjak, Scott, Wang, Yang (2018)

- Recalculation with qT subtraction of total and differential NNLO cross sections

  Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Sargsyan (2019)

Top-pair production at NNLO

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→
in

One of the main technical difficulties in the evaluation of NNLO corrections is the calculation of the two-loop diagrams for the $2 \to 2$ partonic processes

The calculation of the two-loop corrections was carried out numerically, on a grid of point covering the physical region In the s-t plane, for fixed value of the top quark mass

Czakon (2008) Barnreuter, Czakon, Fiedler (2013)

Fu

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2017)

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Czakon, AF, Heymes, Mitov, Pecjak, Scott, Wang, Yang (2018)

- Recalculation with qT subtraction of total and differential NNLO cross sections

Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Sargsyan (2019)

Analytic calculations of the two-loop corrections

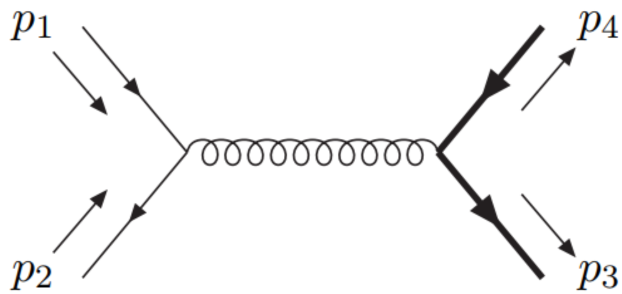
A project aiming to the analytic calculation of the two-loop corrections to top pair production was started a long time ago (2006). Several partial results were obtained and published but a complete analytic calculation of the two-loop corrections is still missing

An analytic calculation of the two-loop corrections to top pair production is still relevant because

- It provides an independent check of the results obtained numerically
- It could provide a faster and cheaper (in terms of CPU time) way to evaluate two-loop corrections needed in order to obtain phenomenological predictions
- It can teach us something interesting about Feynman diagrams

Partonic channels

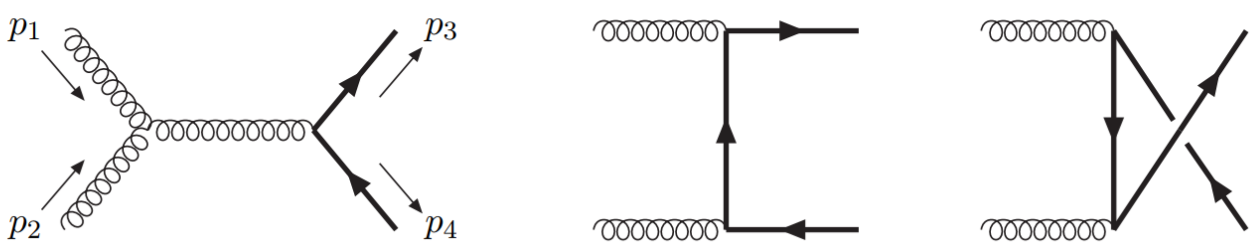
Quark annihilation channel



$$p_1^2 = p_2^2 = 0$$
$$p_3^2 = p_4^2 = m^2$$

$$q(p_1) + \overline{q}(p_2) \longrightarrow t(p_3) + \overline{t}(p_4)$$

Gluon fusion channel



$$g(p_1) + g(p_2) \longrightarrow t(p_3) + \overline{t}(p_4)$$

Mandelstam invariants

$$s = -(p_1 + p_2)^2 \,, \quad t = -(p_1 - p_3)^2 \,, \quad u = -(p_1 - p_4)^2$$
$$s + t + u = 2m^2$$

Loop corrections to the amplitude summed over spins and color depend on three parameters: s, t, and the top quark mass

Two loop gluon-fusion channel

Two-loop X Tree-level interference

16 color factors (A,B,C, …), all known numerically

Barnreuter, Czakon, Fidler, (2013)

$$\mathcal{A}_2^{(2\times0)} = (N_c^2-1)\left\{N_c^3 A + N_c B + \frac{1}{N_c}C + \frac{1}{N_c^3}D + N_c^2 N_l E_l + N_c^2 N_h E_h + N_l F_l + N_h F_h \right.$$
$$+\frac{N_l}{N_c^2}G_l + \frac{N_h}{N_c^2}G_h + N_c N_l^2 H_l + N_c N_h^2 H_h + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l$$
$$\left. + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh} \right\},$$

The color factors are functions of the Mandelstam invariants: A(s,t,m), B(s,t,m), etc.

Analytical calculations:

- All IR poles **AF, Neubert, Pecjak, Yang (2009)**
- Factor A **Bonciani, AF, Gehrmann, von Manteuffel, Studerus (2010)**
- Factor proportional to N_l (E_l, F_l, G_l, H_l, H_lh, I_l, I_lh)

von Manteuffel, Studerus (2013)

Bonciani, AF, Gehrmann, von Manteuffel, Studerus (2013)

Two loop gluon-fusion channel

Only planar diagrams

16 color factors (A,B,C, …), all known numerically

Barnreuter, Czakon, Fidler, (2013)

$$\mathcal{A}_2^{(2\times 0)} = (N_c^2 - 1)\Big\{ N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h + N_l F_l + N_h F_h$$
$$+ \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l$$
$$+ \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh} \Big\},$$

The color factors are functions of the Mandelstam invariants: A(s,t,m), B(s,t,m), etc.

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Two loop gluon-fusion channel

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$$\mathcal{A}_2^{(2\times0)} = (N_c^2-1)\Big\{N_c^3 A + \boxed{N_c B + \frac{1}{N_c}C + \frac{1}{N_c^3}D} + N_c^2 N_l E_l + \boxed{N_c^2 N_h E_h} + N_l F_l + \boxed{N_h F_h}$$

$$+\frac{N_l}{N_c^2}G_l + \boxed{\frac{N_h}{N_c^2}G_h} + N_c N_l^2 H_l + N_c N_h^2 H_h + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c}I_l$$

$$+\frac{N_h^2}{N_c}I_h + \frac{N_l N_h}{N_c}I_{lh}\Big\},$$

Involve elliptic integrals

MIs for the planar topologies involving a closed top loop are known
Adams, Chaubey, Weinzierl (2018)

Analytical calculations:

- All IR poles **AF, Neubert, Pecjak, Yang (2009)**
- Factor A **Bonciani, AF, Gehrmann, von Manteuffel, Studerus (2010)**
- Factor proportional to N_l (E_l, F_l, G_l, H_l, H_lh, I_l, I_lh)

von Manteuffel, Studerus (2013)

Bonciani, AF, Gehrmann, von Manteuffel, Studerus (2013)

Two loop quark-annihilation channel

10 color factors (A,B,C, …), all known numerically

Czakon (2008)

$$\mathcal{A}_2^{(2\times 0)} = N_c C_F \left[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \right],$$

Analytical calculations:

- All IR poles **AF, Neubert, Pecjak, Yang (2009)**
- Factors proportional to N_l and/or N_h (D_l,E_l,D_h,E_h,F_l,F_lh,F_h)
  Bonciani, AF, Gehrmann, Maitre, Studerus (2008)
- Factor A **Bonciani, AF, Gehrmann, Studerus (2009)**

Two loop quark-annihilation channel

10 color factors (A,B,C, …), all known numerically

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$$\mathcal{A}_2^{(2\times 0)} = N_c C_F \left[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \right]$$

Still unavailable analytically, but calculable in terms of Generalized Harmonic Polylogarithms

Analytical calculations:

- All IR poles **AF, Neubert, Pecjak, Yang (2009)**
- Factors proportional to N_l and/or N_h (D_l,E_l,D_h,E_h,F_l,F_lh,F_h)
  Bonciani, AF, Gehrmann, Maitre, Studerus (2008)
- Factor A **Bonciani, AF, Gehrmann, Studerus (2009)**

Quark-annihilation channel: Coefficients B and C

It is possible to calculate analytically the coefficients B and C in the quark annihilation channel: This, together with the results already obtained in the literature, would complete the analytic calculation of the two-loop corrections in quark-annihilation

- The calculation requires the evaluation of the MIs for a few non-planar seven denominator topologies which do not enter in the other color factors
- The result can be written in terms of Generalized Harmonic Polylogarithms (GPLs) – no elliptic integrals
- The only missing planar topology was can be obtained from a calculation of two-loop corrections to electron muon scattering

Mastrolia, Passera, Primo, Schubert (2017)

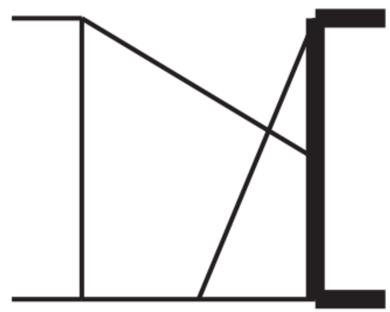
- <u>Here we focus on non-planar topologies</u>

Non-planar topologies for the coefficients B and C

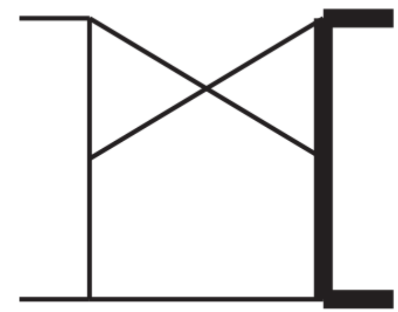
Thin lines → massless propagators,
Thick lines → massive propagators

Von Manteuffel, Studerus (2013)
(first crossed two-loop box with a massive propagator)

Di Vita, Gehrmann Laporta, Mastrolia, Primo, Schubert, (2019)
and 1904.10834
Becchetti, Bonciani, Casconi, AF, Lavacca, von Mantheuffel (2019)



(A)



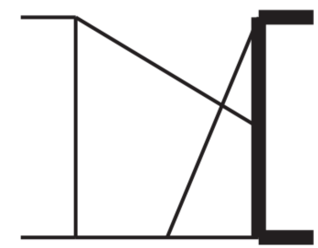
(B)

Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)
Lee, Mingulov (2019)
Recalculated in arXiv:1904.10834

Statement of the problem

Topology A

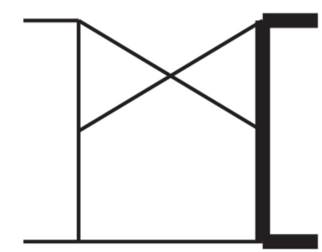
$$\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \, \frac{D_4^{-a_4} \, D_6^{-a_6}}{D_1^{a_1} \, D_2^{a_2} \, D_3^{a_3} \, D_5^{a_5} \, D_7^{a_7} \, D_8^{a_8} \, D_9^{a_9}}$$



(A)

Topology B

$$\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \, \frac{D_5^{-b_5} \, D_6^{-b_6}}{D_1^{b_1} \, D_2^{b_2} \, D_3^{b_3} \, D_4^{b_4} \, D_7^{b_7} \, D_8^{b_8} \, D_9^{b_9}}$$



(B)

Set of denominators

$$D_i = \{-k_1^2, -k_2^2, -(p_1+k_1)^2, -(p_1+k_1+k_2)^2, -(k_1+p_1+p_2)^2, -(k_2+p_1+p_2)^2, -(k_1+k_2+p_1+p_2)^2, m^2-(k_1+k_2+p_3)^2, m^2-(k_2+p_3)^2\}.$$

$$\mathcal{D}^d k_i = \frac{d^d k_i}{i\pi^{\frac{d}{2}}} e^{\epsilon\gamma_E} \left(\frac{m^2}{\mu^2}\right)^{\epsilon}$$

a_i and b_i, with $i = 1, ..., 9$, are integer numbers where $a_4, a_6, b_5, b_6 \leq 0$

Strategy

- Reduction to MIs by means of Integration by Parts as implemented in the public codes **REDUZE** and **FIRE**

  Studerus (2009) von Manteffel, Studerus (2012)

  Smirnov (2008)

- Build the system of differential equations satisfied by the MIs

- Rotation to the canonical basis (i.e. a basis where, at each order in the dimensional regulator, all terms have uniform transcendentality → Sum of Generalized Harmonic Polylogarithms with the same weight)

  Henn (2013)

- Calculation of the MIs with the differential equation method (solution written in GPLs)

- Fix the initial conditions (regularity conditions and known solutions for simple integrals) directly in the physical region

Strategy

- Reduction to MIs by means of Integration by Parts as implemented in the public codes **REDUZE** and **FIRE**

Studerus (2009) von Manteffel, Studerus (2012)

Smirnov (2008)

> The procedure is by now standard, but to find the rotation to a canonical basis when the integral depend on more than one dimensionless parameter remains challenging.
>
> A canonical basis was found by means of the semi-algorithmic approach of **Gehrmann, von Maunteuffel, Tancredi, and Weihs (2014)** for the process qq $\rightarrow$ VV

- Calculation of the MIs with the differential equation method (solution written in GPLs)

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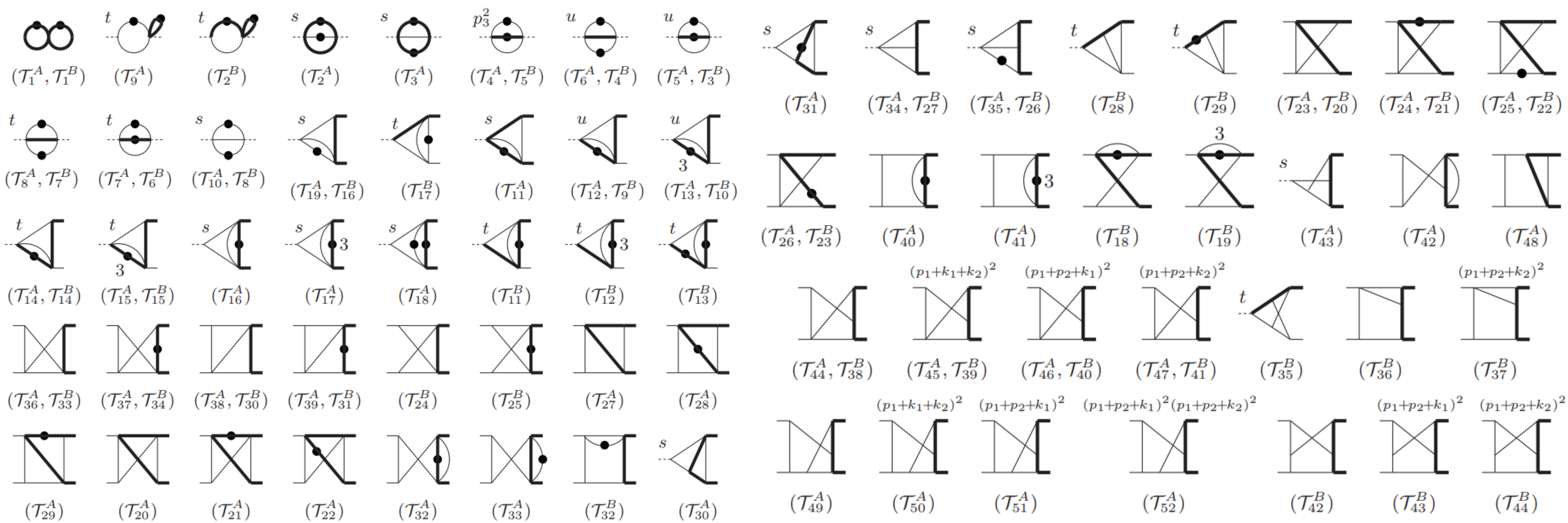
Precanonical MIs



52 MIs for Topology A, 44 MIs for topology B.
Many subtopologies have known MIs; other integrals, among which the 7 denominators boxes of topology A, are new

Canonical basis

Henn (2013)

System of first order DE in the pre-canonical basis

$$\frac{\partial}{\partial x_i}\vec{\tau}(\vec{x},\epsilon) = A_i(\vec{x},\epsilon)\vec{\tau}(\vec{x},\epsilon)$$

A are NXN matrices, Tau, f are vectors of N MIs, x is a vector of dimensionless parameters (2 in our case)

System of first order DE in the canonical basis

$$\frac{\partial}{\partial x_i}\vec{f}(\vec{x},\epsilon) = \epsilon\tilde{A}_i(\vec{x})\vec{f}(\vec{x},\epsilon)$$

Find dimensionless variables so that the dependence of A on x_i is rational

$$d\tilde{A}(\vec{x}) = \sum_{i,k}\tilde{A}^{(k)}d\ln(x_i-\alpha_{k,i})$$

Alphabet

The canonical MIs are written in terms of GPLs and have uniform transcendentality

Generalized Harmonic Polylogarithms (GPLs)

$$G(\alpha_1,\ldots,\alpha_n;z) = \int_0^z \frac{dt}{t-\alpha_1}G(\alpha_2,\ldots,\alpha_n;t)$$

n = weight

$$G(\alpha_1;z) = \int_0^z \frac{dt}{t-\alpha_1} \quad \text{for} \quad \alpha_1\neq 0, \qquad \text{and} \quad G(\vec{0}_n;z) = \frac{\log^n(z)}{n!}$$

Transformation to the canonical basis

MIs in the canonical basis

MIs in the pre-canonical basis

$$f_{42}^A = \epsilon^4\, s\, (s+t-m^2)\, \mathcal{T}_{42}^A\,,$$
$$f_{43}^A = \epsilon^4\, s\, \sqrt{s(s-4m^2)}\, \mathcal{T}_{43}^A\,,$$
$$f_{44}^A = \epsilon^4\, i \sqrt{m^2\, s(m^2-t)(s+t-m^2)}\, \mathcal{T}_{44}^A\,,$$

$$\begin{aligned}
f_{45}^A = {} & \epsilon^4 \sqrt{s(s-4m^2)}\, \mathcal{T}_{45}^A + \epsilon^4 \sqrt{s(s-4m^2)}\, (s+t-m^2)\, \mathcal{T}_{44}^A \\
& + \epsilon^4 \sqrt{s(s-4m^2)}\, \mathcal{T}_{46}^A + \epsilon^4 \sqrt{s(s-4m^2)}\, \mathcal{T}_{47}^A + \frac{\epsilon^2 m^2 s \sqrt{s(s-4m^2)}}{(m^2-t)\,(s+t-m^2)}\, \mathcal{T}_4^A \\
& + \frac{\epsilon^2 s\, \left(12m^4 - m^2(7s+4t) + s(s+t)\right)}{2\sqrt{s(s-4m^2)}\,(s+t-m^2)}\, \mathcal{T}_5^A \\
& - \frac{1}{4}\epsilon^2 \sqrt{s(s-4m^2)}\, \mathcal{T}_6^A - \frac{\epsilon^2 \sqrt{s(s-4m^2)}\,(m^2+t)}{2\,(m^2-t)}\, \mathcal{T}_7^A \\
& - \frac{1}{4}\epsilon^2 \sqrt{s(s-4m^2)}\, \mathcal{T}_8^A + \epsilon^3 \sqrt{s(s-4m^2)}\, \mathcal{T}_{12}^A - \epsilon^2 m^2 \sqrt{s(s-4m^2)}\, \mathcal{T}_{13}^A \\
& + \epsilon^3 \sqrt{s(s-4m^2)}\, \mathcal{T}_{14}^A - \epsilon^2 m^2 \sqrt{s(s-4m^2)}\, \mathcal{T}_{15}^A\,,
\end{aligned}$$

Transformations found with the semi-algoritmic method of **Gehrmann, von Manteuffel, Tancredi, Weihs (2014)**

Sample transformation from the basis change

Transformation to the canonical basis

$$f_{42}^A = \epsilon^4 \, s \, \left(s + t - m^2\right) \, \mathcal{T}_{42}^A \, ,$$
$$f_{43}^A = \epsilon^4 \, s \, \sqrt{s(s-4m^2)} \, \mathcal{T}_{43}^A \, ,$$
$$f_{44}^A = \epsilon^4 \, i \sqrt{m^2 \, s (m^2 - t)(s + t - m^2)} \, \mathcal{T}_{44}^A \, ,$$

$$\begin{aligned}
f_{45}^A = {} & \epsilon^4 \sqrt{s(s-4m^2)} \, \mathcal{T}_{45}^A + \epsilon^4 \sqrt{s(s-4m^2)} \left(s + t - m^2\right) \mathcal{T}_{44}^A \\
& + \epsilon^4 \sqrt{s(s-4m^2)} \, \mathcal{T}_{46}^A + \epsilon^4 \sqrt{s(s-4m^2)} \, \mathcal{T}_{47}^A + \frac{\epsilon^2 m^2 s \sqrt{s(s-4m^2)}}{(m^2-t)(s+t-m^2)} \mathcal{T}_4^A \\
& + \frac{\epsilon^2 s \left(12m^4 - m^2(7s+4t) + s(s+t)\right)}{2\sqrt{s(s-4m^2)} \, (s+t-m^2)} \mathcal{T} \ldots \\
& - \frac{1}{4} \epsilon^2 \sqrt{s(s-4m^2)} \, \mathcal{T}_6^A - \frac{\epsilon^2 \sqrt{s(s-4m^2 \ldots}}{2\,(m^2 \ldots} \\
& - \frac{1}{4} \epsilon^2 \sqrt{s(s-4m^2)} \, \mathcal{T}_8^A + \epsilon^3 \sqrt{s(s-4m^2 \ldots} \\
& + \epsilon^3 \sqrt{s(s-4m^2)} \, \mathcal{T}_{14}^A - \epsilon^2 m^2 \sqrt{s(s-4m^2)} \, \mathcal{T}_{15}^A \, ,
\end{aligned}$$

Two types of square roots enter in the calculation, one needs to choose dimensionless variables that allow one to write the square roots as rational functions

Dimensionless variables

The integrals depend on two dimensionless parameters

$$x = -\frac{s}{m^2}\,, \quad y = -\frac{t}{m^2}$$

The result is more conveniently expressed in terms of w and z

$$x = \frac{(1-w)^2}{w}\,,$$

$$y = \frac{1-w+w^2-z^2}{z^2-w}$$

Physical region

$$0 < -w < z < 1$$

$$\vec{x} \equiv \{w, z\}$$

This choice of variables eliminates the square roots from the differential equations

$$\sqrt{s(s-4m^2)} = m^2\frac{w^2-1}{w}$$

$$\sqrt{m^2 s(m^2-t)(s+t-m^2)} = m^4\frac{(w-1)^3 z}{w(z^2-w)}$$

Rational functions of w and z

Integration

Matrices that define the differential equations in the canonical basis

$$\tilde{A}(\vec{x}) = \sum_k \tilde{A}^{(k)} \ln(l_k)$$

where the $\tilde{A}^{(k)}$ are rational matrices and the letters l_k form the alphabet

$$\{l_k\} = \Big\{w, w-1, w+1, z, z-1, z+1, w-z, w+z, w-z^2, w^2-w+1-z^2, \\ w^2-z^2(w^2-w+1), w^2-3w+z^2+1\Big\}.$$

Weights of the GPLs of argument w

$$\left\{0, 1, -1, z, -z, z^2, \frac{1-\sqrt{4z^2-3}}{2}, \frac{1+\sqrt{4z^2-3}}{2}, \frac{z(z-\sqrt{4-3z^2})}{2(z^2-1)}, \right. \\ \left. \frac{z(z+\sqrt{4-3z^2})}{2(z^2-1)}, \frac{3-\sqrt{5-4z^2}}{2}, \frac{3+\sqrt{5-4z^2}}{2} \right\},$$

Weights of the GPLs of argument z

$$\{0, -1, 1, -i, i\}$$

Analytic Expressions

Analytic results are provided in ancillary **Mathematica** files included with the paper arXiv submission (~ 2 MB for topology A, 2 MB for topology B)

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(* Matteo Becchetti, Roberto Bonciani, Valerio Casconi, Andrea Ferroglia,
 * Simone Lavacca and Andreas von Manteuffel:
 * "Master Integrals for the two-loop, non-planar QCD corrections to
 *  top-quark pair production in the quark-annihilation channel"
 * April 2019
 *
 * This file contains analytical weight four results for the epsilon
 * expansion of the integrals {fA1,...,fA52} defined in (4.1)-(4.52)
 * in the physical region with s > 0. The integral measure is given by
 * d^d k1 * d^d k2 * (Exp[e*EulerGamma]/(I*Pi^(d/2))*(m^2/mu^2)^e)^2
 * with e=(4-d)/2, see (2.5). The variables (w,z) are defined in (5.9)
 * and (5.14), respectively.
 *)
{1 + (e^2*Pi^2)/6 + (7*e^4*Pi^4)/360 - (2*e^3*Zeta[3])/3,
 -(e*GPL[{0}, w]) + e^2*(Pi^2/6 + 6*GPL[{-1, 0}, w] - 4*GPL[{0, 0}, w] +
    2*GPL[{1, 0}, w]) + e^3*(-(Pi^2*GPL[{-1}, w]) + (Pi^2*GPL[{0}, w])/2 -
    (Pi^2*GPL[{1}, w])/3 - 36*GPL[{-1, -1, 0}, w] + 24*GPL[{-1, 0, 0}, w] -
    12*GPL[{-1, 1, 0}, w] + 24*GPL[{0, -1, 0}, w] - 10*GPL[{0, 0, 0}, w] +
    8*GPL[{0, 1, 0}, w] - 12*GPL[{1, -1, 0}, w] + 8*GPL[{1, 0, 0}, w] -
    4*GPL[{1, 1, 0}, w] + 11*Zeta[3]) +
  e^4*((29*Pi^4)/180 + 6*Pi^2*GPL[{-1, -1}, w] - 3*Pi^2*GPL[{-1, 0}, w] +
    2*Pi^2*GPL[{-1, 1}, w] - 4*Pi^2*GPL[{0, -1}, w] + Pi^2*GPL[{0, 0}, w] -
    (4*Pi^2*GPL[{0, 1}, w])/3 + 2*Pi^2*GPL[{1, -1}, w] -
    Pi^2*GPL[{1, 0}, w] + (2*Pi^2*GPL[{1, 1}, w])/3 +
    216*GPL[{-1, -1, -1, 0}, w] - 144*GPL[{-1, -1, 0, 0}, w] +
    72*GPL[{-1, -1, 1, 0}, w] - 144*GPL[{-1, 0, -1, 0}, w] +
    60*GPL[{-1, 0, 0, 0}, w] - 48*GPL[{-1, 0, 1, 0}, w] +
    72*GPL[{-1, 1, -1, 0}, w] - 48*GPL[{-1, 1, 0, 0}, w] +
    24*GPL[{-1, 1, 1, 0}, w] - 144*GPL[{0, -1, -1, 0}, w] +
    96*GPL[{0, -1, 0, 0}, w] - 48*GPL[{0, -1, 1, 0}, w] +
    60*GPL[{0, 0, -1, 0}, w] - 22*GPL[{0, 0, 0, 0}, w] +
    20*GPL[{0, 0, 1, 0}, w] - 48*GPL[{0, 1, -1, 0}, w] +
    44*GPL[{0, 1, 0, 0}, w] - 16*GPL[{0, 1, 1, 0}, w] +
    72*GPL[{1, -1, -1, 0}, w] - 48*GPL[{1, -1, 0, 0}, w] +
    24*GPL[{1, -1, 1, 0}, w] - 48*GPL[{1, 0, -1, 0}, w] +
    20*GPL[{1, 0, 0, 0}, w] - 16*GPL[{1, 0, 1, 0}, w] +
    24*GPL[{1, 1, -1, 0}, w] - 16*GPL[{1, 1, 0, 0}, w] +
    8*GPL[{1, 1, 1, 0}, w] - 66*GPL[{-1}, w]*Zeta[3] +
    (80*GPL[{0}, w]*Zeta[3])/3 - 22*GPL[{1}, w]*Zeta[3]),
```

Numerical cross checks

Analytic MIs evaluated numerically with **GiNaC**	$\stackrel{?}{=}$	Direct numerical evaluation of the MIs	✓

Pre-canonical MIs evaluated numerically by Sector Decomposition, as implemented in **SecDec** and **Fiesta**

Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke (2015)

Smirnov, Tentyukov (2009-2014)

For six and seven denominator integrals the numerical precision of the Sector Decomposition calculations was not sufficient: Canonical MIs were written in terms of quasi-finite integrals. The latter were evaluated numerically and used to find a numerical value for the canonical MIs

Panzer (2014) von Manteuffel, Panzer, Schabinger (2015)

Numerical test carried out for s > 4 m^2 (physical region) and for s < 0, t<0 (non-physical region)

All integrals were cross-checked against the numerical evaluation of the analytic results obtained by **Di Vita, Gehrmann, Laporta, Mastrolia, Primo, and Schubert (2019)**. Notice that since canonical basis is not unique, it was necessary to rotate one basis into the other

Benchmark point

For the 7 denominator MIs, numerical evaluations of the analytic expressions (at 16 digits) in a benchmark point in the physical region are provided in the paper

$$m = 1\,\mathrm{GeV}\,, \qquad s = 5.1\,\mathrm{GeV}^2\,, \quad \text{and} \quad t = -2.5\,\mathrm{GeV}^2$$

$$\begin{aligned}
f^A_{49} =& -0.8125 \\
& + (1.571461643987763 - i\,1.570796326794896)\epsilon \\
& + (1.869800565465933 + i\,7.871341877028778)\epsilon^2 \\
& - (26.64417846013623 + i\,2.934819494524318)\epsilon^3 \\
& - (5.561888073241050 + i\,69.90392666348392)\epsilon^4\,, \\
f^A_{50} =& \;\; (0.3936751877201319 - i\,1.229555494857724)\epsilon^2 \\
& + (14.12478202913410 - i\,2.239408800880071)\epsilon^3 \\
& + (49.29394916594301 + i\,37.08333857464637)\epsilon^4\,, \\
f^A_{51} =& \;\; 0.02083333333333333 \\
& + 0.07833393820762243\epsilon \\
& + (8.538951737141223 - i\,1.580009353612773)\epsilon^2 \\
& - (4.529079554851615 + i\,2.930479163733208)\epsilon^3 \\
& + (0.1103747892867767 - i\,78.51623866876891)\epsilon^4\,, \\
f^A_{52} =& -0.0625 \\
& + 0.2937522682785850\epsilon \\
& + (7.827416975804789 2 + i\,5.478308035237822)\epsilon^2 \\
& - (26.35732295414653 0 - i\,15.39070197526472)\epsilon^3 \\
& - (121.0171434327693 9 + i\,42.90574414612206)\epsilon^4\,,
\end{aligned}$$

$$\begin{aligned}
f^B_{42} =& -(0.5193031088754503 + i\,0.7853981633974483)\epsilon \\
& - (5.247646105592740 - i\,3.641850161748369 8)\epsilon^2 \\
& - (51.07989282173662 + i\,30.03948566663873 2)\epsilon^3 \\
& - (68.03046563599218 + i\,107.2120345188574 6)\epsilon^4\,, \\
f^B_{43} =& -1 \\
& + (2.702244138720994 - i\,3.926990816987242 3)\epsilon \\
& + (18.05310915519800 + i\,12.79602527636828 8)\epsilon^2 \\
& - (3.231845611282520 + i\,2.772417644395675 0)\epsilon^3 \\
& + (127.3934689436371 - i\,12.98463204885098 1)\epsilon^4\,, \\
f^B_{44} =& -0.625 \\
& + (0.9285563596344188 + i\,3.141592653589792 8)\epsilon \\
& - (6.716934387387509 + i\,10.08990983271162 8)\epsilon^2 \\
& + (50.38595267312016 - i\,44.14987849621522 8)\epsilon^3 \\
& + (146.4134579642496 - i\,86.51246012697473 0)\epsilon^4\,.
\end{aligned}$$

Toward the quark-annihilation two-loop amplitude

To assemble the two-loop amplitude in the quark-annihilation channel is now – in principle - a matter of bookkeeping (which as usual will require some time):

- Write the full squared amplitude (summed over spin and color) for the color factors B and C in terms of the integrals in the canonical basis
- Insert the analytic expression of the canonical MIs in the amplitude
- Subtract the UV counter term
- Evaluate numerically the expression with GiNaC
- Check that the residual IR poles are the known ones
- Check the result against the numerical calculation in some benchmark point
- Check the limit for vanishing top-quark velocity

Conclusions and Outlook

- We evaluated analytically the last unknown Master Integrals needed for two-loop amplitude in the quark-annihilation channel in top-pair production
- These MIs belong to non-planar box topologies. They can be evaluated by a via IBPs, rotation to the canonical basis, differential equation method
- The results can be written in terms of Generalized Harmonic Polylogarithms (up to weight 4)
- Everything is in place for the complete analytic evaluation of the two-loop QCD corrections to the quark-annihilation channel

Back up material

The Laporta Algorithm

The set of denominators $\mathcal{D}_1, \cdots, \mathcal{D}_t$ defines a topology; for each topology

- The scalar integrals are related via Integration By Parts identities (10 identities per integral for a two-loop four-point function)

$$\int \mathfrak{D}^d k_1 \mathcal{D}^d k_2 \frac{\partial}{\partial k_i^\mu} \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right] = 0 \quad v^\mu = k_1, k_2, p_1, \cdots, p_3$$

- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster that the number of unknown: one finds a system of equations which is apparently over-constrained
- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.

Calculation of the MIs: Differential Equation Method

For each Master Integral belonging to a given topology
$F_l^{(q)} \to \{\mathcal{D}_1, \cdots, \mathcal{D}_q\}$

- Take the derivative of a given integrals with respect to the external momenta p_i
- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs
- Rewrite the diff. eq. in terms of derivatives with respect to s and t
- Fix somehow the initial condition(s) (ex. knowing the behavior of the integral at $s = 0$) and solve the system of DE(s)

Harmonic Polylogarithms (HPLs)

E. Remiddi, J. Vermaseren (1999)

E. Remiddi, T. Gehrmann (2001)

Functions of the variable x and a set of indices $\vec{a}$ with weight w; each index can assume values $1, 0, -1$

$$H(\mathbf{a}; x)$$

Definitions: $w = 1$

$$\begin{aligned} H(1;x) &= \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\ H(0;x) &= \ln x \\ H(-1;x) &= \int_0^x \frac{dt}{1+t} = \ln(1+x) \end{aligned}$$

$$\frac{d}{dx} H(a;x) = f(a;x) \quad f(1;x) = \frac{1}{1-x} \quad f(0;x) = \frac{1}{x} \quad f(-1;x) = \frac{1}{1+x}$$

HPLs: Definitions

Definitions: $w > 1$

$$\text{if } \vec{a} = 0, 0, \ldots, 0 \ (w \text{ times}) \quad H(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\text{else} \quad H(i, \vec{a}; x) = \int_0^x dt f(i; t) H(\vec{a}; t)$$

consequences: $\frac{d}{dx} H(i, \vec{a}; x) = f(i; x) H(\vec{a}; x) \quad H(\vec{a} \notin \vec{0}; 0) = 0$

a few examples @ $w = 2$

$$H(0, 1; x) = \int_0^x dt f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln(1 - t) = \mathrm{Li}_2(x)$$

$$H(1, 0; x) = \int_0^x dt f(1; t) H(0; t) = \int_0^x dt \frac{1}{1 - t} \ln t$$

$$= -\ln x \ln(1 - x) + \mathrm{Li}_2(x)$$

The HPLs Algebra

- Shuffle Algebra:

$$H(\vec{p};x)H(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} H(\vec{r};x)$$

some examples

$$\begin{aligned} H(a;x)H(b;x) &= H(a,b;x) + H(b,a;x) \\ H(a;x)H(b,c;x) &= H(a,b,c;x) + H(b,a,c;x) + H(b,c,a;x) \end{aligned}$$

- Product Ids:

$$\begin{aligned} H(m_1,\ldots,m_q;x) &= H(m_1;x)H(m_2,\ldots,m_q;x) \\ &- H(m_2,m_1;x)H(m_3,\ldots,m_q;x) \\ &+ \cdots + (-1)^{q+1}H(m_q,\ldots,m_1;x) \end{aligned}$$

2-dimensional Harmonic Polylogarithms (2dHPLs)

E. Remiddi, T. Gehrmann (2000)

As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

$$f(-y;x) = \frac{1}{x+y} \quad f(-1/y;x) = \frac{1}{x+1/y}$$

$$G(i,\vec{a};x) = \int_0^x dt\, f(i;t)\, G(\vec{a};t)$$

a few examples:

$$G(-y;x) = \int_0^x \frac{dz}{z+y} = \ln\left(1+\frac{x}{y}\right) \quad G(-1/y;x) = \int_0^x \frac{dz}{z+1/y} = \ln(1+xy)$$

$$G(-y,0;x) = \ln x \ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$

Rewrite the amplitude in terms of multiple polylogarithsm

It is important to have a fast and stable way of evaluating numerically the analytic squared amplitude. This is more easily achieved when the results are written in terms of multiple polylogarithm

$$\mathrm{Li}_2, \mathrm{Li}_3, \mathrm{Li}_4, \mathrm{Li}_{2,2}$$

$$\mathrm{Li}_{m_1,\cdots,m_k}(x_1,\cdots,x_k) = \sum_{i_k=1}^{\infty} \frac{x_k^{i_k}}{i_k^{m_k}} \sum_{i_{k-1}=1}^{i_k-1} \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}^{m_{k-1}}} \cdots \sum_{i_1=1}^{i_2-1} \frac{x_1^{i_1}}{i_1^{m_1}}$$

The arguments of the Li functions are in general complicated rational functions of the dimensionless parameters

$$\mathrm{Li}_n(x) = -G(\underbrace{0,\cdots,0,1}_{n};x)\,, \qquad \mathrm{Li}_{2,2}(x_1,x_2) = G\left(0,\frac{1}{x_1},0,\frac{1}{x_1x_2};1\right)$$