Master integrals for all cuts of 4-loop propagators

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(based on work with Andrei Pikelner)

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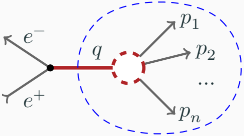
Motivation: semi-inclusive cross sections

The distant goal: NNLO corrections to time-like splitting functions.

- Calculated by an analytic continuation from the space-like splitting functions. [Mitov, Moch, Vogt '06; Moch, Vogt '07; Almasy, Moch, Vogt '11]
  - An uncertainty leaves one of the terms in $P_{gq}^{(2)T}$ undetermined.
  - A direct calculation is required to fix the missing terms.
- Can be extracted from α_s^3 terms (N^{3}LO) of e^+e^- annihilation cross-section differential in the energy fraction of one of the outgoing partons ("x").
  - Master integrals for this differential cross-section are not known!
    - They can be calculated via differential equations (in x).
    - How to fix the integration constants?
    - An integral over all x values should turn a differential cross-section master into a fully inclusive one! [Gituliar, Moch '15]

The goal at hand: master integrals for fully inclusive cross-sections at α_s^3 (N^{3}LO, correspond to 4-loop propagators).

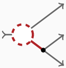
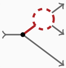
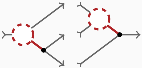
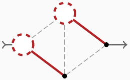
What are cut integrals?



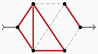
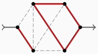
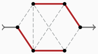
For a fully inclusive decay cross-section of an off-shell particle:

$$\sigma \sim \sum_n \int d\mathrm{PS}_n \left|\langle p_1, \dots, p_n|S|q\rangle\right|^2 = \sum_n \int d\mathrm{PS}_n$$ 

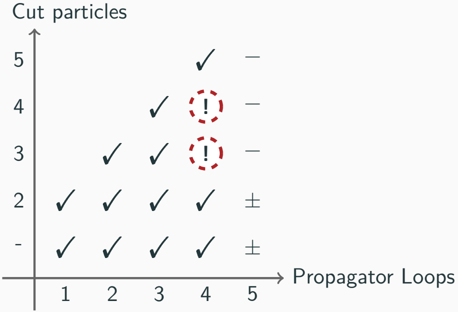
Expanding the module squared, each term becomes a cut integral:

$$\int d\mathrm{PS}_3$$  $\Big($  $\Big)^*$ $= \int d\mathrm{PS}_3$  $=$ 

For α_s^3 we need 4-loop propagator cuts:

2-particle cuts	3-particle cuts	4-particle cuts	5-particle cuts
			

Propagators and their cuts: state of the art



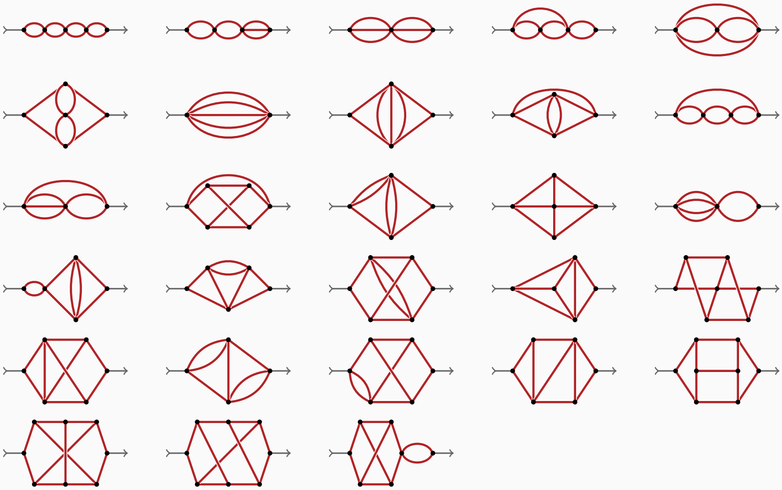
At four loops we know:

- Virtual integrals (4-loop propagators). [Baikov, Chetyrkin '10]
- Two-particle cuts (3-loop form-factors). [Heinrich, Huber, Maitre '07; Heinrich et al. '09; Lee, Smirnov, Smirnov '10]
- Five-particle cuts (phase-space integrals). [Gituliar, V.M., Pikelner '18]
- Three- and four-particle cuts: completed now, this talk.

Identifying cut masters

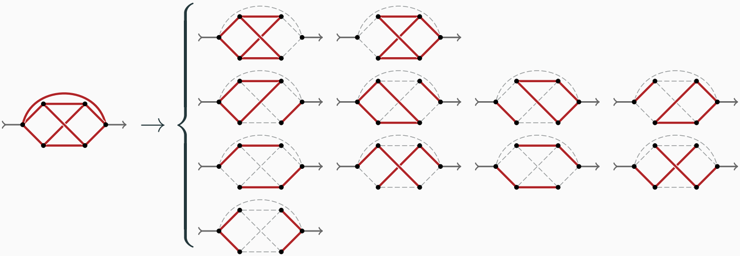
Start with the 28 master integrals for 4-loop propagators.

[Baikov, Chetyrkin '10]



Identifying cut masters, II

Cut each virtual master in all possible ways.

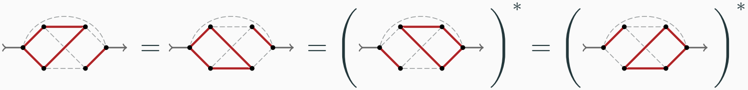


3-cut

4-cut

5-cut

Remove symmetric duplicate integrals.

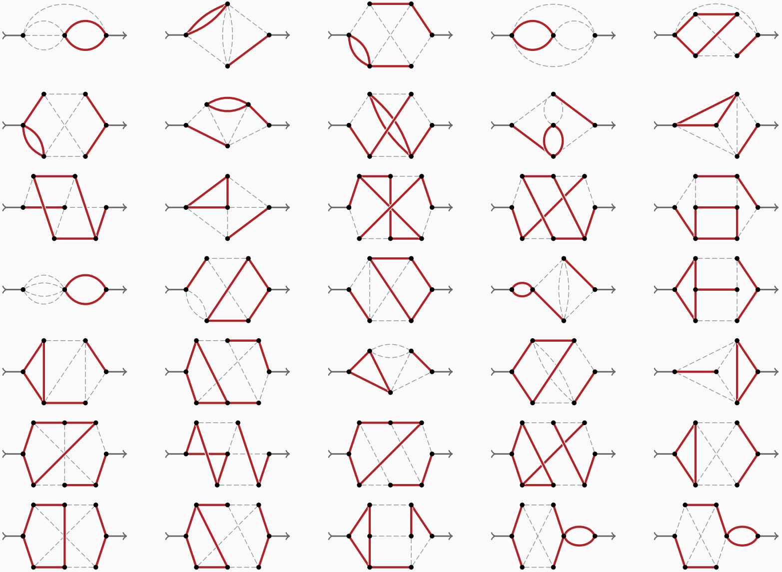


Done! There are no additional IBP relations between the remaining integrals. No additional master integrals either.

4-particle cut masters

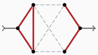
4-particle cut masters

There are 35 master integrals for 4-particle cuts.

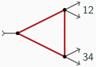


Direct integration?

Integrate the 1-loop amplitude over the phase space?



$$= \int$$



$$\frac{1}{s_{13}s_{24}} d\mathrm{PS}_4$$

4-particle phase-space has 5 degrees of freedom and nontrivial shape:…………

$$d\mathrm{PS}_4 = \left(q^2\right)^{\frac{3d-8}{2}} 2^{1-2d} \left(2\pi\right)^{4-3d} \Omega_{d-1}\Omega_{d-2}\Omega_{d-3} \left(\Delta_4\right)^{\frac{d-5}{2}} \times$$
$$\theta(\Delta_4)\, \delta\Big(1 - \sum s_{ij}\Big)\, ds_{12}\, ds_{13}\, ds_{14}\, ds_{23}\, ds_{24}\, ds_{34}$$

$$\Delta_4 = \det \begin{vmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{24} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{24} & s_{34} & 0 \end{vmatrix}$$

There exists a “tripole parameterization” that maps this to a hypercube, but is contains square roots, making analytic integration difficult.

[Gehrmann-De Ridder et al. ’03]

Dimensional recurrence relations

Using a parametric representation of integrals and the IBP tables, obtain dimension recurrence relations (DRR) for each master I_i: [Tarasov '96; Lee '09]

$$I_i(d+2) = M_{ij}(d)\, I_j(d)$$

For 4-particle cuts, the DRR matrix M_{ij} is:

- Triangular! No coupled blocks, can be solve one by one.
- Has the diagonal of the form $M_{ii} = C \prod_k \left(d/2 - a_k\right)^{n_k}$.

General solution:

$$I_i(d) = H_i(d)\, \omega(d) + R_i(d)$$

- H_i is the homogeneous solution, $H_i = C^{d/2} \prod_k \Gamma^{n_k}(d/2 - a_k)$;
- R_i is a particular solution, constructed from lower integrals as an indefinite nested sum;
- ω is an arbitrary periodic function, $\omega(d+2) = \omega(d)$.

The challenge: find $\omega(d)$.

4-particle cut masters: fixing $\omega(d)$

1. Find d_0, such that $I_i(d)$ is finite if $\mathrm{Re}(d) \in (d_0, d_0 + 2]$.
    - 4-particle masters I_i mostly diverge at even d.
    - But $J_i \equiv I_i /$  do not! All J_i are finite if $\mathrm{Re}(d) \in [6, 8]$.
    - All homogeneous solutions H_i can be chosen to be finite too.

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2. Find the poles of the particular solution $R_i(d)$ by evaluating it numerically for many values of d.
   - Use DREAM for the evaluation. [Lee, Mingulov '17]
   - All R_i appear to be smooth.

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3. Construct an ansatz for ω_i looking at the poles of J_i, H_i, and R_i.
   - We mostly use $\sum C_i \cot\left(\frac{\pi}{2}(d-d_i)\right)$ or modifications.
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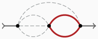
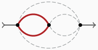
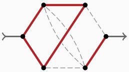
   - For smooth J_i, H_i, and R_i only one choice is possible: a constant.
4. Fix the constants in the ansatz from various considerations.
   - The leading pole of I_i can be calculated by inserting a mass in the loop, and looking at large mass expansion of the integral—this is enough to fix one constant.
   - Turns out $\omega_i(d) = 0$ for all integrals, except for , , and .

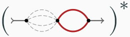
4-particle cut masters, wrap up

Once ω_i are fixed, the resulting sums can be evaluated with `SummerTime` or `DREAM` numerically to arbitrary precision, and then restored in terms of MZVs via PSLQ.

Example result:



$=$



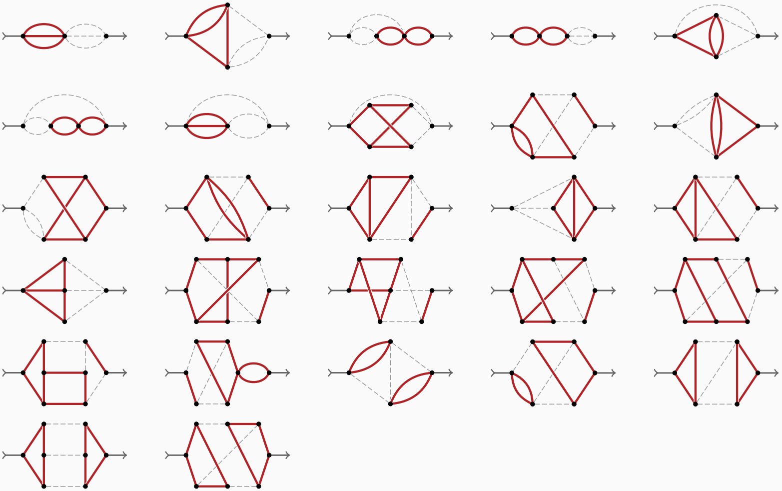
$$= \frac{\left(\cdots \right)^*}{\left(q^2\right)^3} \Big[-6\zeta_2 \frac{1}{\epsilon^2} + \Big(59\zeta_2 - 60\zeta_3\Big)\frac{1}{\epsilon} + \Big(-203\zeta_2 + 590\zeta_3 - 129\zeta_2^2\Big) + \Big(288\zeta_2 - 2030\zeta_3 + \frac{2537}{2}\zeta_2^2 + 192\zeta_2\zeta_3 - 1806\zeta_5\Big)\epsilon + O\left(\epsilon^2\right)\Big]$$

Overall, series up to MZVs of weight 12. Poles up to $1/\epsilon^5$. Zetas up to weight 6 in the ϵ-finite part.

3-particle cut masters

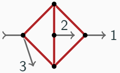
3-particle cut masters

We've found 27 master integrals among 3-particle cuts.



How to calculate 3-particle cuts?

Split integrals into the 2-loop 1→3 amplitude and 3-particle phase space:

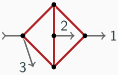
 $= \int$ 

dPS_3

Parameterize the phase-space via $s_{ij} = \left(p_i + p_j\right)^2 / q^2$:

$$dPS_3 = \left(q^2\right)^{d-3} \frac{2^{4-3d} \pi^{\frac{3}{2}-d}}{\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-1}{2}\right)} \left(s_{12} s_{13} s_{23}\right)^{\frac{d-4}{2}} \delta\left(1 - \sum_{i<j} s_{ij}\right) \prod_{i<j} ds_{ij}$$

Insert 2-loop 1→3 amplitudes as series in $\epsilon = (4-d)/2$:

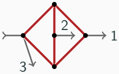


$$= \frac{\mathcal{C}}{\epsilon^4} \left(s_{12}\right)^{d-6} = \frac{\mathcal{C}}{s_{12}^2} \left(\frac{1}{\epsilon^4} - \frac{2 \ln s_{12}}{\epsilon^3} + O\left(\frac{1}{\epsilon^2}\right)\right)$$

Multiply series and integrate order by order?

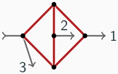
How to calculate 3-particle cuts, II

Integration of the series does not converge:

$$\int$$ 

$$dPS_3 = \int \frac{\mathcal{C}}{s_{12}^2}\left(\frac{1}{\epsilon^4} - \frac{2\ln s_{12}}{\epsilon^3} + O\Big(\frac{1}{\epsilon^2}\Big)\right) dPS_3 = \frac{\infty}{\epsilon^4} + \frac{\infty}{\epsilon^3} + \ldots$$

Taking series after integration does:

$$\int$$ 

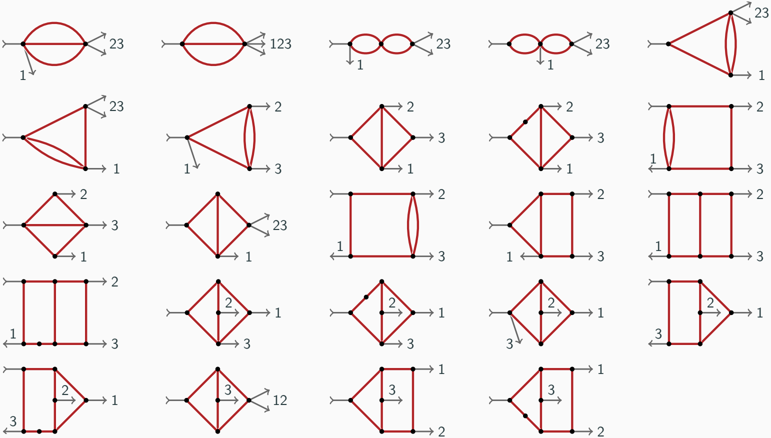
$$dPS_3 = \int \frac{\mathcal{C}}{\epsilon^4}\left(s_{12}\right)^{d-6} dPS_3 = \frac{\mathcal{K}}{\epsilon^5} + \frac{\mathcal{K}'}{\epsilon^4} + \ldots$$

Problem: the integral is not infrared-finite (in 4 dimensions).

Solution: increase d to 6 where it is.

Aside: 2-loop 1→3 amplitudes

28 in total, not counting $\{p_1, p_2, p_3\}$ permutations.



Aside: 2-loop 1→3 amplitudes, II

First computed up to weight 4 by Gehrmann and Remiddi, recomputed to weight 8 by us.

- Higher weights are needed because the ϵ-finite part of the cut integrals contain MZVs up to weight 6.

To compute: [Gehrmann, Remiddi '00, '01]

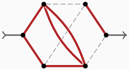
1. Write down differential equations in s_{12} and s_{13}.
2. Find the ϵ-form (using `Fuchsia`) and construct the general solution. [Gituliar, V.M. '17]
   - The general form is $\sum G(\{0,1,1-z,-z\};y)\, G(\{0,1\};z)\, \mathcal{R}(y,z,\mathcal{C})$.
3. Fix the integration constants $\mathcal{C}$ by:
   - matching with known single-scale integrals;
   - enforcing regularity at $s_{ij} \to 1$ (because the integrals are massless);
   - enforcing regularity at $s_{ij} \to 0$ for planar integrals if i and j are not adjacent.

How to calculate 3-particle cuts, III

In summary:

1. Use dimensional recurrence for the 1→3 amplitudes to get them as series around $d = 6 - 2\epsilon$.
2. Multiply by ϵ-expansion of $d\mathrm{PS}_3$, integrate order by order.
3. Use dimensional recurrence for the cut integrals to lower the series to $4 - 2\epsilon$.

Example result:

 =  / $\left(q^2\right)^2$

$$\left[2\zeta_2 \frac{1}{\epsilon} + \Big(-13\zeta_2 + 16\zeta_3 \Big) + \Big(27\zeta_2 - 104\zeta_3 + \frac{156}{5}\zeta_2^2 \Big)\epsilon + \Big(-18\zeta_2 + 216\zeta_3 - \frac{1014}{5}\zeta_2^2 - 90\zeta_2\zeta_3 + 448\zeta_5 \Big)\epsilon^2 + O(\epsilon^3)\right]$$

Overall, series up to MZVs of weight 8. Poles up to $1/\epsilon^6$. Zetas up to weight 6 in the ϵ-finite part.

Crosschecks

Numerical checks

3-particle cuts:

- Feynman parameterization for the loop amplitude.
- Direct parameterization of the phase space.
- Sector decomposition with `FIESTA`. [Smirnov '15]
- Checked in $d = 6 - 2\epsilon$, and $8 - 2\epsilon$. Too slow in $4 - 2\epsilon$.

4-particle cuts:

- Feynman parameterization of the loop amplitudes.
    - 1-loop amplitude—all the divergence is in the prefactor of the parameterization, the integral part is finite.
- Tripole parameterization of the phase space. [Gehrmann-De Ridder et al. '03]
- Direct Monte-Carlo integration of the result.
    - Always converges, because the divergent prefactor is separated!

Cutkosky rules check

Cutkosky rules for each Feynman diagram F:

$$F + F^* = -\sum_i \mathrm{Cut_i} F$$

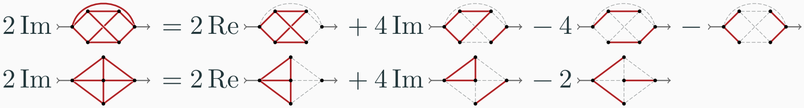
To use this simple form, add Feynman rules to the integrals:

$= \cdots = i$

$= \dfrac{i}{p^2 + i0}$

$= 2\pi\delta^+(p^2)$

Then, for each propagator master:



...

All cuts are now known! Insert them here to check the result consistency.

Bonus: use these relations and the previously derived values of 3-particle cuts (to weight 8) to solve DRR for them, and obtain them to weight 12!

Summary

- Master integrals for:
  - 3-particle ($+$ 2-loop) cuts of 4-loop propagators;
  - 4-particle ($+$ 1-loop) cuts of 4-loop propagators.
- Results as series in ϵ, up to MZVs of weight 12.
- Bonus:
  - `SummerTime` files to expand around any d with arbitrary precision.
  - Master integrals for 2-loop $1 \to 3$ amplitudes up to weight 8.
- On the arXiv soon.

Thank you for your attention.