Global Geometry of Integrable Dynamics in IOTA with Applications to Dynamic Aperture

FAST/IOTA Collaboration Meeting

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Acknowledgments and Collaborators

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- *LBNL – Robert Ryne, Kilean Hwang (especially for FMA results)*
- *Fermilab/NIU – Alexander Valishev, Jeffrey Eldrid, Alexander Romanov, Ben Freemire, Eric Stern, Sebastian Szustkowski*
- *RadiaSoft – David Bruhwiler, Chris Hall, Stephen Webb, Nathan Hall, Jonathan Edelen*

Outline

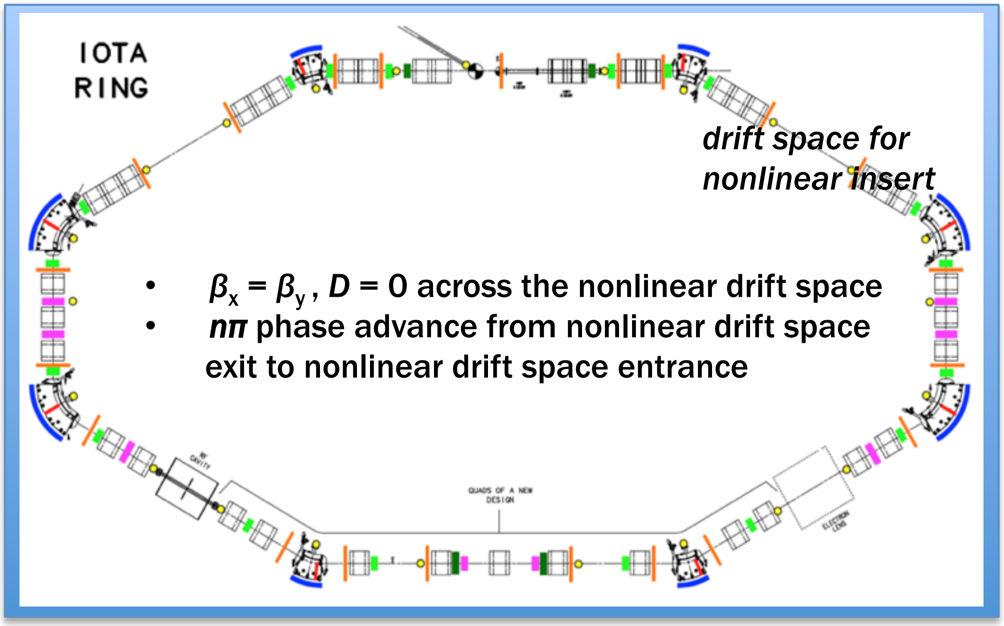
- *Introduction and Motivation*
- *Momentum Mapping of an Integrable Hamiltonian System*
  - *- critical points, critical values, and bifurcation diagrams*
  - *- prototypical example: the nonlinear pendulum*
  - *- application to ideal integrable optics in IOTA*
- *Classification of Integrable Orbits in IOTA: a Visual Tour*
- *Dynamic Aperture in the Presence of a Perturbation*
  - *- studies of the IOTA toy lattice with non-integer tune advance*
  - *- studies of the physical IOTA lattice with space charge*
- *Conclusions*

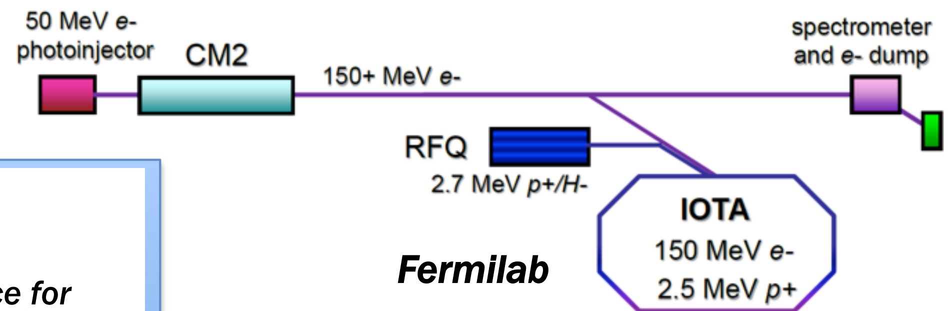
- **Introduction and Motivation**

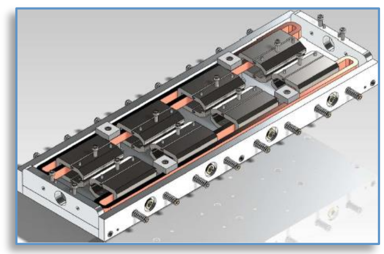
The IOTA ring : a test bed for first-principles accelerator science, nonlinear dynamics, & space charge mitigation.

- **Integrable Optics Test Accelerator (IOTA)**

  - Novel accelerator physics: strongly nonlinear design
  - Experimental test bed for SC mitigation schemes
  - Run first with electrons, then low-energy protons







nonlinear magnetic insert

- **Nonlinearity** ➡ **tune spread "washes out" coherent space charge instabilities**
- **Integrability** ➡ **ensures orbits are regular and remain bounded (no chaos)**

S. Antipov *et al*, Journal of Instrumentation **12**, T03002 (2017);
V. Danilov and S. Nagaitsev, Phys Rev Accel Beams **13**, 084002 (2010)

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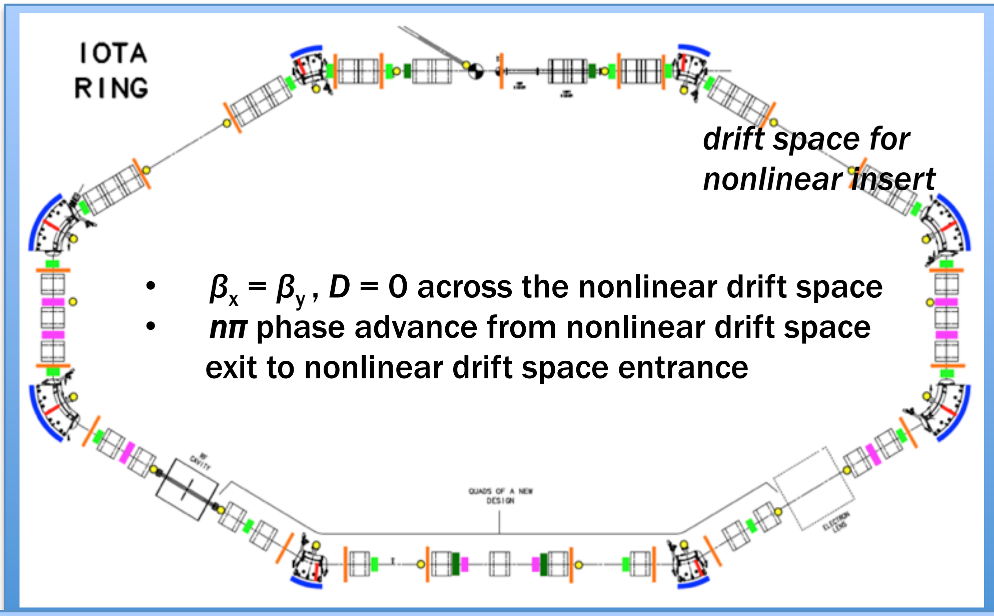

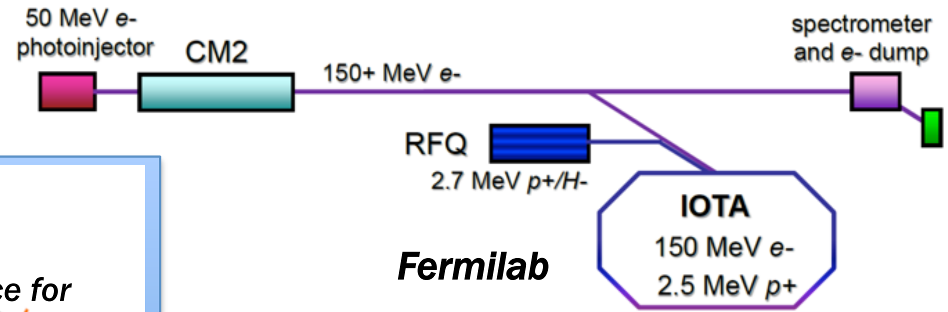

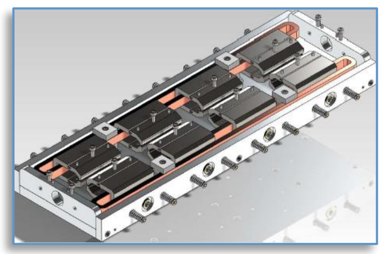

nonlinear magnetic insert

- Integrability holds for on-energy orbits in the two transverse degrees of freedom.
- After a linear canonical transformation (Courant-Snyder), dynamics is equivalent to a nonlinear s-independent Hamiltonian system with 2 invariants of motion: *H*, *I*.

S. Antipov *et al*, Journal of Instrumentation **12**, T03002 (2017);
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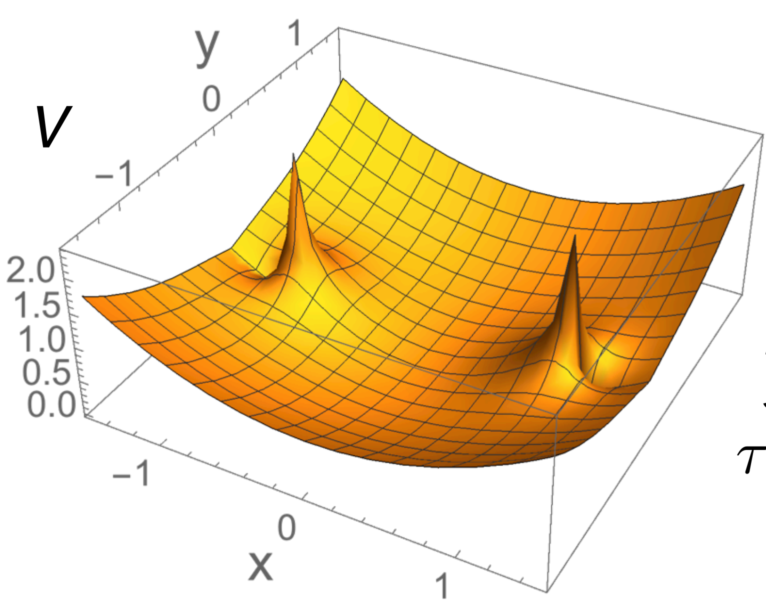
Basic Properties of the IOTA Nonlinear Potential

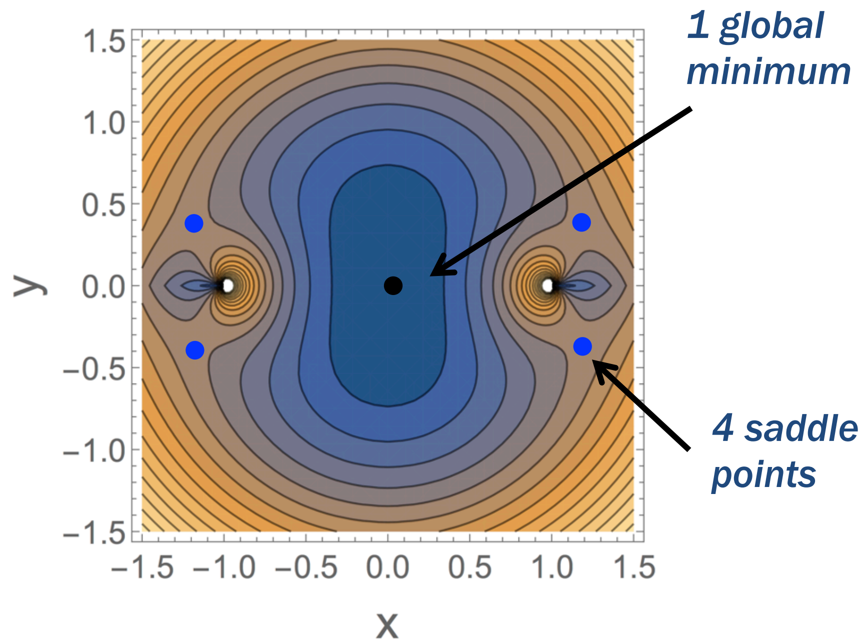
The potential (for the ideal integrable system in normalized coordinates) takes the form (-1/2 < τ < 0):

$$V(x,y) = \frac{1}{2}(x^2+y^2) - \tau \mathcal{R}eF(x+iy) \qquad F(z) = \frac{z}{\sqrt{1-z^2}}\arcsin(z)\,.$$

1) Singular points occur at (1,0) and (-1,0), where V diverges.
2) *V* is smooth on the plane minus the set of points with y=0, |x|≥1 (branch cuts).
3) *V* is continuous everywhere except at the singular points. (This includes the branch cuts).
4) *V* is symmetric under reflection about either the *x* or *y* axis.



Potential shown for $\tau = -0.4$



Motivation for studying the geometry of integrable dynamics in IOTA

- While the dynamics of an integrable system is simple in action-angle variables, a general integrable system *cannot* be described using a global set of action-angle variables. There may be several systems of local action-angle coordinates, with separatrices, fixed points, etc.

- We want to obtain as much information about the dynamics as possible without knowledge of any local sets of action-angle variables (which we may not know/are difficult to obtain).

- Geometric methods from the theory of dynamical systems provide a global view of the integrable dynamics and can be applied knowing *only* the invariants of motion.

- An improved understanding of the integrable dynamics—including stable and unstable fixed points, unstable periodic orbits, and phase space separatrices—might allow one to search experimentally for some of these dynamical behaviors.

- Studies of *dynamic aperture* in the presence of a perturbation (non-integer tune advance in the arc) suggest that the dynamic aperture is intimately connected with the geometry of orbits in the ideal integrable system.

- **Momentum Mapping of an Integrable Hamiltonian System**

Geometry of the Regular Level Sets of an Integrable Hamiltonian System

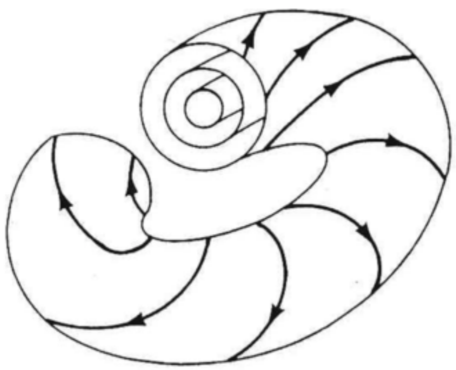
Liouville-Arnold Theorem

Suppose *H* is a time-independent Hamiltonian for an *n* degree-of-freedom system, and *H* = $f_1, f_2, \ldots, f_n$ are *n* smooth functions on the phase space *M* such that:

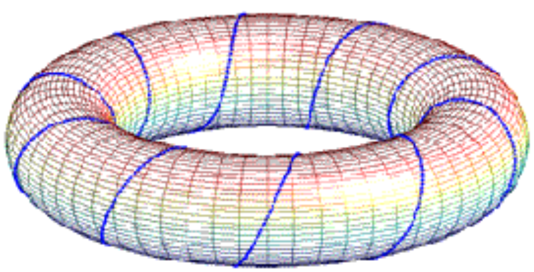
1) $\nabla f_1, \ldots, \nabla f_n$ are linearly independent (the f_j are *independent*)
2) $\{f_i, f_j\} = 0 \quad (i, j = 1, \ldots, n)$ (the f_j are *in involution*)

$\Longrightarrow$ The motion is confined to a set $M_z = \{p \in M | f_i(p) = z_i, i = 1, \ldots, n\}$.

If M_z is compact and connected, then M_z is diffeomorphic to the *n*-torus.



smooth coordinate transformation



linear flow

Geometry of the Regular Level Sets of an Integrable Hamiltonian System

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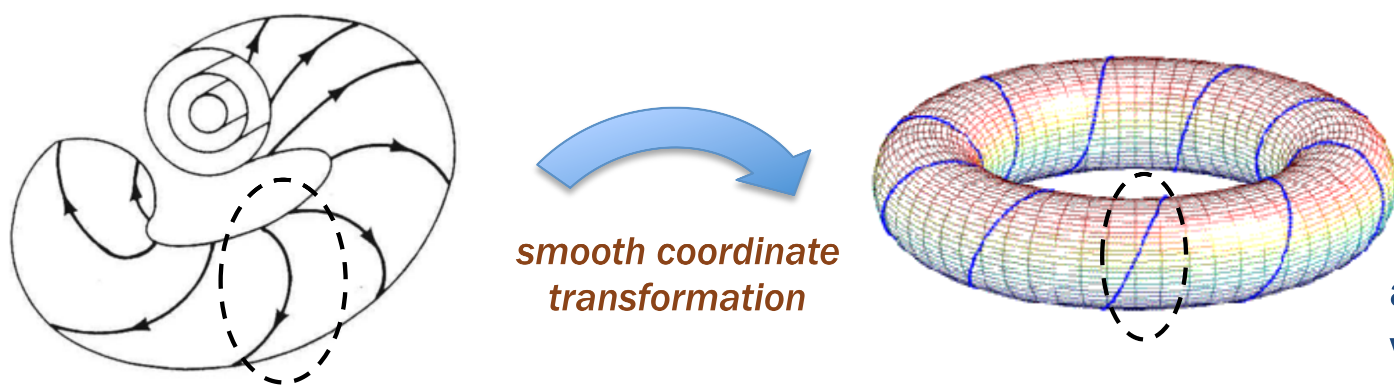
original variables



action-angle variables

Geometry of the Regular Level Sets of an Integrable Hamiltonian System

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$\Longrightarrow$ The motion is confined to a set $M_z = \{p \in M | f_i(p) = z_i, i = 1, \ldots, n\}$.

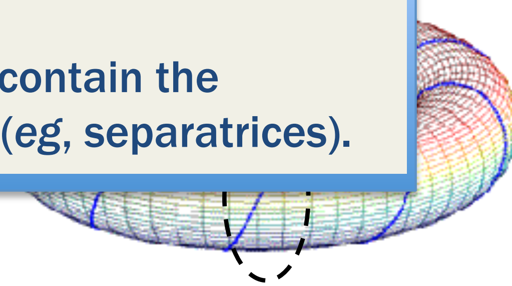
If M_z is compact and connected, then M_z is diffeomorphic to the n-torus.

What happens on level sets where condition 1) fails?

These level sets are *degenerate*, and contain the most interesting orbits of the system (eg, separatrices).

original variables

smooth coordinate transformation



linear flow

action-angle variables

Concept of the Momentum Mapping and Its Critical Points

Suppose H is an integrable Hamiltonian for an n degree-of-freedom system on a phase space M, and suppose $H = f_1,\ldots,f_n$ denote its n invariants of motion (independent a.e. and in involution).

The ***momentum mapping*** is the smooth map given by:

$$\mathcal{F} : M \to \mathbb{R}^n \qquad \mathcal{F}(p) = (f_1(p), \ldots, f_n(p)), \quad p \in M$$

At a point $p = (x_1,\ldots,x_{2n})$ in the phase space, the Jacobian matrix of $\mathcal{F}$ is given by:

$$[D\mathcal{F}(p)]_{jk} = \frac{\partial f_j}{\partial x_k} \qquad (j = 1, \ldots, n, \quad k = 1, \ldots, 2n)$$

A point p in M is a ***critical point*** of the momentum mapping if: $\mathrm{rank}(D\mathcal{F}(p)) < n$

If p is a critical point, its image $\mathcal{F}(p)$ in $\mathbb{R}^n$ is called a ***critical value***.

The set of all critical values of the momentum mapping is called the ***bifurcation diagram***.

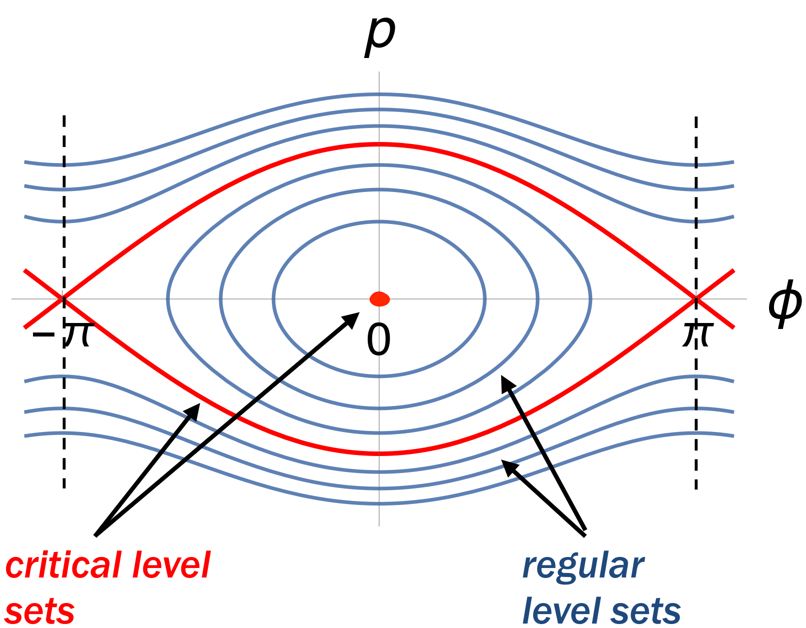
- The (critical) level sets of $\mathcal{F}$, corresponding to critical values, give unusual (degenerate) tori.
- All other (regular) level sets of $\mathcal{F}$ give the smooth n-tori of the Liouville-Arnold theorem.

A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems: Geometry, Topology, Classification*, Chapman & Hall/CRC Press, NY, 2004; Section 1.7.

Prototypical Example: The Nonlinear Pendulum

In this simple case, n=1, we have a single invariant f_1=H, and the momentum mapping is:

$$\mathcal{F}:\mathbb{S}^1\times\mathbb{R}\rightarrow\mathbb{R}\,,\qquad \mathcal{F}(\phi,p)=H(\phi,p)=\frac{p^2}{2}-\cos\phi\,.$$



Phase space (cylinder)

Jacobian matrix:

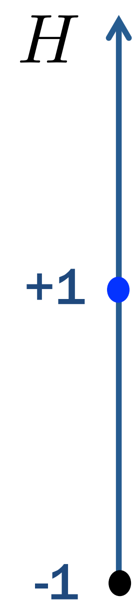
$$D\mathcal{F}(\phi,p)=(\sin\phi,p)$$

Critical points: *where* $D\mathcal{F}=0$

$$p=0,\quad \phi=n\pi,\quad n\in\mathbb{Z}$$

Critical values: $H=-1,+1$

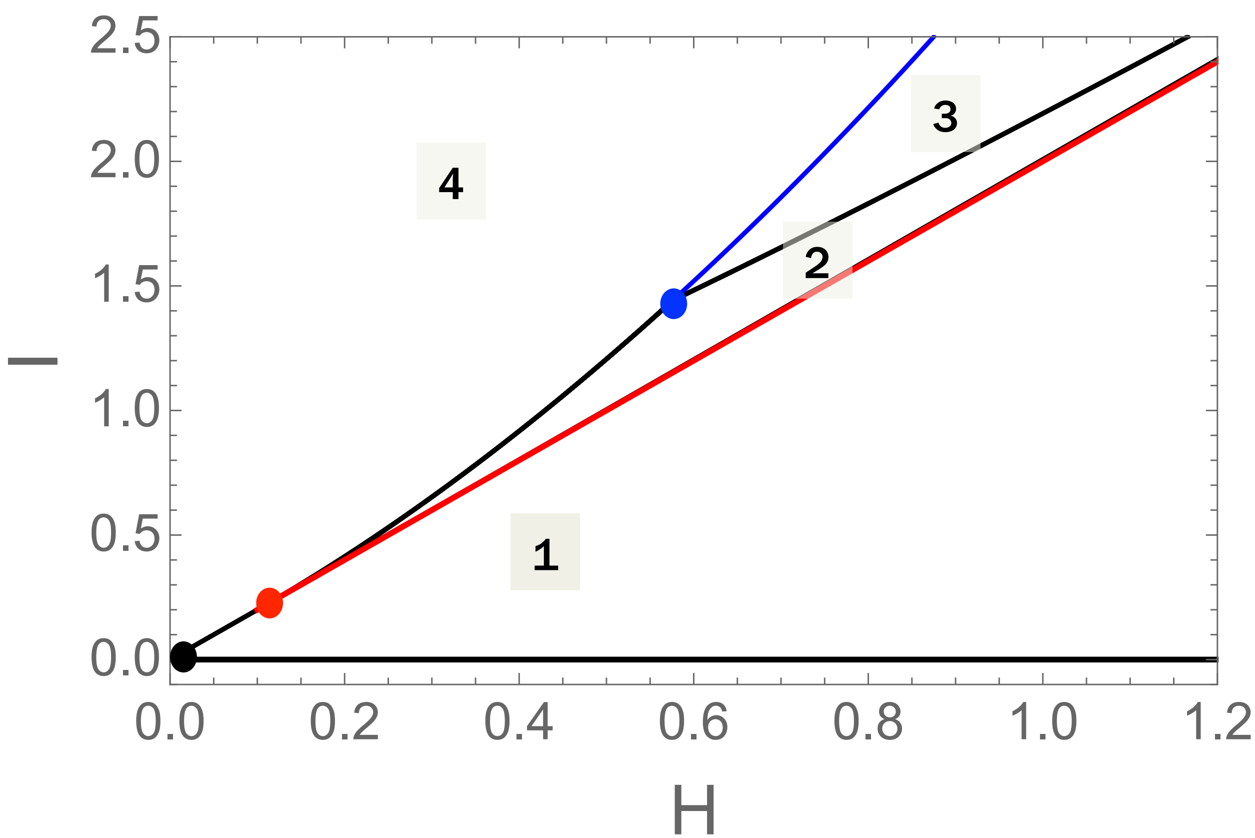
Bifurcation diagram:



In this case, the critical points are the fixed points. Critical level sets are the origin (H = -1) and the separatrix (H = +1).

Application to the Ideal Integrable Dynamics in IOTA (Shown for τ = -0.4)

Bifurcation diagram showing critical values of (H,I)



Note four distinct regions.

Level sets within these four regions differ qualitatively.

Points on the boundaries correspond to critical level sets (fixed points, separatrices, etc.)

Large dots indicate where these boundary curves split.

1) one simply-connected component containing the origin, 2) four distinct components
3) one component with a hole that excludes the origin, 4) empty (not allowed)

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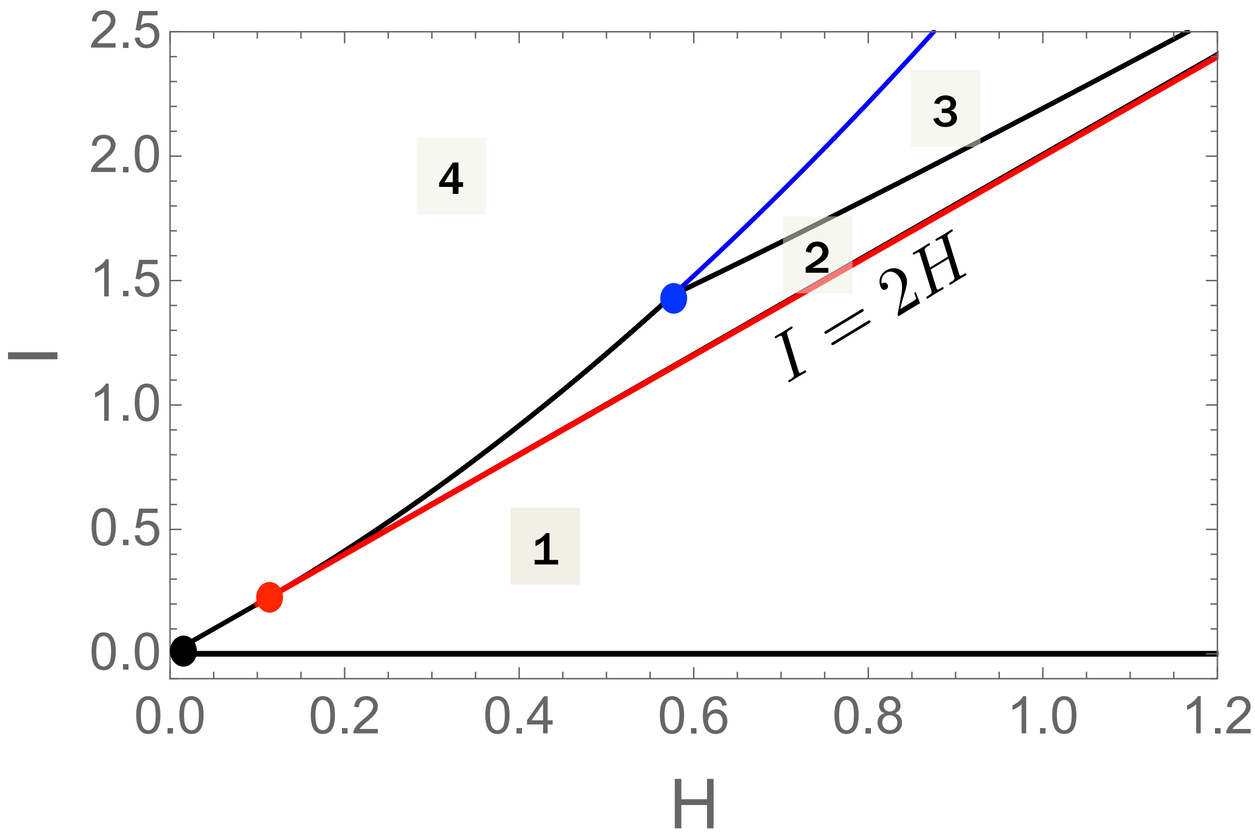

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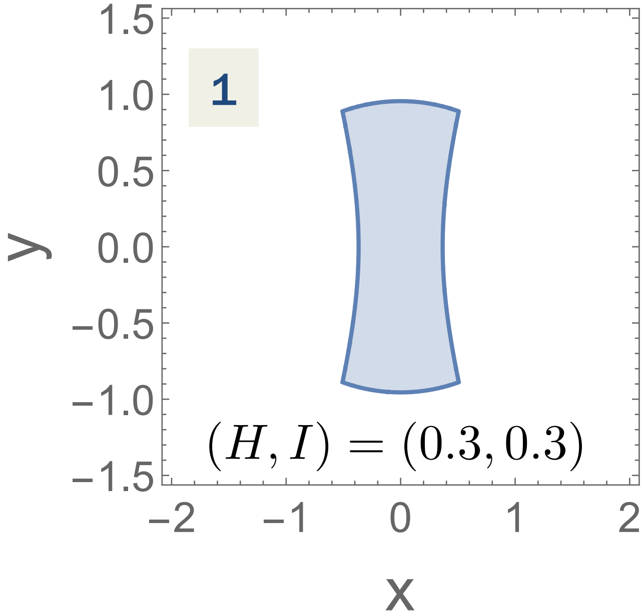
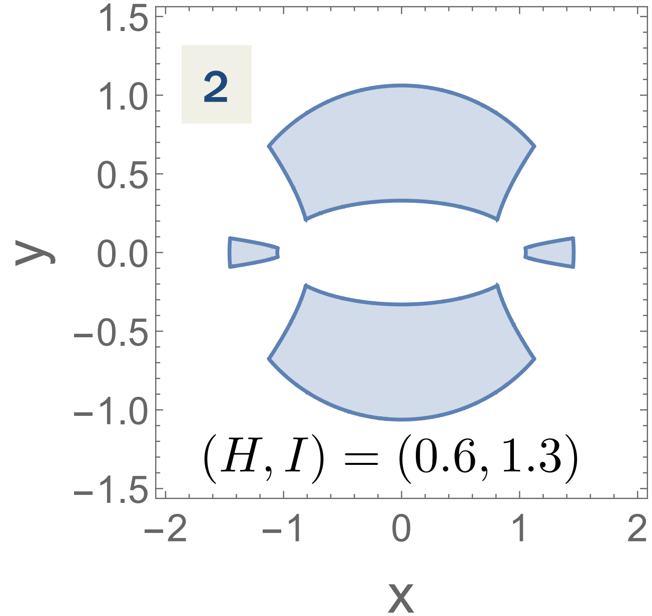
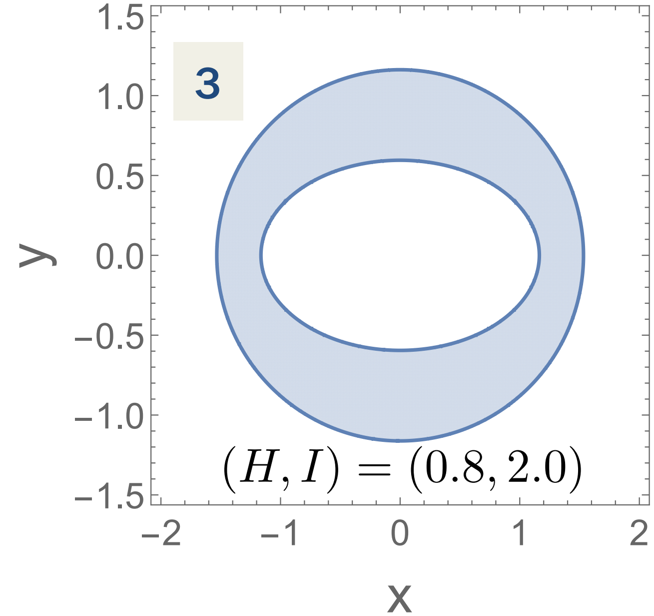
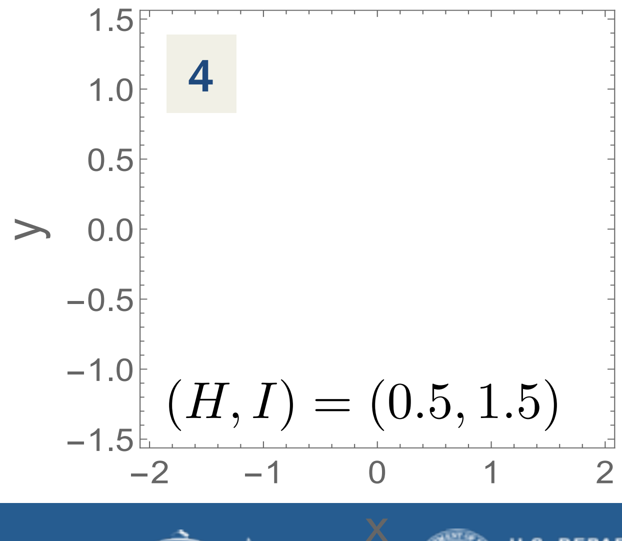
Examples of Regular Level Sets (τ = -0.4): Projections into the (X,Y) Plane







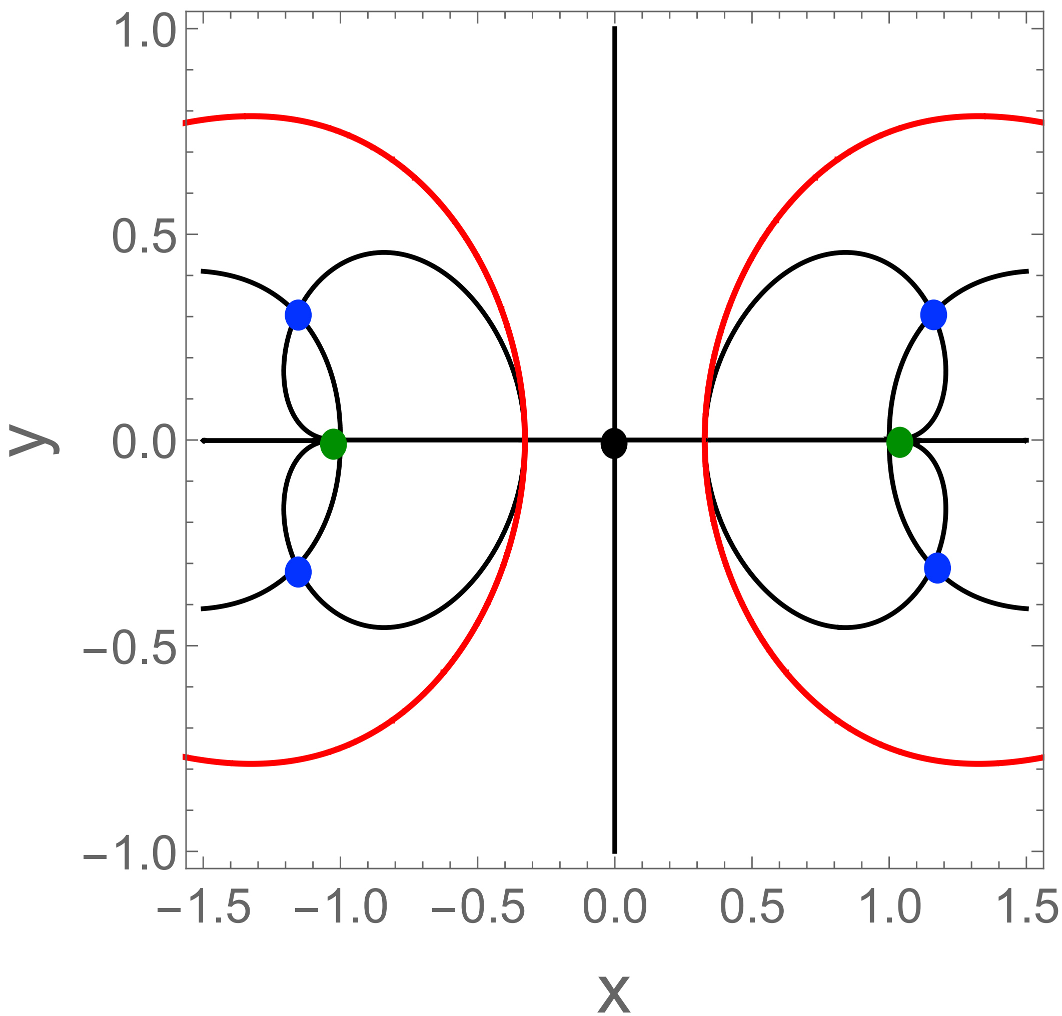


More interesting are the boundary cases, corresponding to critical values of (H,I), where transitions occur.

We will see many more examples in what follows.

Here it is simplest to visualize the level sets using projection onto the transverse (X,Y) plane.

Network of Critical Initial Conditions in the (X,Y) Plane Shown for P_x=P_y=0 (with τ = -0.4)



We show all points in the plane (x,y) such that the initial condition (x,0,y,0) lies on a critical level set.

These curves naturally divide the plane into regions with qualitatively distinct dynamical behavior.

Black point: stable fixed point

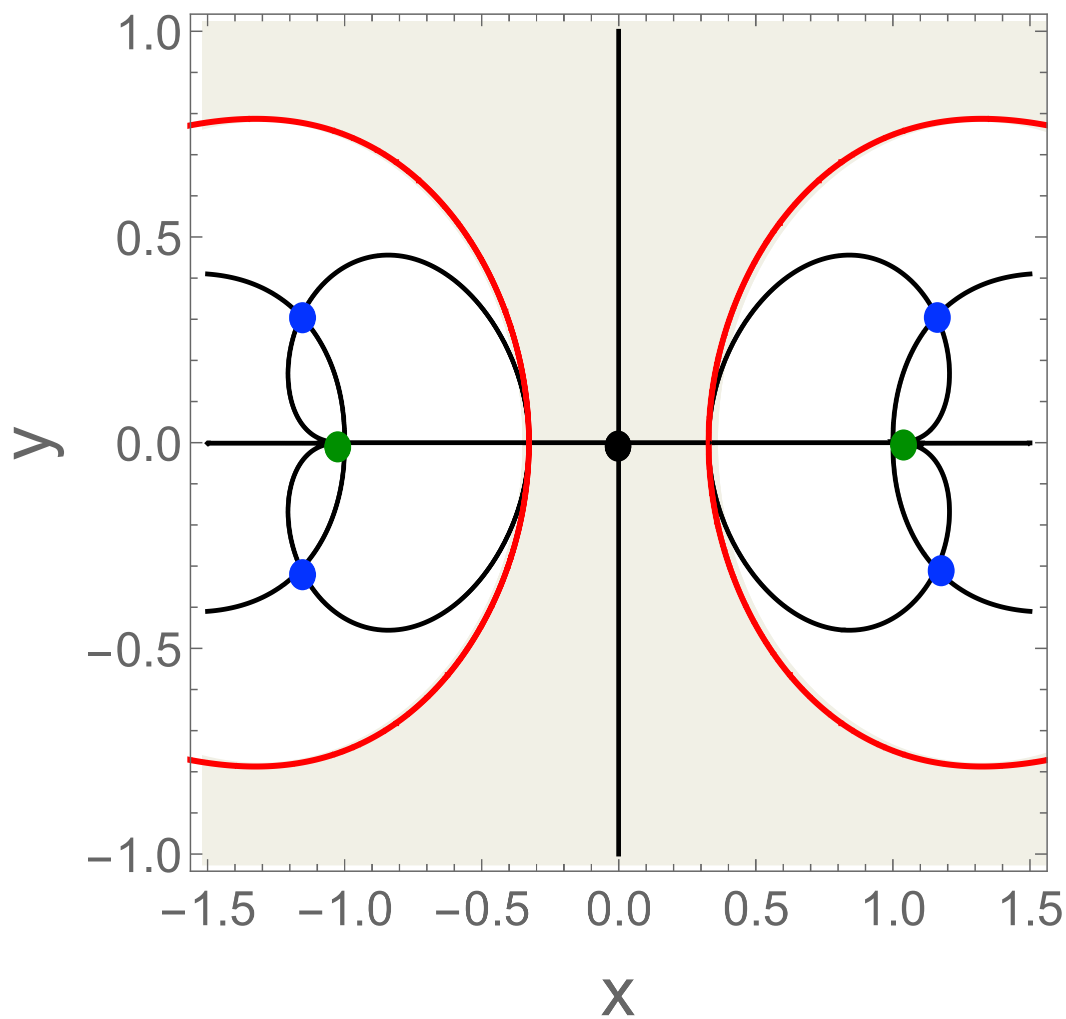
Blue points: 4 saddle points

Green points: singular points

Red and black curves correspond to the red and black curves in the bifurcation diagram.

- **Classification of Integrable Orbits: A Visual Tour**

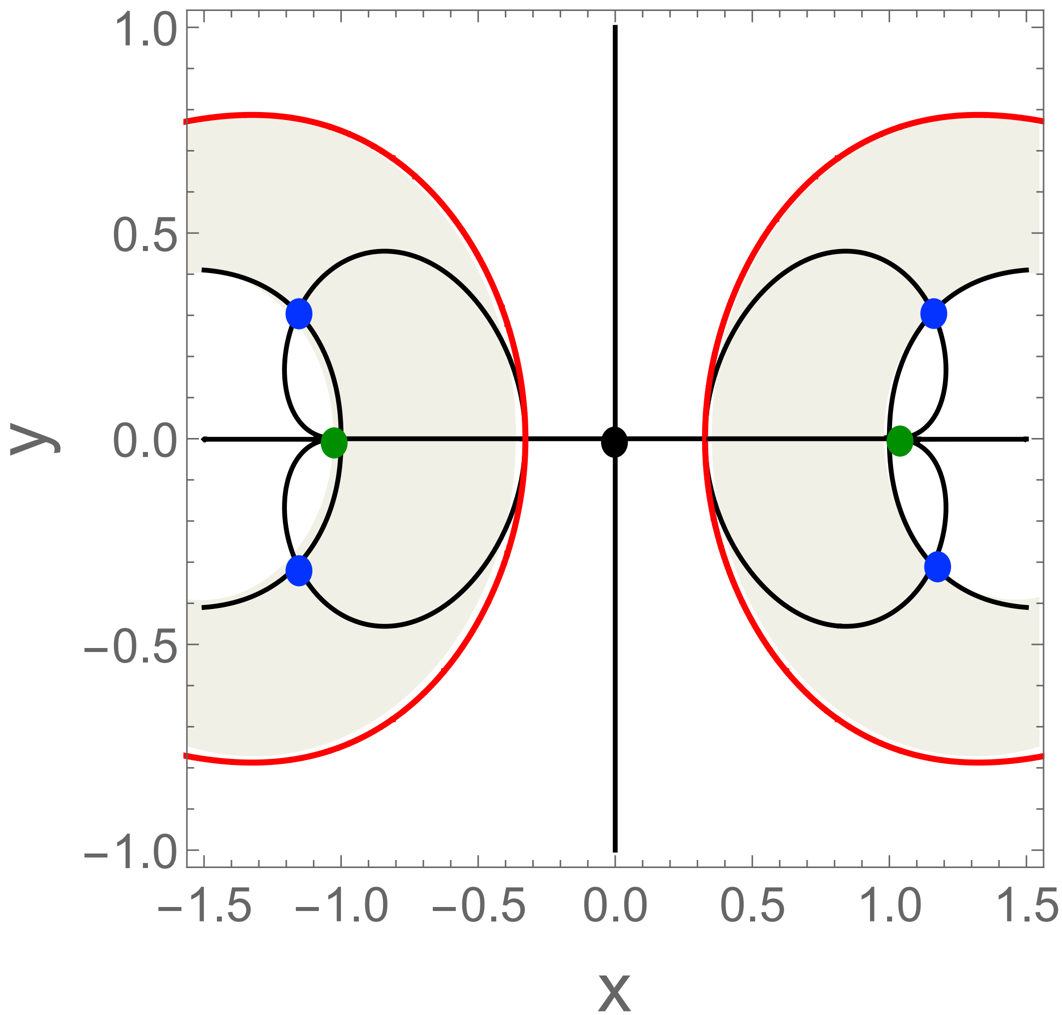
Classification of Orbits Initialized in the (X,Y) Plane with $P_x = P_y = 0$



For initial conditions in the highlighted region:

- Orbits move over level sets containing the origin.
- Stable periodic orbits occur for initial conditions on the black curves.
- A fixed point occurs at the origin.

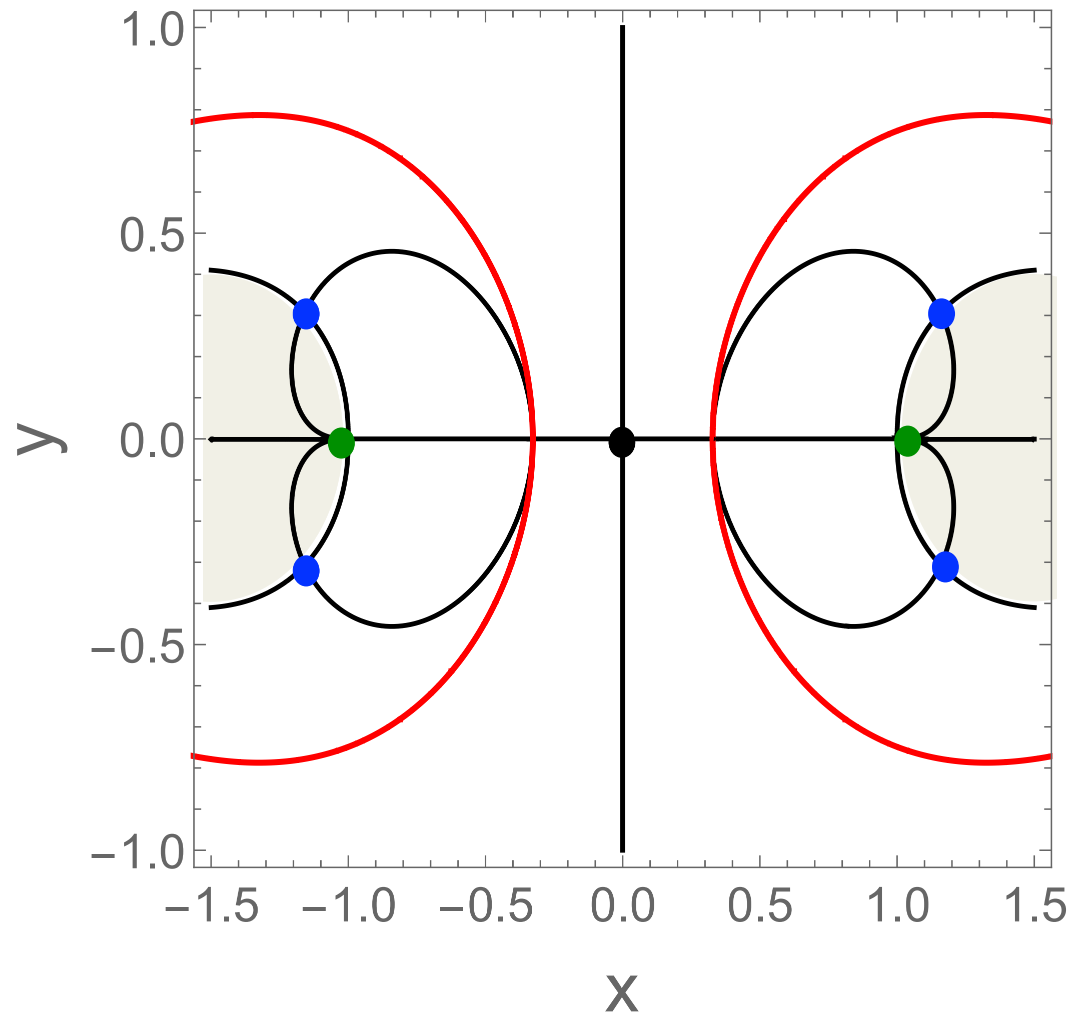
Classification of Orbits Initialized in the (X,Y) Plane with $P_x = P_y = 0$



For initial conditions in the highlighted region:

- **Orbits "bounce" across the Y axis within level sets confined above (or below) the midplane.**
- **Periodic orbits occur for initial conditions on the black arcs (midplane unstable, others stable).**
- **A fixed point occurs at the blue dots (saddle). The singular point is in green.**

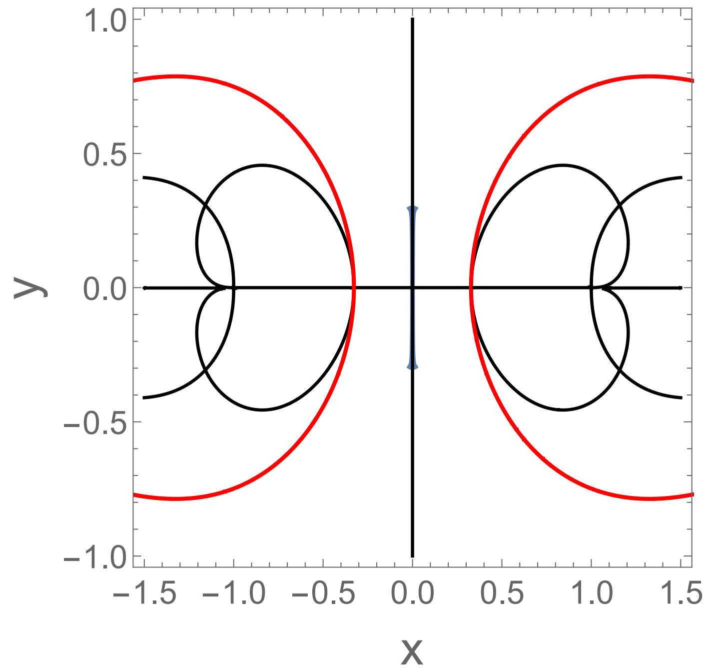
Classification of Orbits Initialized in the (X,Y) Plane with $P_x=P_y=0$



For initial conditions in the highlighted region:

- **Orbits move back and forth across the X axis within level sets confined to the right (or left) half-plane.**
- **Periodic orbits occur for initial conditions on the black arcs (some stable, some unstable).**
- **A fixed point occurs at the blue dot (saddle). The singular point is in green.**

Scan of initial conditions in X-Y (starting from rest)

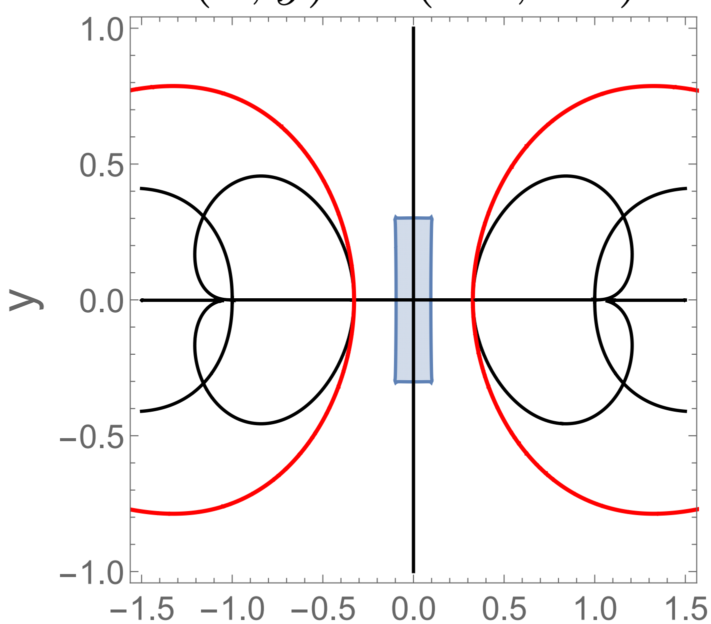


$$(x, y) = (0, 0.3)$$

initial condition on the y-axis: periodic orbit confined to the y-axis (degenerate line segment)

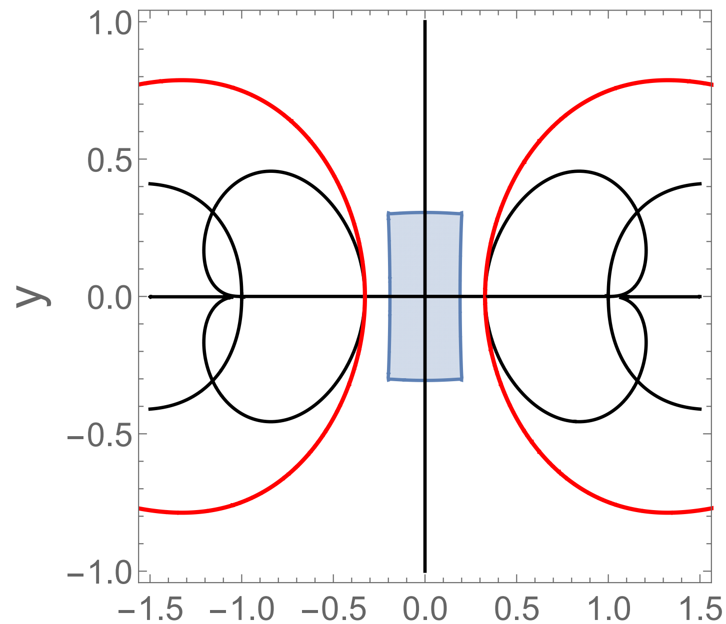
initial condition lies on the corner of the shaded region

$$(x, y) = (0.1, 0.3)$$

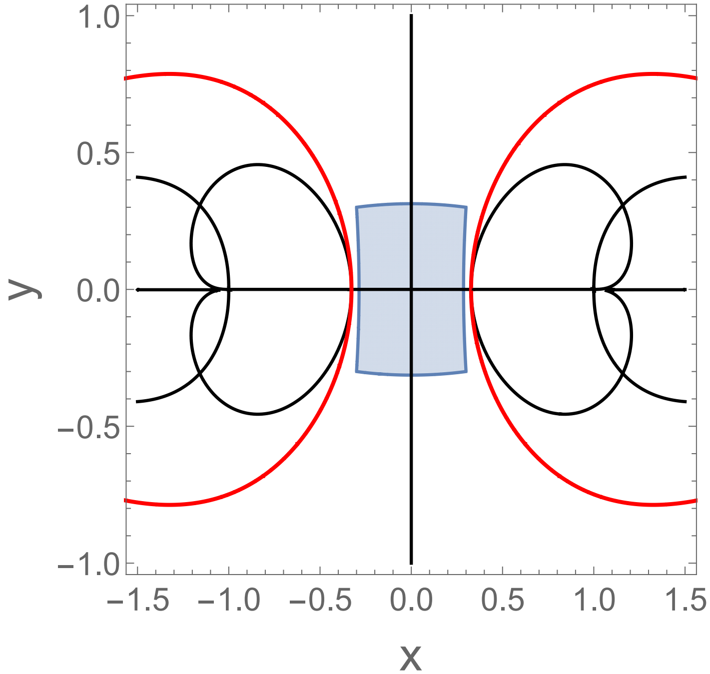


orbit fills a rectangle of increasing horizontal size

$$(x, y) = (0.2, 0.3)$$



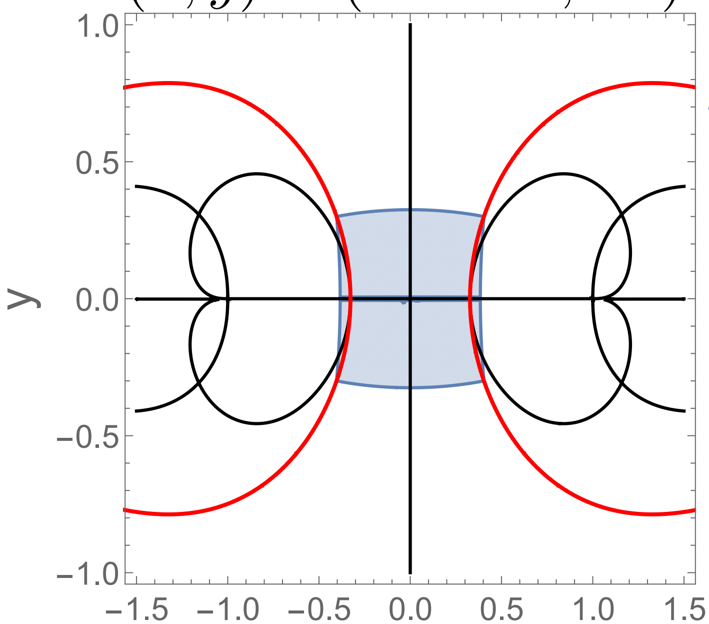
$$(x, y) = (0.3, 0.3)$$



initial condition is approaching the red arc

initial condition on the red arc:
level set *splits* along midplane, &
2 small islands appear

$$(x, y) = (0.40375, 0.3)$$

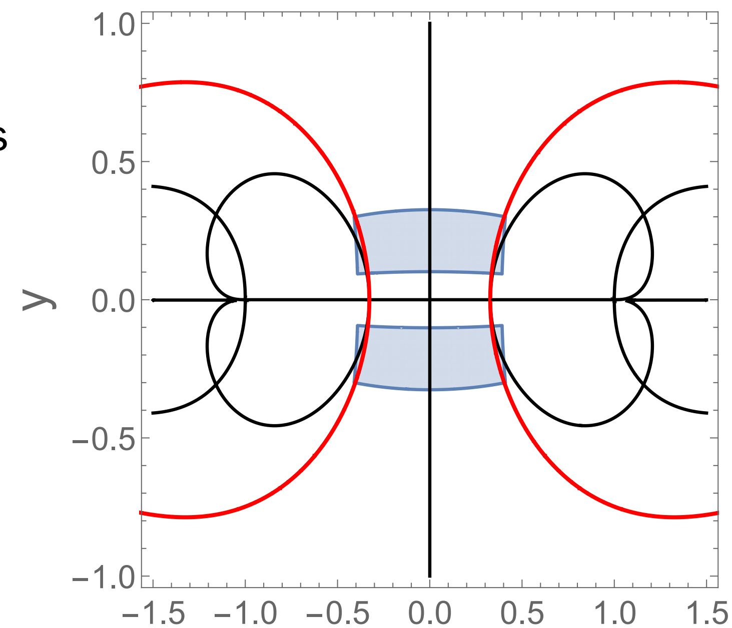


four components

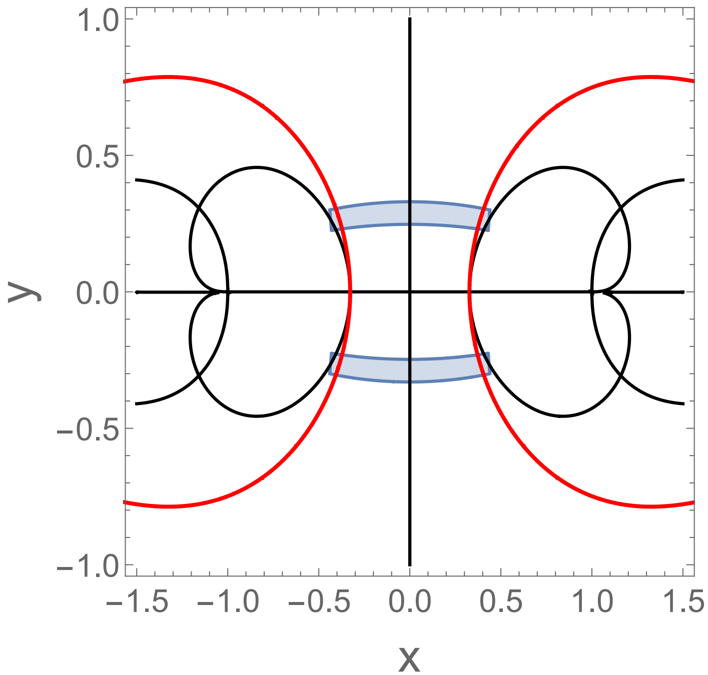
orbit lies in the component above the midplane

“bouncing orbit”

$$(x, y) = (0.41, 0.3)$$



$$(x, y) = (0.44, 0.3)$$

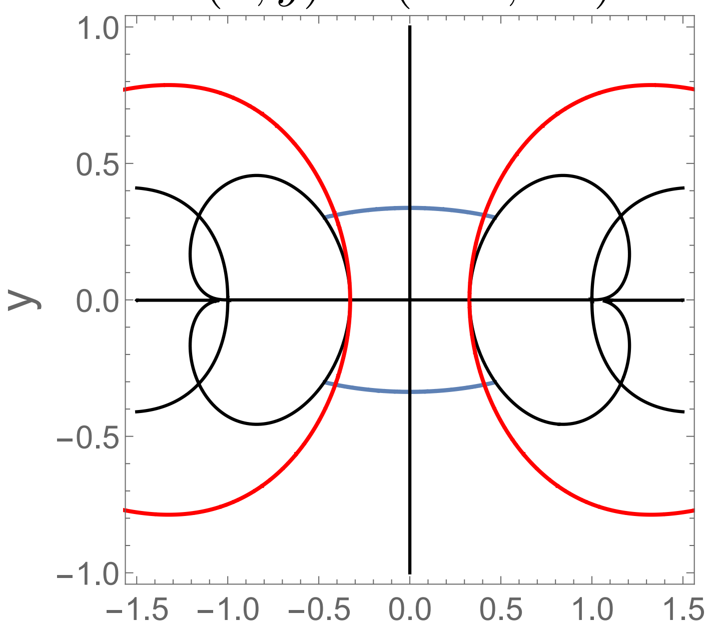


two visible components shrink in the vertical direction

initial condition approaches the black arc

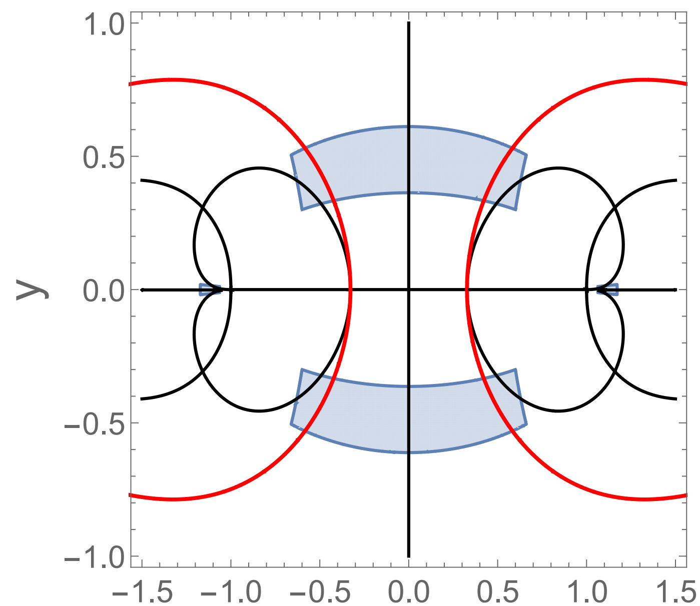
initial condition lies on the black arc: degenerate curves (periodic orbit)

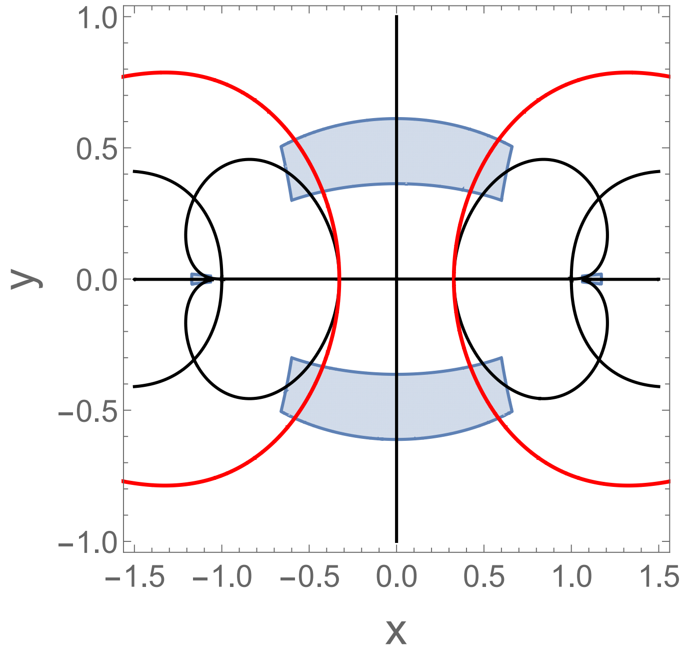
$$(x, y) = (0.47, 0.3)$$



components increase in size after crossing the black arc

$$(x, y) = (0.55, 0.3)$$



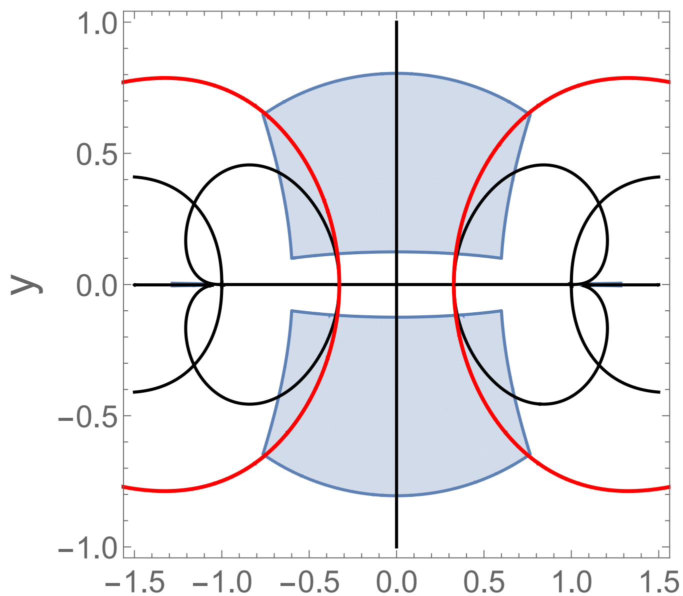


$$(x, y) = (0.6, 0.3)$$

Now we fix X and begin to decrease Y, moving downward toward the midplane.

visible components increase in vertical size

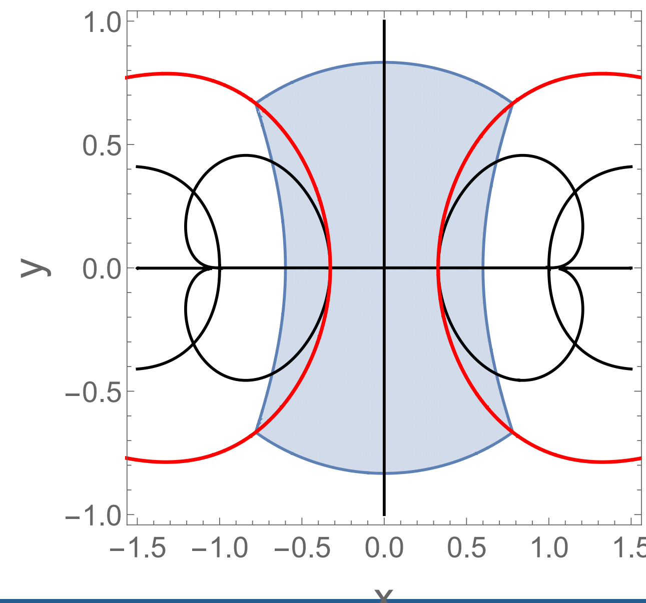
$$(x, y) = (0.6, 0.1)$$



initial condition in the midplane

two large components merge; unstable periodic orbit in the midplane

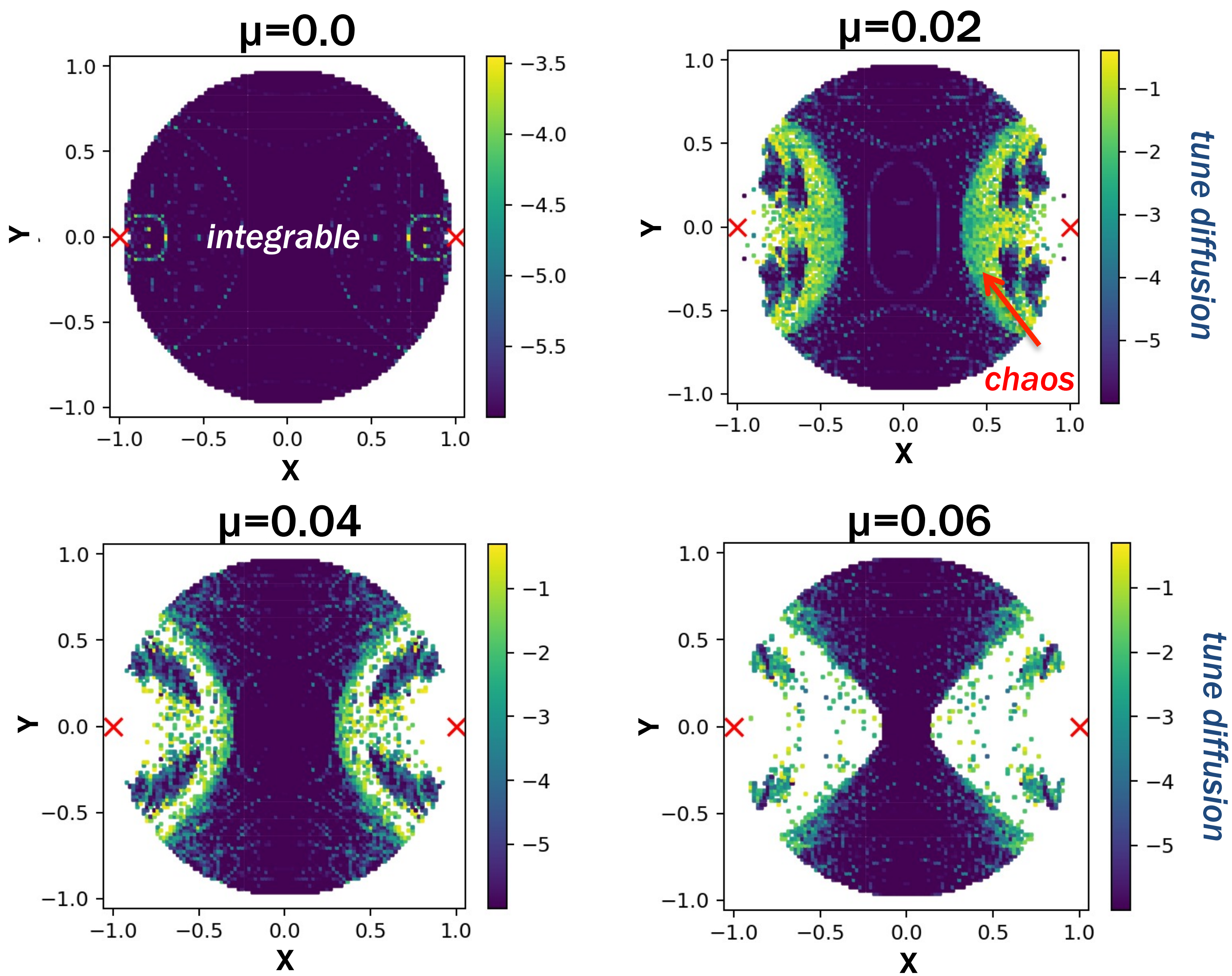
$$(x, y) = (0.6, 0)$$



- **Dynamic Aperture in the Presence of a Perturbation**

IOTA ideal lattice plus tune advance error – dynamic aperture with increasing tune error (Frequency Map Analysis)

~8K distinct initial conditions (*x*,*0*,*y*,*0*) in a disk, 2048 turns. Particles are lost if *R* > 2.83 cm.



Measure of tune diffusion: $\log(\Delta)$

$$\Delta = \sqrt{\Delta\nu_x^2 + \Delta\nu_y^2}$$

NL Insert parameters:
***L* = 1.8 m, *τ* = -0.40,**
***c* = 0.01 m$^{1/2}$, *ν* = 0.303**

Singular points are located at:

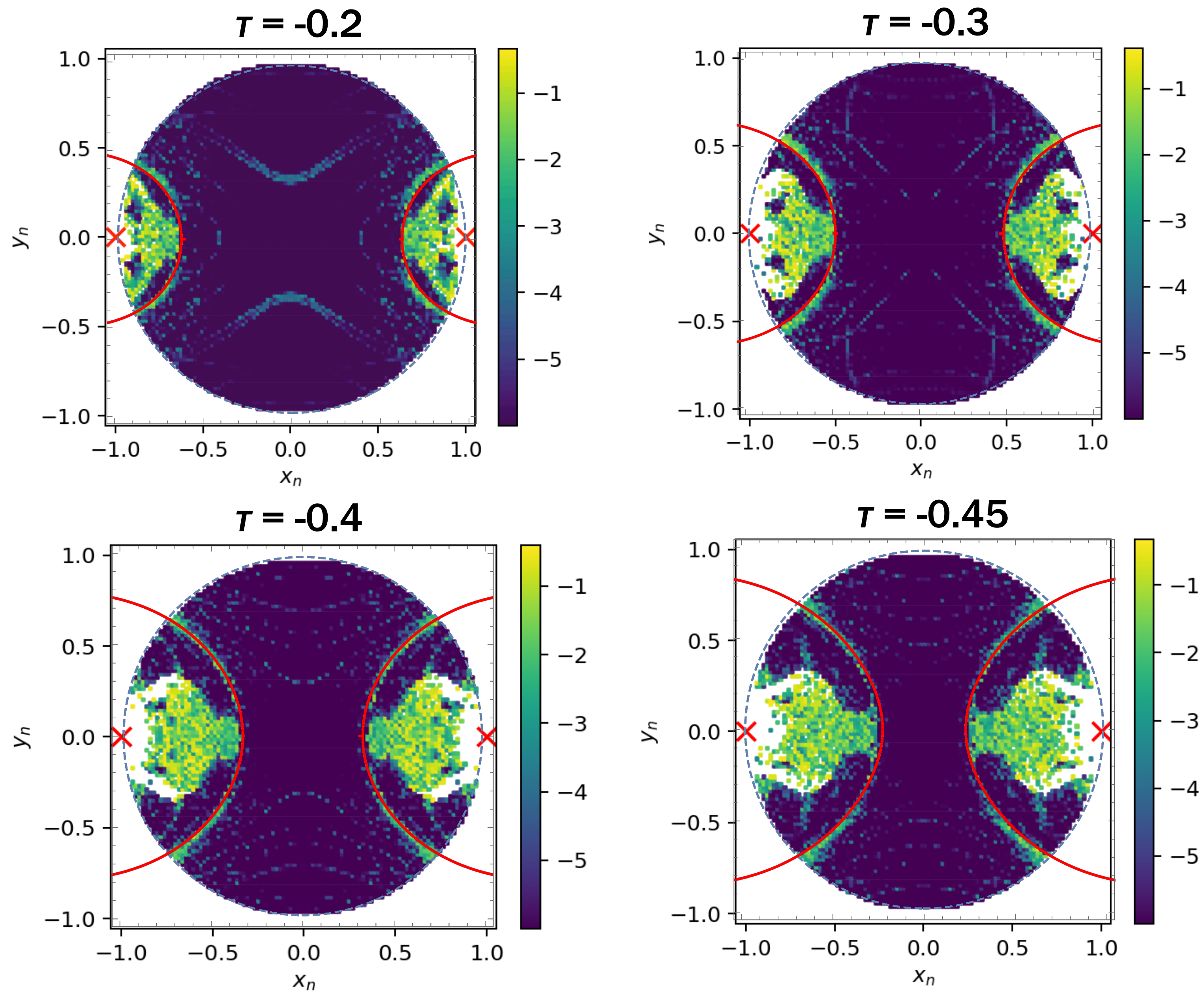
$$(\pm 1.38, 0)\ \mathrm{cm}$$

Limited primarily by horizontal aperture.

DA shrinks with increasing tune error.

K. Hwang *et al*, "Chaos Indicators for Studying Dynamic Aperture in the IOTA Ring with Protons," WEPTS078, IPAC 2019.

Boundary of the dynamic aperture for small tune error is well-described by the primary separatrix of integrable motion



Red arcs: primary separatrix, I = 2H

Comparison for several values of insert strength.

NL Insert parameters:
L = 1.8 m, various τ,
c = 0.01 m$^{1/2}$, ν = 0.3

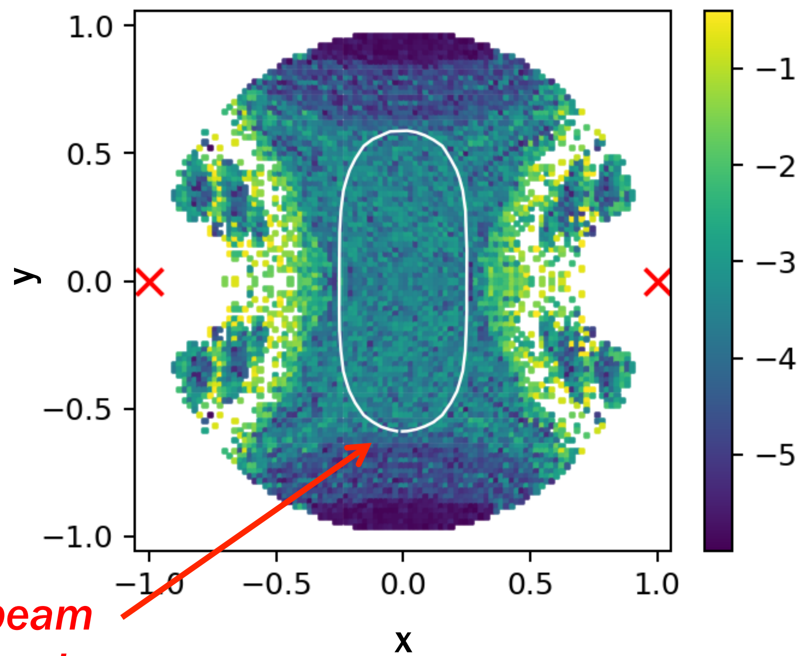
Tune advance error:
μ = 0.01 in each case

Innermost chaotic orbits coincide with the separatrix.

IOTA ring with 0.03 space charge tune depression – dynamic aperture (Frequency Map Analysis)

- 8K distinct initial conditions (*x,0,y,0*) in a disk, 2048 turns. Particles are lost if *R* > 2.83 cm.
- Note the large beam tune spread (beam is stable despite crossing many low-order resonances).
- Evidence of diffusion in the beam core. Test particles with *y*=0 outside the beam are lost.

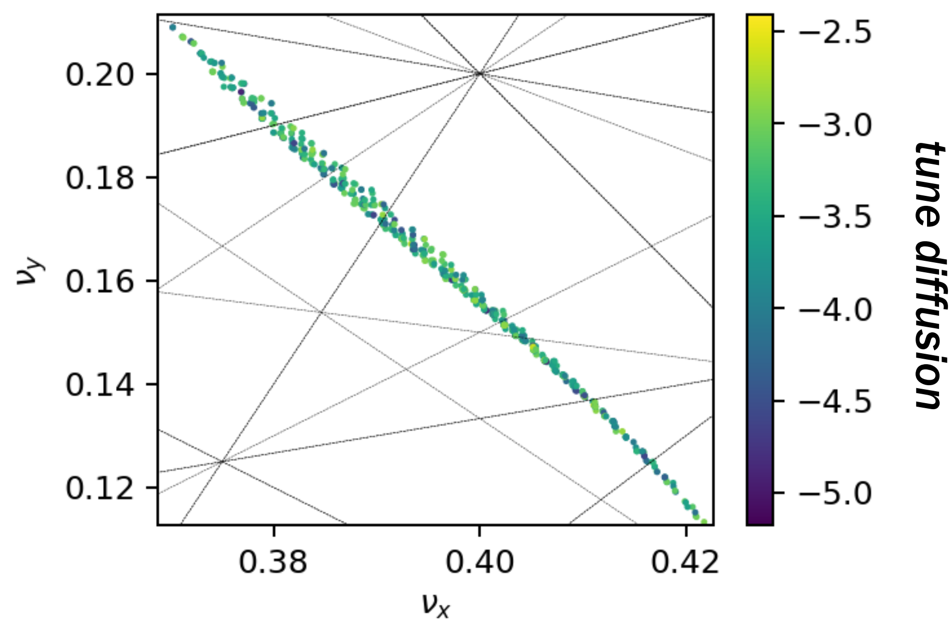
Dynamic aperture



beam boundary

Singular points are located at: $(\pm 1.38, 0)$ cm

Beam core tune footprint

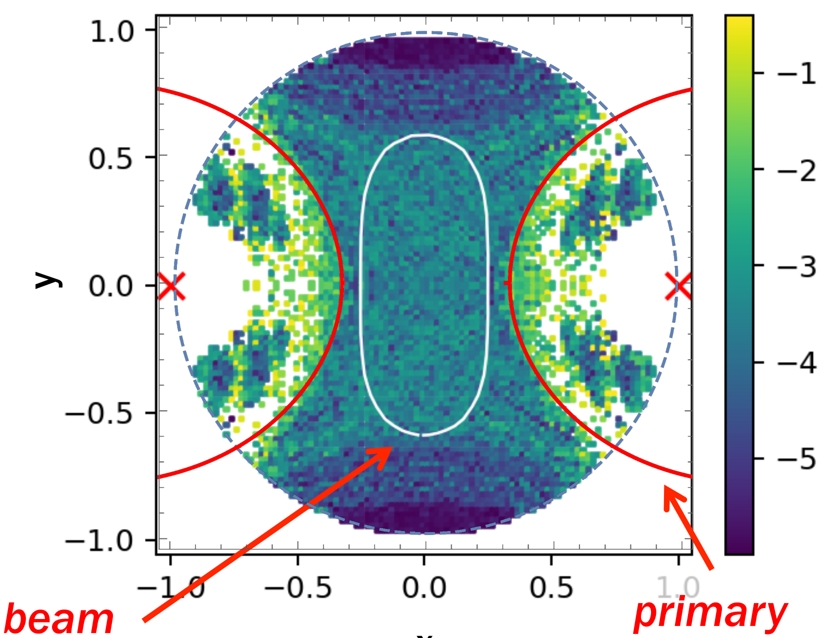


Measure of tune diffusion: $\log(\Delta)$ $\Delta = \sqrt{\Delta\nu_x^2 + \Delta\nu_y^2}$

IOTA ring with 0.03 space charge tune depression – dynamic aperture (Frequency Map Analysis)

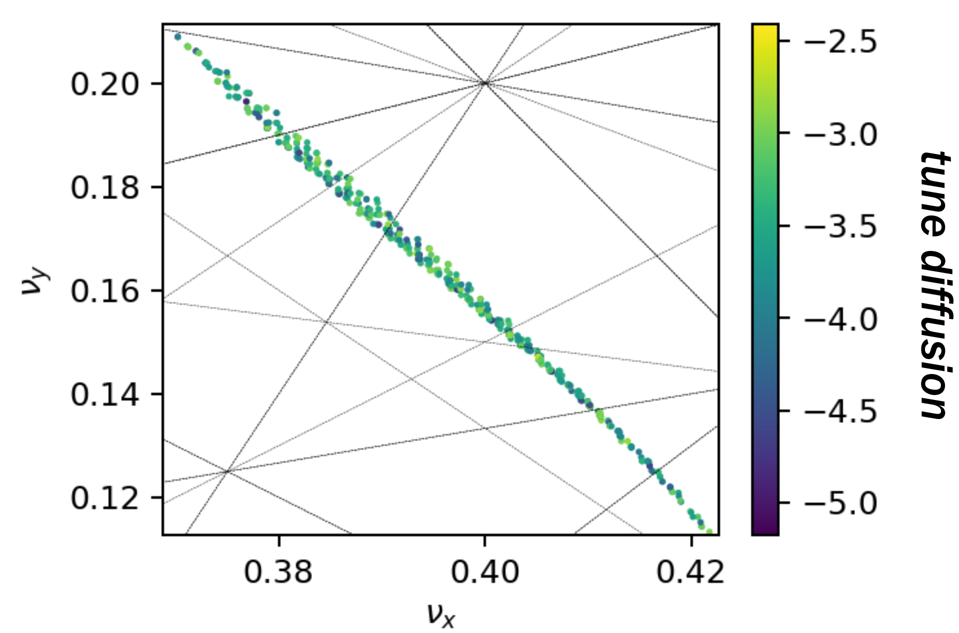
- 8K distinct initial conditions (*x,0,y,0*) in a disk, 2048 turns. Particles are lost if *R* > 2.83 cm.
- Note the large beam tune spread (beam is stable despite crossing many low-order resonances).
- Evidence of diffusion in the beam core. Test particles with *y*=0 outside the beam are lost.

Dynamic aperture



Singular points are located at: $(\pm 1.38, 0)$ cm

Beam core tune footprint



Measure of tune diffusion: $\log(\Delta)$ $\quad \Delta = \sqrt{\Delta\nu_x^2 + \Delta\nu_y^2}$

Conclusions

- General techniques exist for finding critical points—including fixed points, separatrices, and unstable periodic orbits—of integrable Hamiltonian systems. These were applied to study the ideal dynamics in the IOTA nonlinear integrable optics experiment.

- The unique structure of the nonlinear potential allows for unusual orbits—for example, orbits that are confined to the upper half-plane, bouncing off of a potential barrier in the midplane.

- The system has 5 fixed points and several families of periodic orbits. The global behavior of the level sets is well-illustrated by using a bifurcation diagram. The network of critical initial conditions in the (X,Y) plane gives a global picture of orbit behavior.

- The boundary between "bouncing" and ordinary orbits lies at *I=2H (primary separatrix).* This separatrix coincides with the inner boundary of stable dynamic aperture when the integrable system is subject to a small perturbation in tune advance.

- This behavior appears to persist when we consider the dynamic aperture of a population of test particles moving in the potential of the beam core, for small space charge tune shift.

- **Backup Material**

Application to the Ideal Integrable Dynamics in IOTA

In the case of IOTA, the momentum mapping is given explicitly by (here *z* = *x*+*iy*):

$$\mathcal{F}:\mathbb{R}^4\rightarrow\mathbb{R}^2, \qquad \mathcal{F}(x,p_x,y,p_y)=(H(x,p_x,y,p_y),I(x,p_x,y,p_y)),$$

$$H=\frac{1}{2}(p_x^2+p_y^2+x^2+y^2)-\tau U(x,y), \qquad U=\mathcal{R}e\left(\frac{z}{\sqrt{1-z^2}}\arcsin(z)\right)$$

$$I=(xp_y-yp_x)^2+p_x^2+x^2-\tau W(x,y), \qquad W=\mathcal{R}e\left(\frac{z+\bar{z}}{\sqrt{1-z^2}}\arcsin(z)\right)$$

Critical points occur where any of the following (equivalent) conditions is satisfied:

1) $\mathrm{rank}(D\mathcal{F})<2$

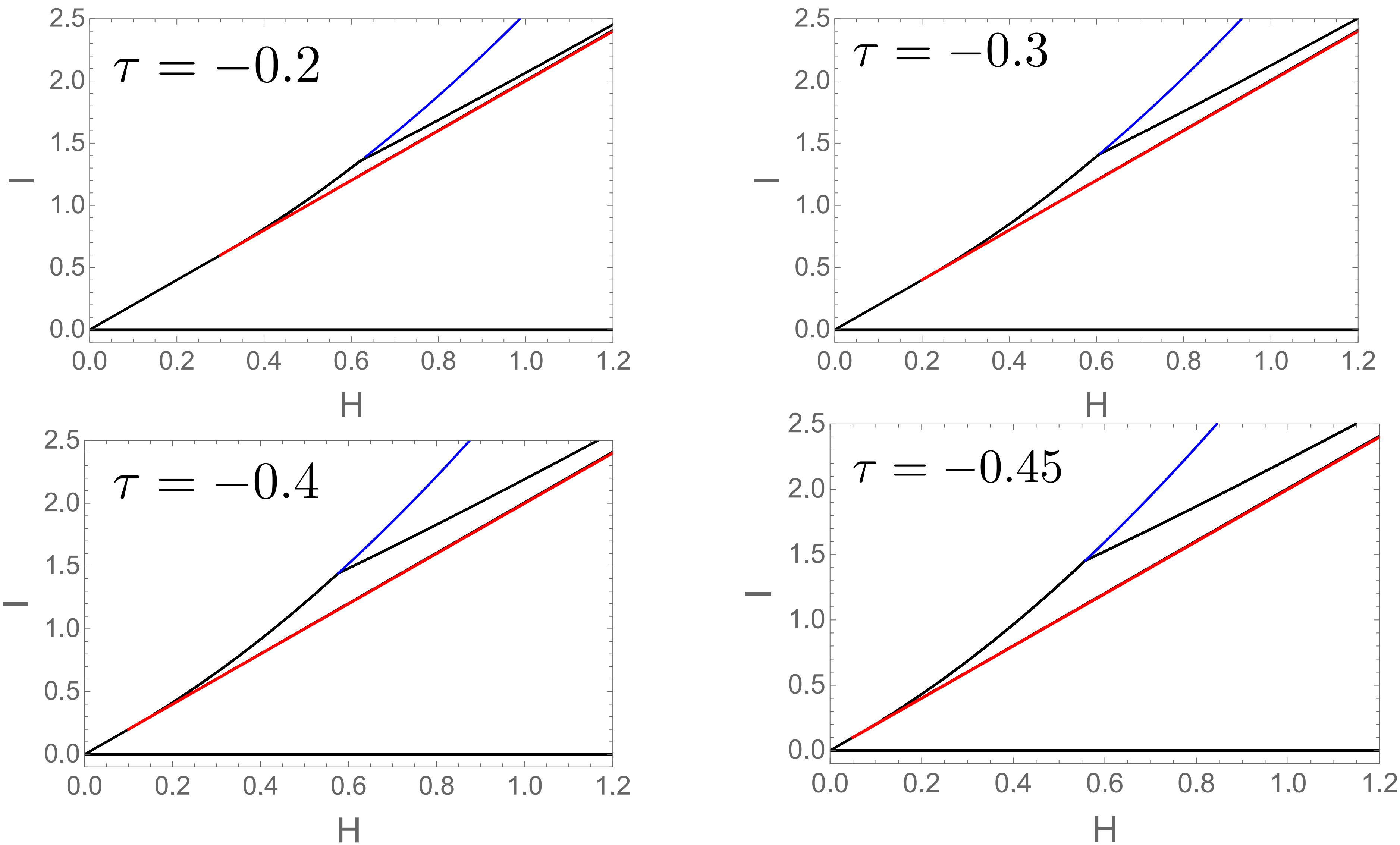
2) $\det(D\mathcal{F})^T(D\mathcal{F})=0$

3) $\{\nabla H,\nabla I\}$ *fails to be linearly independent*

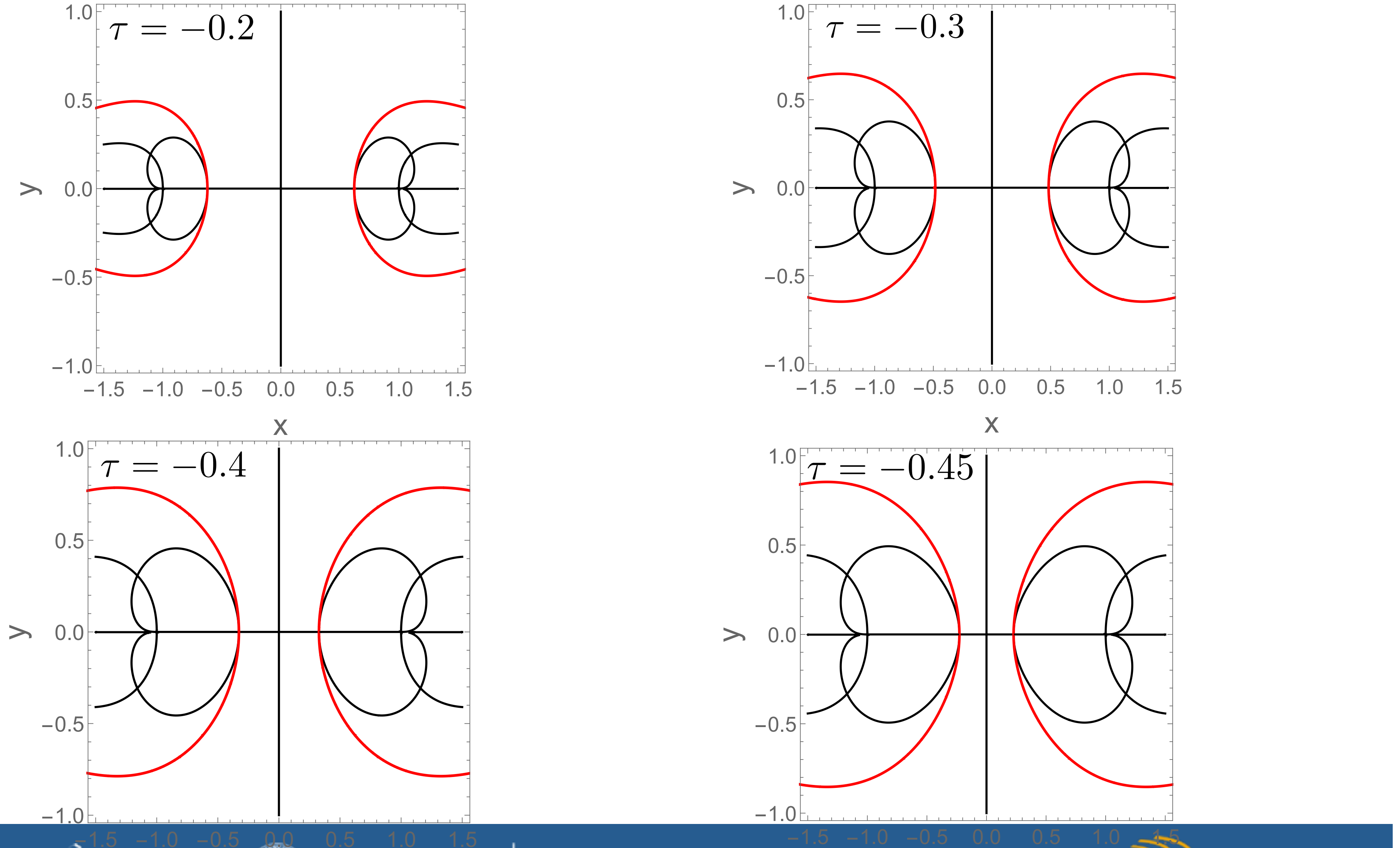
4) $dH\wedge dI=0$

- Finding these points requires searching for the zeros of one or more functions of 4 variables.
- This is numerically challenging, but with effort this can be done using, *eg*, *Mathematica*.

IOTA Bifurcation Diagram: Dependence on Nonlinear Insert Strength

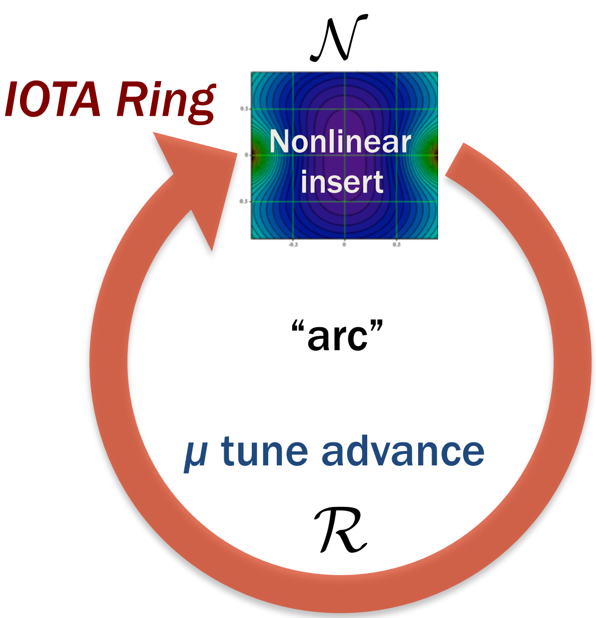


Network of Critical Initial Conditions in the (X,Y) Plane: Dependence on Nonlinear Insert Strength



The IOTA ideal lattice plus tune advance error – a model of sensitivity to perturbation (eg, space charge tune shift)

- IOTA ideal lattice requires integer or half-integer tune advance between the exit of the nonlinear insert and its entrance (the arc section) to ensure integrability.
- Model the sensitivity to space charge tune shift by introducing a small non-integer tune advance μ (equal in x and y) in the linear map of the arc.
- This gives a nonlinear one-turn map that can used to study the breakdown of integrability.



One-turn map (at NLI entrance): $\mathcal{M} = \mathcal{A}^{-1}\mathcal{N}\mathcal{R}\mathcal{A}$

Courant-Snyder normalizing map: $\mathcal{A}$

Map for the nonlinear insert: $\mathcal{N} = e^{-2\pi\nu:H_N:}$

Map for the arc: $\mathcal{R} = e^{-2\pi\mu:H_R:}$

$$H_R = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2)$$ **generates isotropic phase advance**

$$H_N = \frac{1}{2}(p_x^2 + p_y^2) + V(x,y)$$ **nonlinear insert Hamiltonian**

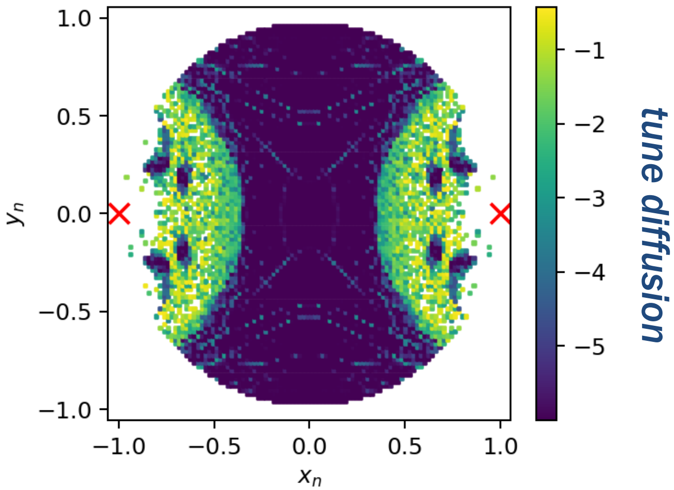
Reliably computing dynamic aperture

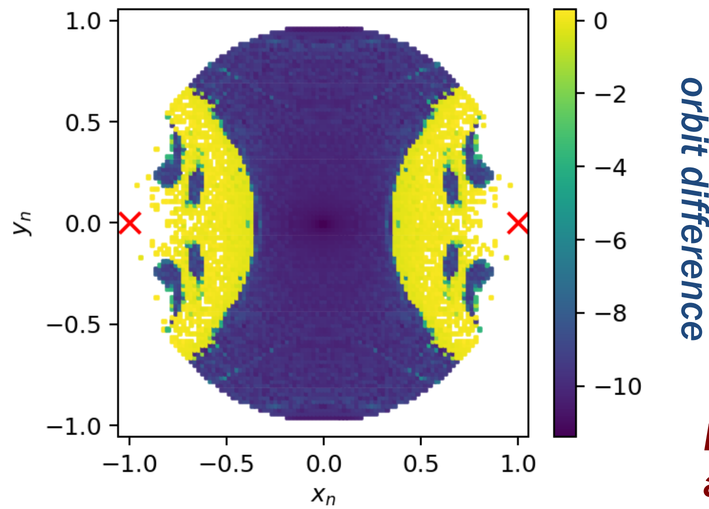
~8K initial conditions (*x,0,y,0*) in a disk, showing agreement between two distinct chaos detection methods.

Using FMA | **Using FB Integration**

Number of turns

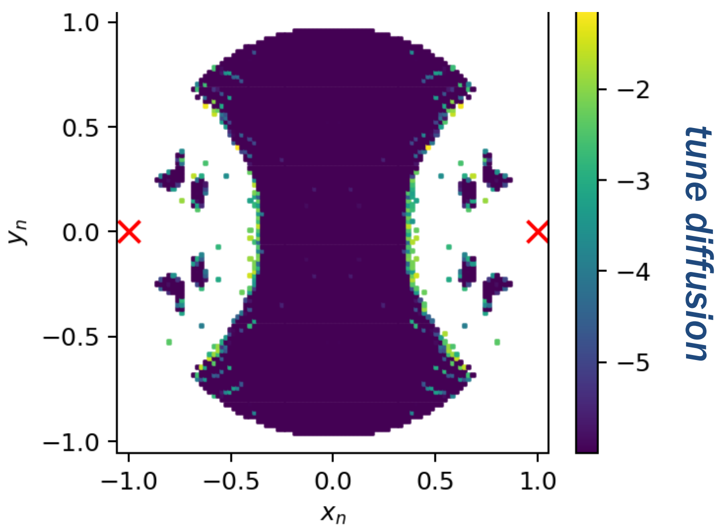
***T*=1025**

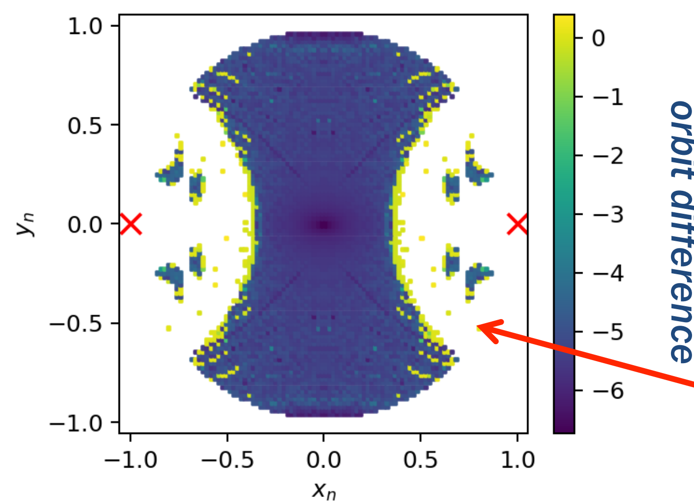




DA boundary appears to be well-defined

***T*=65536**





nearly all chaotic orbits are lost

Tracking in the IOTA Lattice with Space Charge - Assumptions and Simulation Parameters

Objective: To understand the perturbative effects of space charge on the ideal integrable single-particle dynamics at weak-moderate space charge tune depression.

- Elements external to the nonlinear insert are sliced longitudinally and treated as symplectic maps alternating with space charge momentum kicks (split-operator approach): *linear order*.

- Space charge is included self-consistently throughout the lattice using the symplectic spectral solver with a rectangular boundary of large aperture to emulate free-space boundary conditions.

- We consider a long, unbunched beam with zero energy spread to remain near the ideal integrable working point.

- Quadrupole settings are retuned to provide $n\pi$ phase advance across the arc after including the linearized space charge fields at the desired value of beam current (A. Romanov, [1]).

- Twiss functions with linearized space charge included must be appropriately matched to the nonlinear insert. *See also [2].*

Lattice parameters: $\tau = -0.4$, $c = 0.01\ \mathrm{m}^{1/2}$, $\mu_0 = 0.30345$, $L = 1.8$ m, $\Delta Q_x = \Delta Q_y = -0.03$
Beam parameters: $KE = 2.5$ MeV, $I = 0.4113$ mA, $\langle H \rangle = 0.04$, $\varepsilon_{x,n} = 0.12\ \mu\mathrm{m}$, $\varepsilon_{y,n} = 0.28\ \mu\mathrm{m}$

[1] A. Romanov *et al*, THPOA23, NAPAC 2016. [2] C. Hall *et al*, WEA4CO02, NAPAC 2016.