A Study of Beam Equilibria in Strongly Nonlinear Focusing Channels with Applications to IOTA

FAST/IOTA Collaboration Meeting

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ACCELERATOR TECHNOLOGY & APPLIED PHYSICS DIVISION ATAP

Acknowledgments and Collaborators

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- *LBNL – Robert Ryne, Kilean Hwang (especially for FMA results)*
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- *RadiaSoft – David Bruhwiler, Chris Hall, Stephen Webb, Nathan Hall, Jonathan Edelen*

Outline

- *Introduction and Motivation*

- *Vlasov Equilibria in a Nonlinear Constant Focusing Channel*
  - *construction of Hamiltonian and stationary beam distributions*
  - *nonlinear PDE for the 2D equilibrium space charge potential*

- *Numerical Tests Using the IOTA Nonlinear Potential*
  - *preservation of 0, 60 mA, and 120 mA beams*
  - *tracking results in the total constant-focusing potential*

- *Self-Consistent Matching to a Nonlinear Periodic Channel*
  - *thoughts on an approximate matching procedure (ongoing)*

- *Conclusions*

Questions regarding space charge and nonlinear integrable optics in IOTA (using 2.5 MeV protons)

1) Will the presence of space charge destroy the integrability of single-particle motion in IOTA?
2) What are the primary (resonance) mechanisms by which this occurs?
3) How does space charge affect the structure of the beam distribution at high current?
4) What consequences will space charge have for beam stability, halo, and losses?
5) How can we address 1)-4) accurately in the presence of numerical artifacts (particle noise)?

- Use ***fully symplectic*** tracking methods (including self-consistent space charge*).
- Use modeling with high spatial resolution and a large number of particles (≥ 1M).
- Study ***reduced dynamical models*** to aid in understanding the novel dynamics.
- Use multiple methods to distinguish between integrability and chaos (preservation of invariants, sensitive dependence on initial conditions, frequency map analysis).

* J. Qiang, Phys. Rev. ST Accel. Beams 20, 014203 (2017).

Motivation for studying an IOTA constant focusing channel

- Existence of Vlasov equilibria (matched periodic solutions) in a general s-dependent lattice is a deep and difficult problem, closely connected to the existence of invariants of motion.

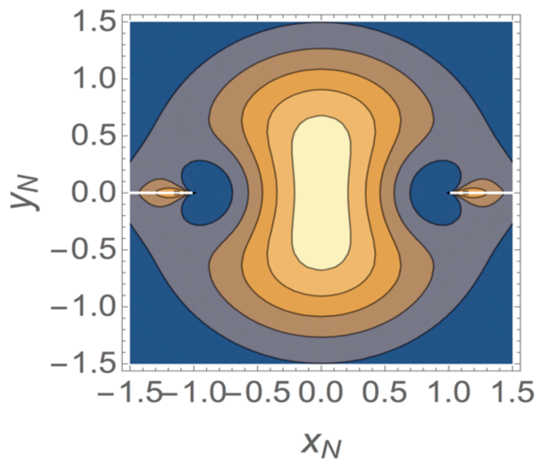
- Constant focusing channels are well-studied standard tools for studying intense beam equilibria in the presence of linear external focusing.

- It is known that, in some cases[1] (such as a periodic solenoid channel) constant-focusing equilibria can also be used to construct approximate equilibria of the periodic lattice.

- We would like to use *nonlinear* constant-focusing equilibria to investigate how space charge is expected to affect the beam distribution in IOTA as the beam intensity is varied.

Example:

Density contours of an intense beam in self-consistent 4D thermal equilibrium in a strongly nonlinear IOTA channel



density contours

$\Lambda = 10$

$\tau = -0.45$

$\kappa_x = \kappa_y = 1$

$H_0 = 0.3$

[1] J. Struckmeier and I. Hofmann, Particle Accelerators 39, 219 (1992).

- **Vlasov Equilibria in a Nonlinear Constant Focusing Channel**

Construction of an IOTA Constant Focusing Channel (1)

We begin with the *s*-dependent Hamiltonian of the IOTA ring (for on-energy orbits in the paraxial approximation):

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(k_x^2 x^2 + k_y^2 y^2) - \frac{\tau c^2}{\beta} U\left(\frac{x}{c\sqrt{\beta}}, \frac{y}{c\sqrt{\beta}}\right) + \frac{q\phi(x,y,s)}{\beta_0^2 \gamma_0^3 m c_0^2}$$

space charge potential

nonlinear insert potential A_z

$$U(x,y) = \mathcal{R}eF(x+iy), \quad F(z) = \frac{z}{\sqrt{1-z^2}} \arcsin(z) \; .$$

The relativistic factors contain a subscript *0* to distinguish them from the twiss *β* and nonlinear insert parameter *c*.

This assumes a coasting beam, and all momenta are normalized by the design momentum $p^0 = \gamma_0 \beta_0 m c_0$. The beam is assumed to be longitudinally uniform, so that space charge is 2D and in the laboratory frame:

$\nabla^2 \phi = -\rho/\epsilon_0$ with $\phi = 0$ on the boundary of the domain (pipe).

Note that k_x, k_y, τ, β, and ϕ all contain *s*-dependence.

Construction of an IOTA Constant Focusing Channel (2)

We can construct an approximately "equivalent" *s*-independent Hamiltonian using methods to be described in the final section of the talk (on matching to periodic lattices).

For simplicity, we will assume here that the *s*-dependence of all quantities in *H* is ignored. Then we perform a Courant-Snyder transformation and scale by *c* to give the dimensionless variables:

$$x_N = x/c\sqrt{\beta}, \quad y_N = y/c\sqrt{\beta}, \quad p_{xN} = \sqrt{\beta}p_x/c, \quad p_{yN} = \sqrt{\beta}p_y/c$$

With the phase advance $\psi = s/\beta$ as the new independent variable, the Hamiltonian in the new variables is:

$$H_N = \frac{1}{2}(p_{xN}^2 + p_{yN}^2) - \tau U(x_N, y_N) + \frac{1}{2}(\kappa_x^2 x_N^2 + \kappa_y^2 y_N^2) + \Phi_N(x_N, y_N)$$

where:

$$\kappa_x = k_x\beta, \quad \kappa_y = k_y\beta, \quad \Phi_N(x_N, y_N) = \frac{\beta}{c^2}\left[\frac{q\phi(x,y)}{\beta_0^2\gamma_0^3 mc_0^2}\right]$$

nominal integrable optics when $\kappa_x = \kappa_y = 1, \quad \Phi = 0$

Construction of a Stationary Beam Distribution

We define a stationary distribution function *f* in normalized coordinates by setting $f = G \circ H_N$ for some specified function *G*, so that:

$$f(x_N, p_{xN}, y_N, p_{yN}) = G(H_N(x_N, p_{xN}, y_N, p_{yN})), \quad \int f dx_N dp_{xN} dy_N dp_{yN} = 1 .$$

Then projecting onto the spatial coordinates gives the spatial density in the form:

$$P_{xy}(x_N, y_N) = \int f(x_N, p_{xN}, y_N, p_{yN}) dp_{xN} dp_{yN} = 2\pi \int_{V(x_N, y_N)}^{\infty} G(h) dh .$$

total potential in H_N

Examples:

1) KV beam: $G(h) = f_0 \delta(H_0 - h), \quad P_{xy} = 2\pi f_0 \Theta(H_0 - V)$

2) Waterbag beam: $G(h) = f_0 \Theta(H_0 - h), \quad P_{xy} = 2\pi f_0 (H_0 - V)\Theta(H_0 - V)$

3) Thermal beam: $G(h) = f_0 \exp(-h/H_0), \quad P_{xy} = 2\pi f_0 H_0 \exp(-V/H_0)$

Nonlinear PDE for the Equilibrium Potential

Expressed in our normalized coordinates, the Poisson equation becomes:

$$\nabla_N^2 \Phi_N = -\left(\frac{\Lambda}{2\pi}\right) P_{xy}, \qquad \Lambda = \frac{(2\pi)^2\beta}{c^2} K \quad \text{where} \quad K = \frac{2I}{\beta_0^3 \gamma_0^3 I_A}$$

generalized perveance

Using our expression for the spatial density gives the PDE that must be satisfied by the self-consistent potential:

$$\nabla_N^2 \Phi_N = -\Lambda \int_{V_0+\Phi_N}^{\infty} G(h)dh \ \text{ on } \ \Omega \qquad \Phi_N = 0 \ \text{ on } \ \partial\Omega$$

boundary condition (★)

Here V_0 is the external focusing potential: $V_0(x_N, y_N) = \frac{1}{2}(\kappa_x^2 x_N^2 + \kappa_y^2 y_N^2) - \tau U(x_N, y_N)$.

If one is able to solve for Φ_N , then the Hamiltonian H_N and the distribution function *f* are determined for a given *G*.

Nonlinear PDE for the Equilibrium Potential

Expressed in our normalized coordinates, the Poisson equation becomes:

$$\nabla_N^2 \Phi_N = -\left(\frac{\Lambda}{2\pi}\right) P_{xy}, \qquad \Lambda = \frac{(2\pi)^2\beta}{c^2}K \quad \text{where} \quad K = \frac{2I}{\beta_0^3\gamma_0^3 I_A}$$

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boundary condition (★)

Numerical solution is obtained using a **spectral Galerkin algorithm** implemented in parallel Fortran. For simplicity, we assumed a rectangular domain Ω.

- The code produces: 1) 2D Fourier coefficients of the space charge potential, 2) the potential and beam density on a 2D grid in coordinates *x-y*, 3) the difference between left and right-hand sides of (★) on the same grid, and 4) a sampled 4D equilibrium particle distribution.

- **Numerical Tests Using the IOTA Nonlinear Potential**

Numerical Example: Tracking of an Equilibrium Beam in an IOTA Constant Focusing Channel

Physical parameters:

$$G(h) \propto \exp(-h/H_0)$$

Beam energy: 2.5 MeV protons

Thermal beam with <*H*> = 0.125 (norm. emittances $\varepsilon_{x,n}$ = 0.4 μm, $\varepsilon_{y,n}$ = 0.8 μm)

Constant focusing nonlinear insert: τ = -0.4, *c* = 0.01 $m^{1/2}$, *L* = 1.8 m

Twiss beta: 1.27 m (Based on the IOTA ring circumference and tune.)

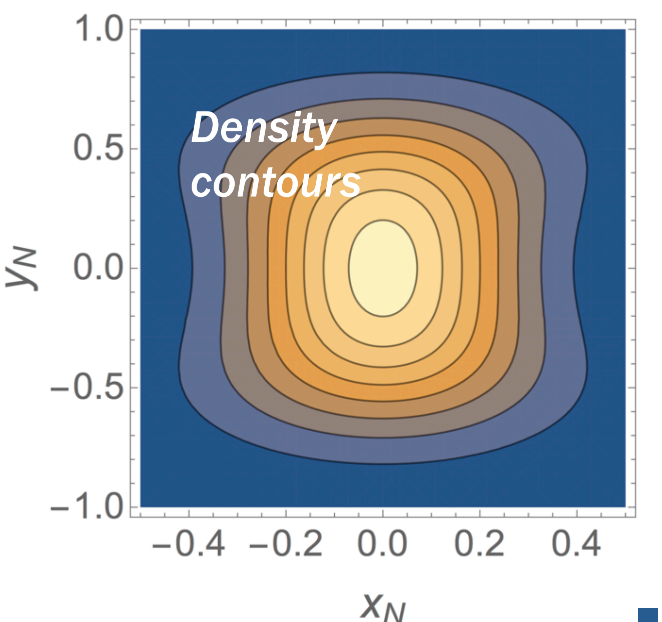
Numerical parameters:

1M particles, with 1K numerical steps per 1.8 m

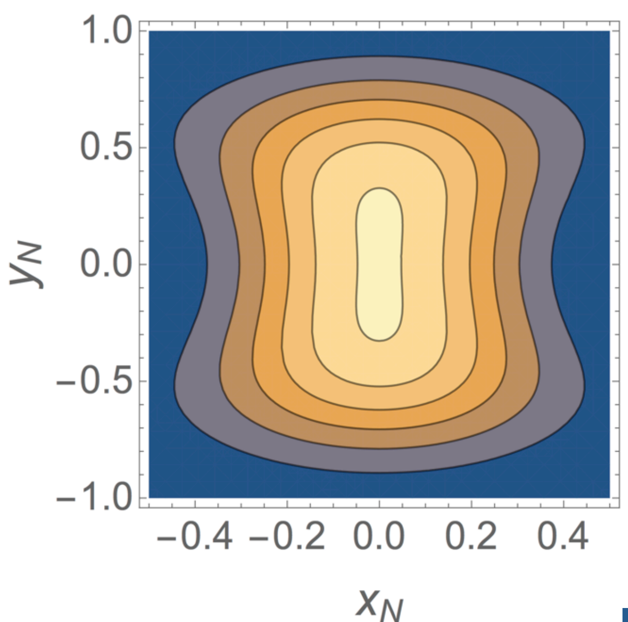
symplectic spectral space charge solver, 128x128 modes

rectangular domain w/ a = b = 3.39 cm

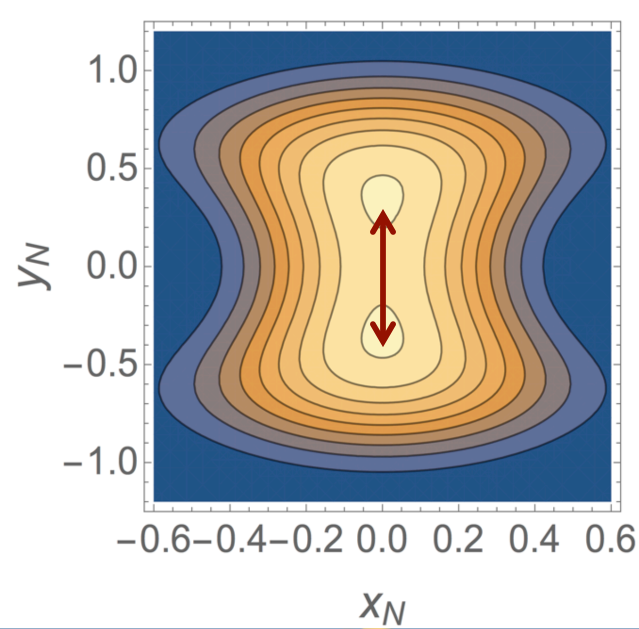
Zero current (Λ=0)



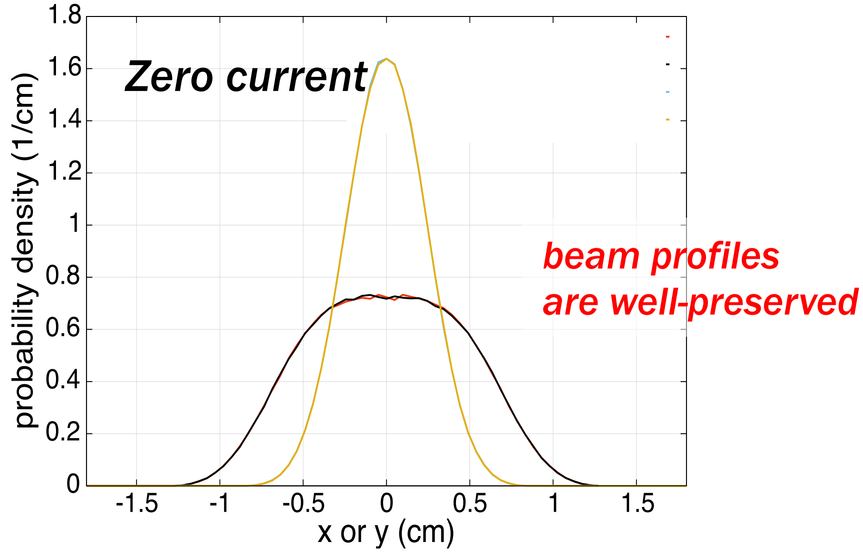
60.7 mA current (Λ=5)

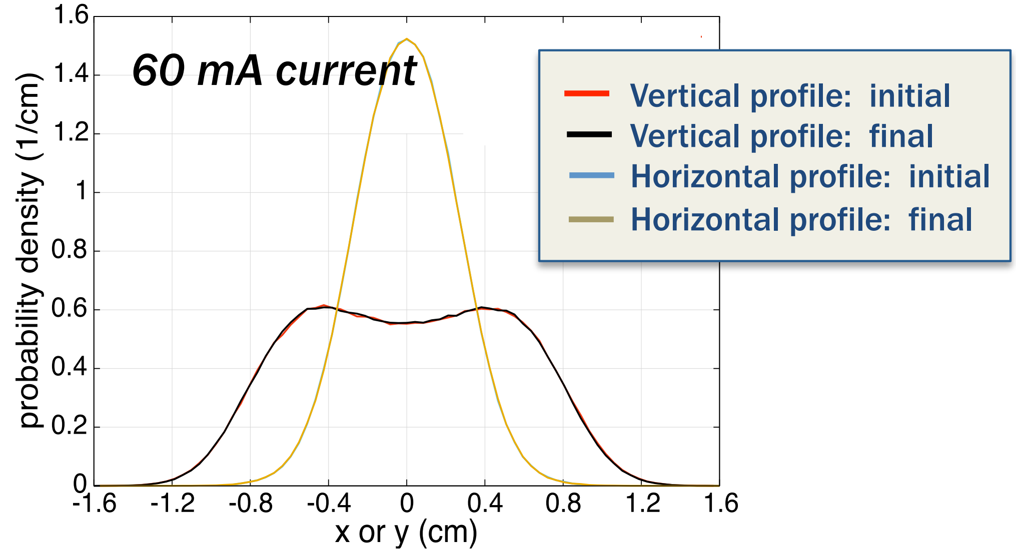


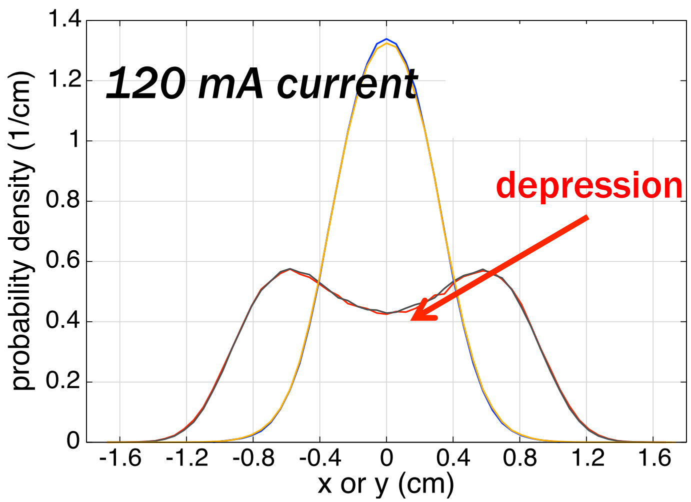
121.4 mA current (Λ=10)



Tracking an Equilibrium Beam in an IOTA Constant Focusing Channel: Preservation of the Beam Distribution







Properties of beam equilibria with increasing current:

- increase in vertical beam size
- depression of the density in the beam core

After 22 betatron periods of the bare lattice (180 m)

- change horizontal, vertical beam size: < 1.5, 0.7%
- change horizontal, vertical emittance: < 0.5, 0.15%

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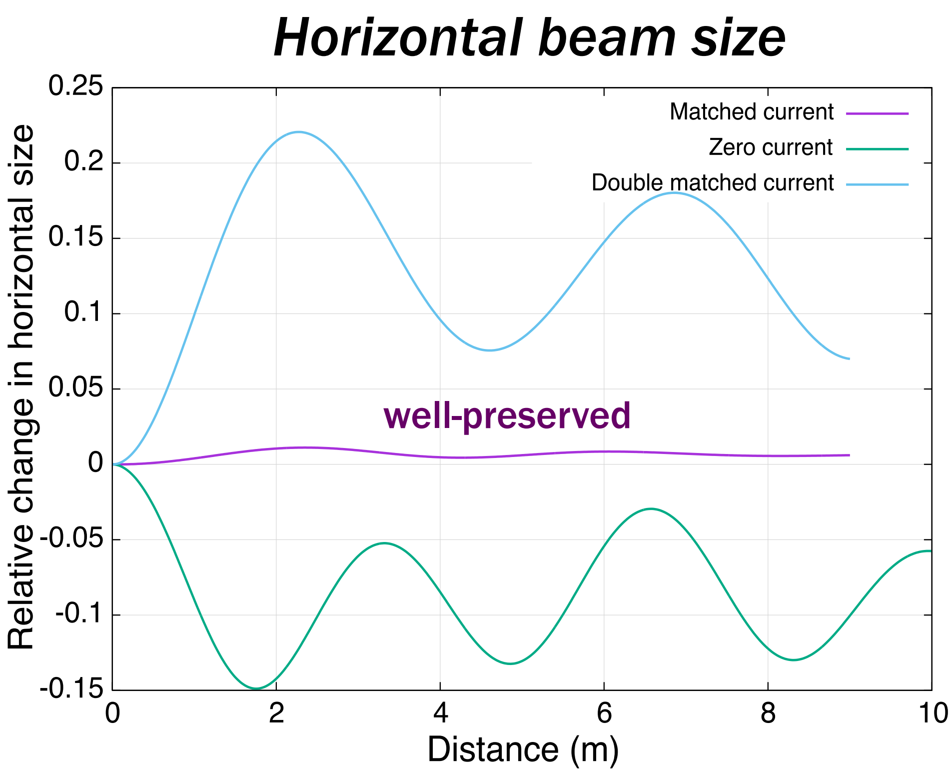
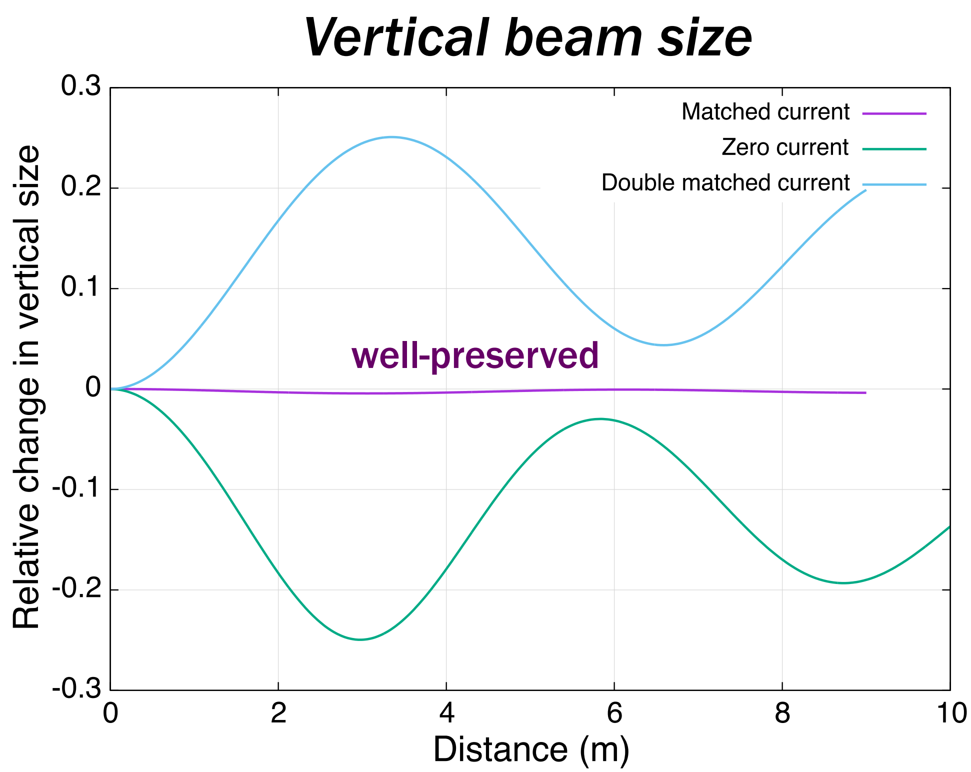
60 mA Equilibrium Beam Propagating at 3 Values of Beam Current: Evolution of RMS Beam Sizes (First 10 m)





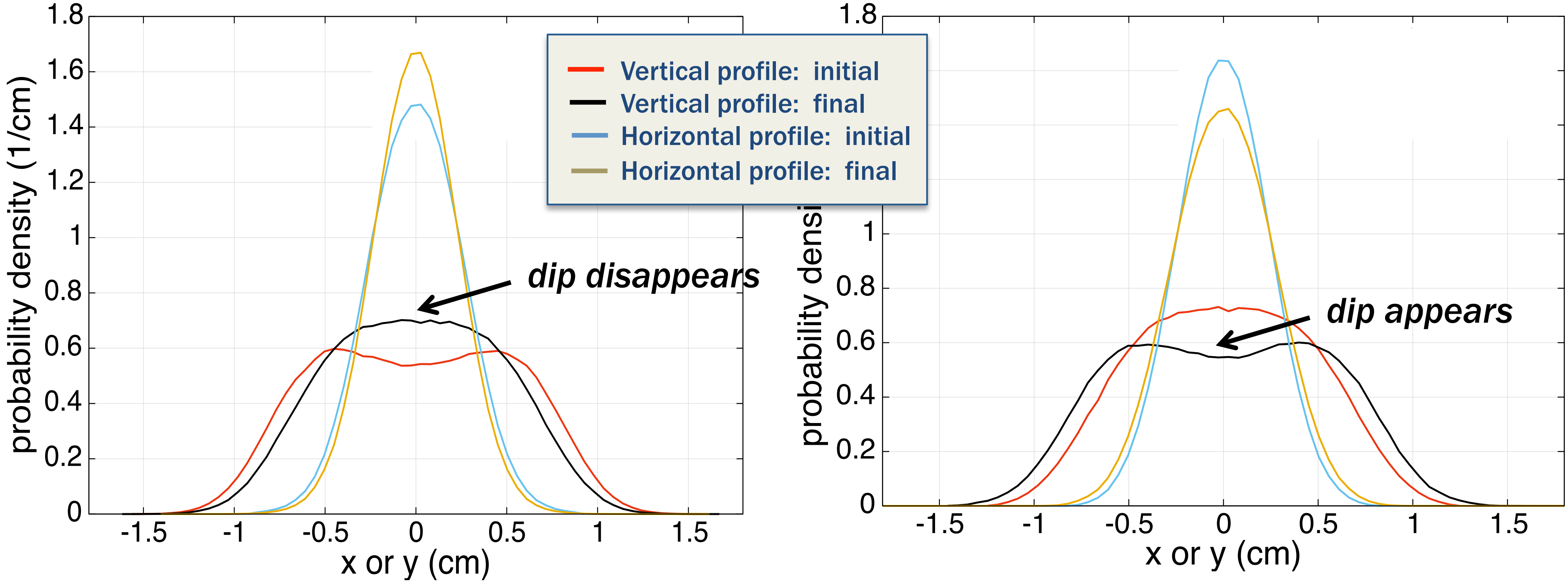
- Results are shown for a 60 mA equilibrium beam propagating at 0, 60, 120 mA current.
- Visible sensitivity to current illustrates the strength of space charge at these settings.

Tracking an Equilibrium Beam in an IOTA Constant Focusing Channel: Observing Transition to Equilibrium

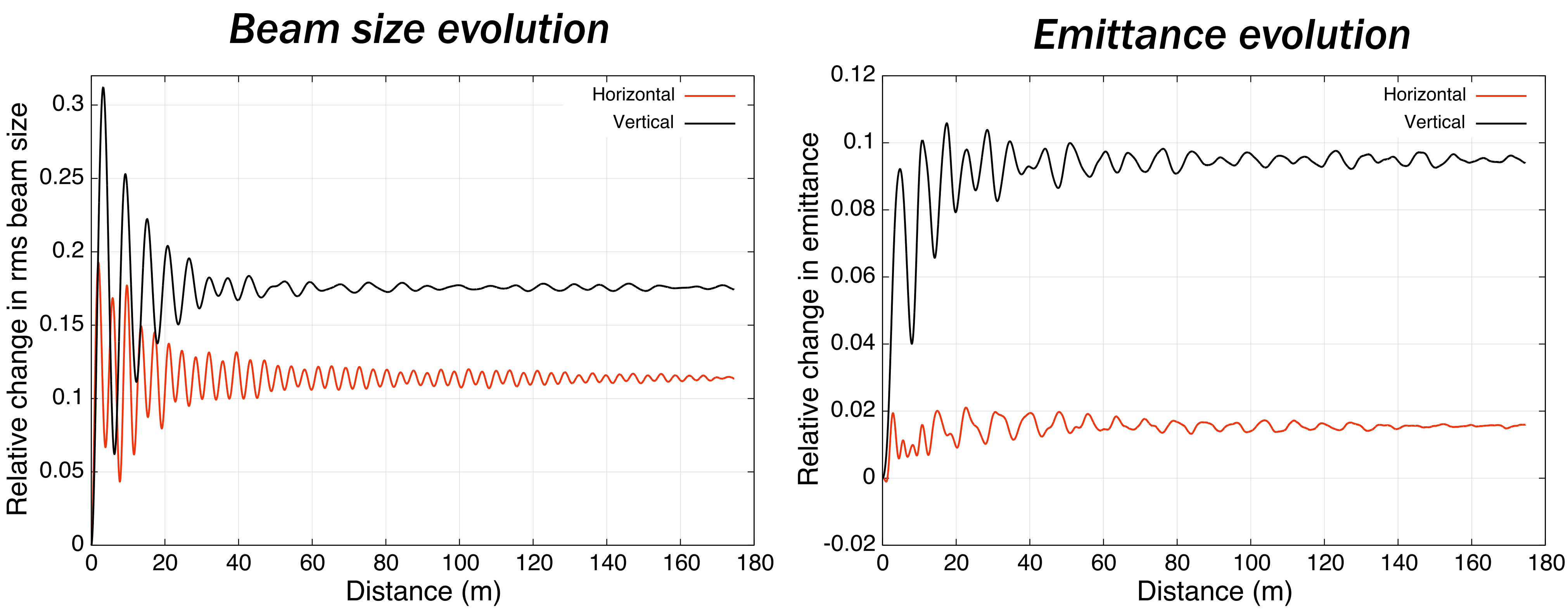
By generating an equilibrium beam at one value of current, and tracking at a different value of current, we can observe transition between the corresponding beam equilibria (here after 180 m).

60 mA equilibrium beam propagating at zero current

0 current equilibrium beam propagating at 60 mA current



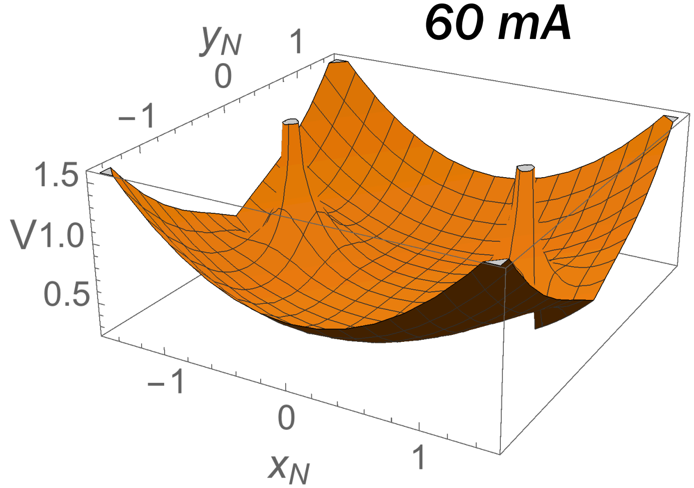
Tracking an Equilibrium Beam in an IOTA Constant Focusing Channel: Observing Transition to Equilibrium



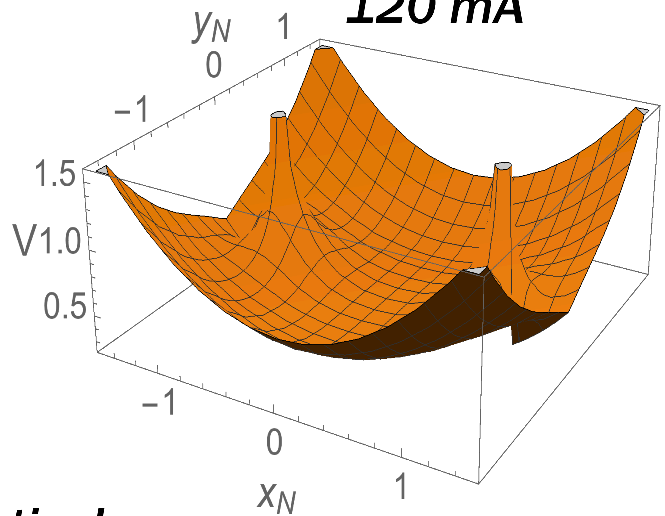
- **Results are shown for a 0 current equilibrium beam propagating at 60 mA current.**
- **The rate of approach to equilibrium is likely enhanced due to rapid filamentation caused by strong nonlinear phase mixing.**

Properties of the Total Constant Focusing Potential (Space Charge Computed Using 15x15 Spectral Modes)

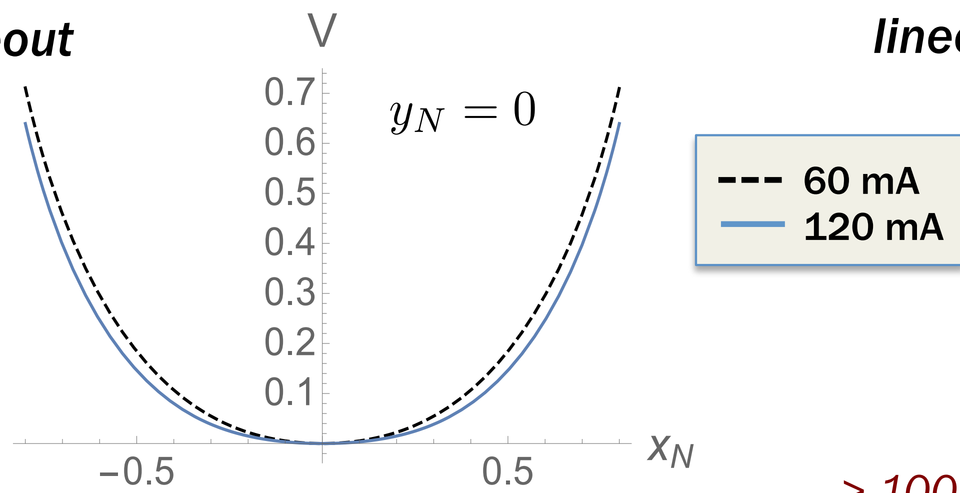
60 mA



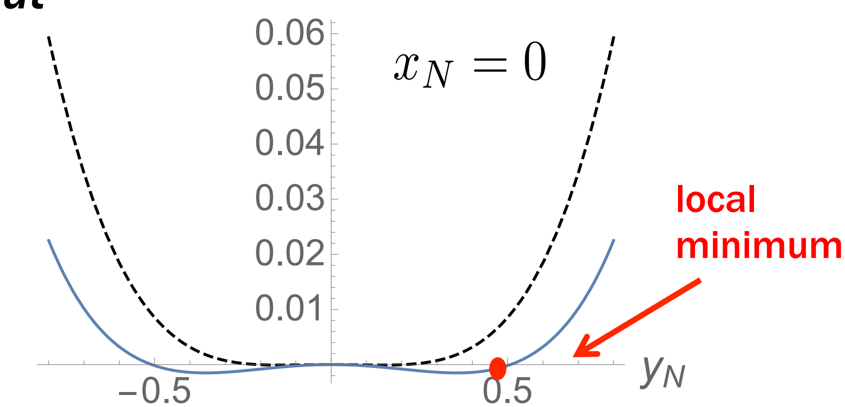
120 mA



Horizontal lineout



Vertical lineout



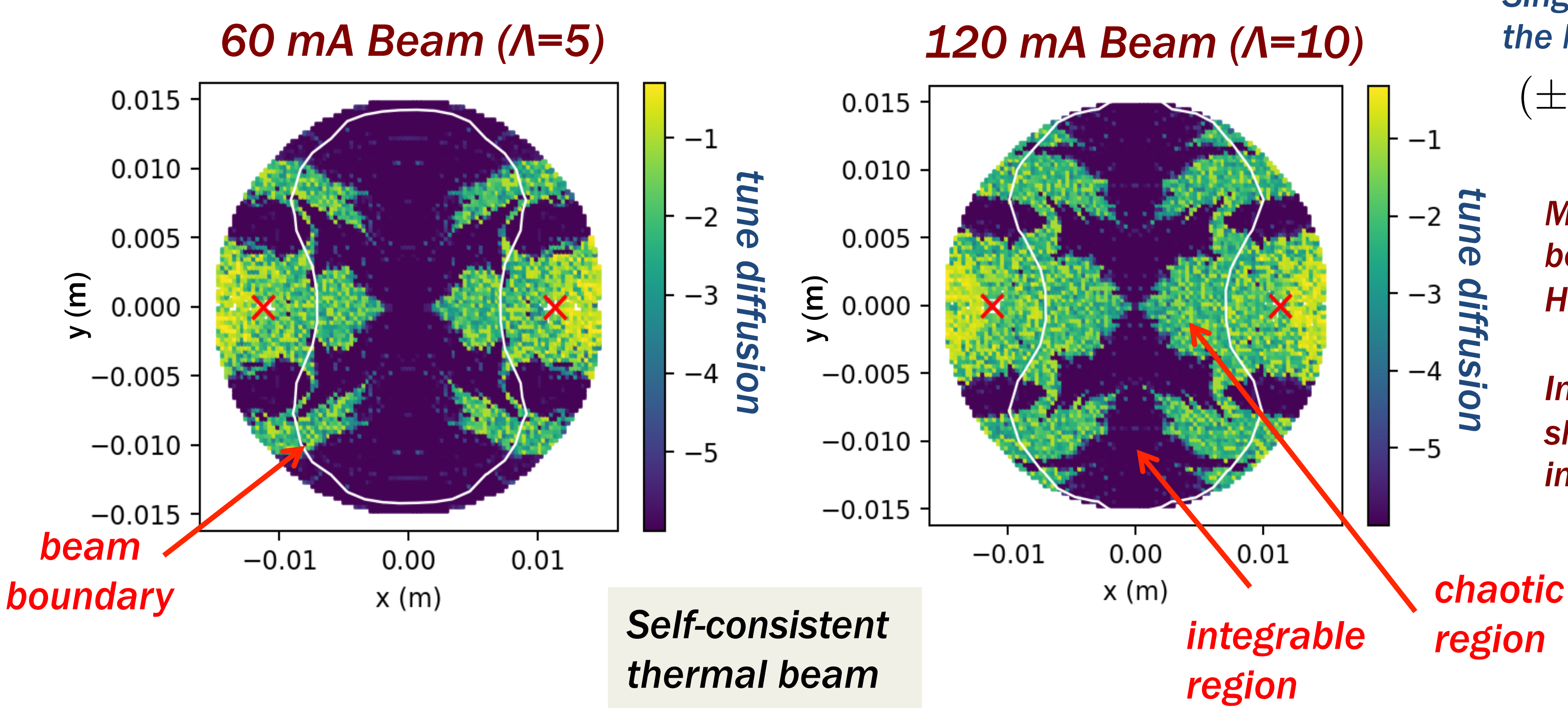
> 100% vertical tune depression near the origin

Frequency Map Analysis of Orbits in the Total Constant Focusing Potential (These beams are extreme cases, for illustration.)

- 8K distinct initial conditions (*x,0,y,0*) in a disk, at the entrance to the nonlinear insert.
- Orbits are tracked in the sum of the external potential and the equilibrium space charge potential (using 15x15 modes) for 2048 passes through the 1.8 m nonlinear constant focusing section.

Singular points of the NL potential at:

$$(\pm 1.13\ \mathrm{cm}, 0)$$



Self-consistent thermal beam

Motion is bounded due to H conservation.

Integrable region shrinks with increasing current.

- **Matching in a Nonlinear Periodic Channel**

Comments on Periodic Equilibria in s-Dependent Lattices

For a lattice of period *L*, we would like a self-consistent distribution function *f* satisfying:

$$f(x, p_x, y, p_y, s) = f(x, p_x, y, p_y, s + L) \qquad (\star)$$

Even without space charge, such periodic equilibria need not exist unless the one-turn map possesses an invariant of motion. Can we approximately satisfy (★)?

- In the limit of zero current using the IOTA integrable or quasi-integrable (octupole) optics design, exactly matched solutions exist (provided the dynamics external to the nonlinear insert is treated as linear).[1]

- In the limit of a purely linear lattice with a KV beam, an exactly matched solution exists.

- In the limit of a purely linear axisymmetric lattice with a non-KV beam, near-equilibria can be constructed by combining the rms envelope equations with the use of constant-focusing equilibria.[2]

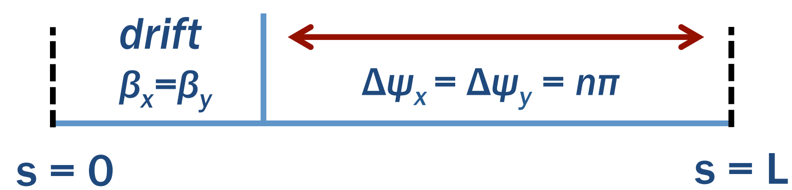
We would like an approximate matching procedure that allows both nonlinear optics and space charge, and reduces to these special cases.

[1]S. Webb, WEPPR012, IPAC2012 (2012).
[2]J. Struckmeier and I. Hofmann, Particle Accelerators 39, 219 (1992). Also R. D. Ryne, Los Alamos technical note.

Proposed Procedure for Matching to a Periodic Nonlinear Integrable Lattice

Input: lattice, current, rms emittances

Periodic lattice[1]



matched Twiss functions from rms envelope equations w/SC

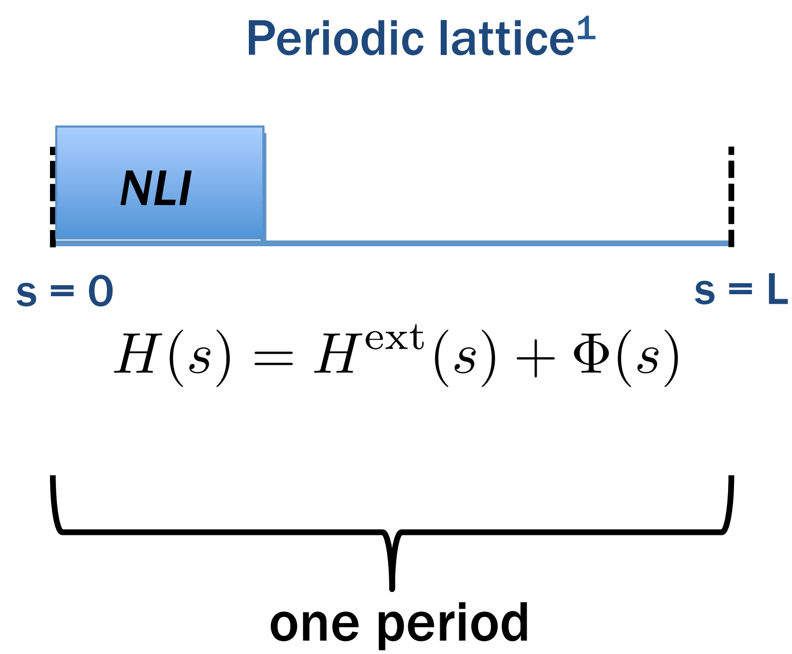
$$H(s) = H^{\text{ext}}(s) + \Phi_{KV}(s)$$

one period

[1]A. Romanov *et al*, THPOA23, NAPAC2016

Proposed Procedure for Matching to a Periodic Nonlinear Integrable Lattice

Input: lattice, current, rms emittances



$$H(s) = H^{\text{ext}}(s) + \Phi(s)$$

nonlinear insert follows the Twiss functions of the bare lattice after rematching for SC

Proposed Procedure for Matching to a Periodic Nonlinear Integrable Lattice

Input: lattice, current, rms emittances

Periodic lattice

Courant-Snyder transformation + average w/r/t betatron phase

$$H(s) = H^{\text{ext}}(s) + \Phi(s)$$

$$\begin{pmatrix} x_N \\ p_{xN} \end{pmatrix} = \begin{pmatrix} 1/c\sqrt{\beta} & 0 \\ \alpha/c\sqrt{\beta} & \sqrt{\beta}/c \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

phase advance new independent variable

Proposed Procedure for Matching to a Periodic Nonlinear Integrable Lattice

Input: lattice, current, rms emittances

Periodic lattice

$$H(s) = H^{\text{ext}}(s) + \Phi(s)$$

Courant-Snyder transformation
+
average w/r/t betatron phase

"Equivalent" constant focusing lattice

$$\langle H_N \rangle = \langle H_N^{\text{ext}} \rangle + \langle \Phi_N \rangle$$

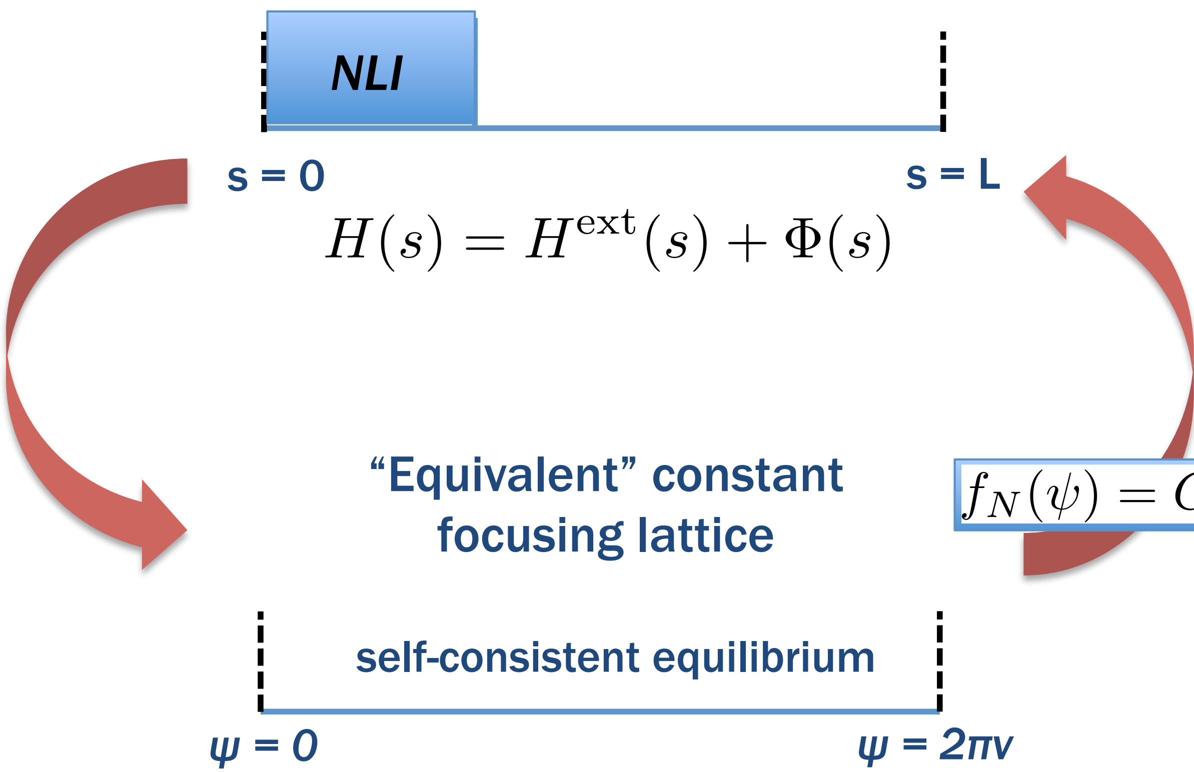
solve the nonlinear PDE for the space charge potential of a Vlasov equilibrium beam

Proposed Procedure for Matching to a Periodic Nonlinear Integrable Lattice

Input: lattice, current, rms emittances

Output: nearly-matched beam at $s = 0$

Periodic lattice



$$H(s) = H^{\rm ext}(s) + \Phi(s)$$

Courant-Snyder transformation + average w/r/t betatron phase

Courant-Snyder transformation^{-1} + use equilibrium SC potential

$$f_N(\psi) = G(H_N^{\rm ext}(\psi) + \langle \Phi_N \rangle)$$

"Equivalent" constant focusing lattice

$$\langle H_N \rangle = \langle H_N^{\rm ext} \rangle + \langle \Phi_N \rangle$$

Conclusions

- A new PDE solver was applied to study Vlasov equilibria in a nonlinear channel constructed from the IOTA nonlinear insert potential. Numerical tests verify that the resulting beam equilibria are indeed stationary. Transition from non-equilibrium to equilibrium was investigated.

- Nonlinear self-consistent beam equilibria at high intensity exhibit unusual features, including a bimodal vertical beam profile and an "hourglass" contour in the x-y plane.

- In general, the dynamics at high current reveals complex regions of integrable and bounded chaotic motion, with the size of the integrable region decreasing as current is increased.

- Suggested a procedure to use rms envelope equations and constant focusing equilibria to improve a procedure for matching with space charge to the IOTA ring (tests in progress).

- **Backup Material**

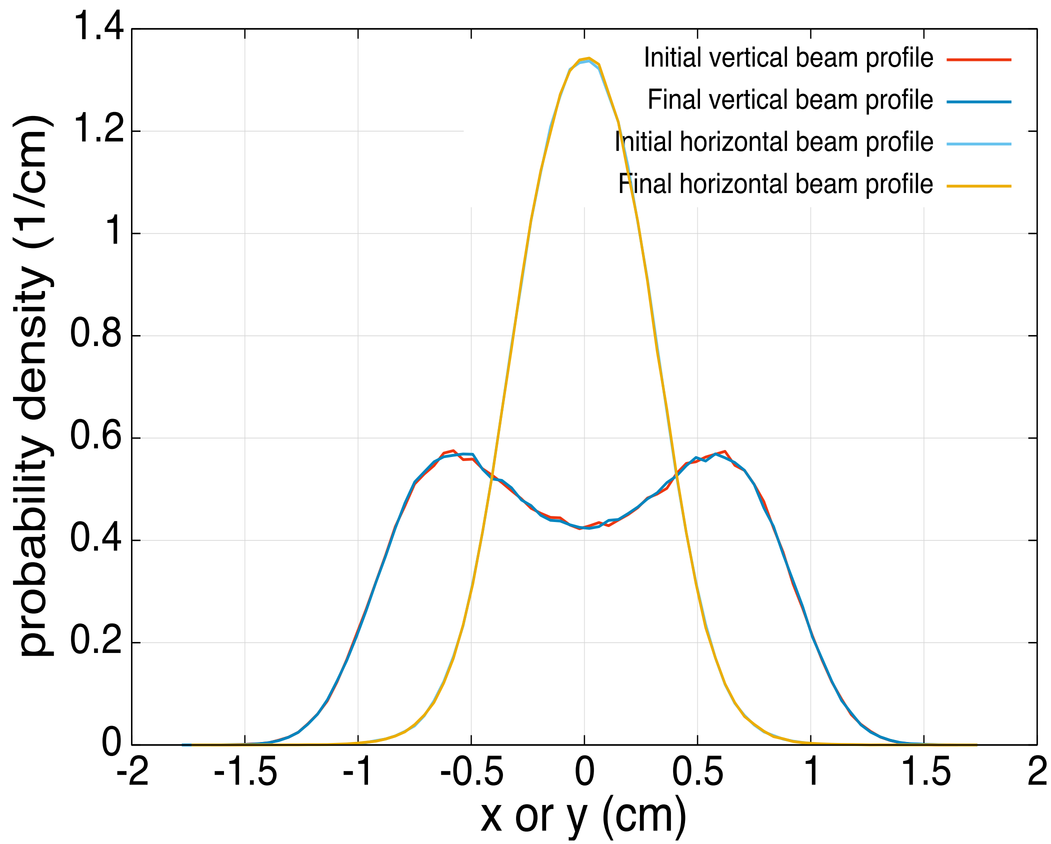
120 mA Equilibrium Beam Propagating in the Total Constant Focusing Potential: Preservation of Spatial Beam Profile, 74 m

The same thermal equilibrium beam: 120 mA, 1M particles, was used as in the prior tracking study.

The spectral Garlerkin PDE solver produces as output a set of 15x15 Fourier coefficients for the equilibrium space charge potential.

Instead of tracking particles using the symplectic spectral space charge solver, particles are tracked using the potential reconstructed from these Fourier coefficients.

(No space charge solver is used.)



Proposed Nonlinear Integrable Lattice Matching Procedure

1. Choose desired values of the beam emittances and current (perveance K).
2. Use the rms envelope equations in the bare lattice to find a set of matched envelopes and the corresponding Twiss functions and phase advances over one period.
3. Tune the bare lattice settings to produce $n\pi$ phase advance across the arc (from NLI exit to entrance) and to match the design Twiss parameters at the NLI entrance and exit.
4. Transform the Hamiltonian of the physical lattice into C-S normalized coordinates associated with the bare lattice Twiss functions, yielding a Hamiltonian H_N.
5. Average H_N over the bare betatron phase, to yield a constant-focusing Hamiltonian.
6. Solve the PDE for the equilibrium space charge potential using the Hamiltonian H_N.
7. Use the equilibrium space charge potential in the original (non-averaged) H_N.
8. Generate a distribution f_N of the desired current and emittances by taking the desired function G of H_N.
9. Transform the sampled particles from the distribution f_N from normalized coordinates to physical coordinates using the bare lattice Twiss functions at the lattice location of interest.