Eigenvalues & eigenstates of the many-body collective neutrino oscillation problem

Michael J. Cervia

Department of Physics,
University of Wisconsin-Madison

Thursday, October 10, 2019

References

Basis for this talk:


Amol V. Patwardhan, Michael J. Cervia, and A. B. Balantekin
Phys. Rev. D 99, 123013 (2019), arXiv:1905.04386


Michael J. Cervia, Amol V. Patwardhan, A. B. Balantekin, S. N. Coppersmith, and Calvin W. Johnson
Phys. Rev. D 100, 083001 (2019), arXiv:1908.03511


Michael J. Cervia, Amol V. Patwardhan, and A. B. Balantekin
IJMPE 28 (2019) 1950032, arXiv:1905.00082

Background Reading
Bedtime Reading


Ermal Rrapaj
arXiv:1905.13335


Savas Birol, Y. Pehlivan, A. B. Balantekin, and T. Kajino
Phys. Rev. D 98, 083002 (2018), arXiv:1805.11767


Y. Pehlivan, A. B. Balantekin, Toshitaka Kajino, and Takashi Yoshida
Phys. Rev. D 84, 065008 (2011), arXiv:1105.1182


Alexandre Faribault, Omar El Araby, Christoph Sträter, and Vladimir Gritsev
Phys. Rev. B 83, 235124 (2011), arXiv:1103.0472


Pieter W. Claeys
Ph.D. thesis, Ghent University (2018), arxiv:1809.04447

Neutrino Flavor/Mass Isospin

SU(2) Notation

- Fermionic ops for ν flavor and mass states $a_\alpha(\mathbf{p})$ and $a_j(\mathbf{p})$, respectively, for $\alpha = e, x$ and $j = 1, 2$ with mixing angle θ

$$\begin{pmatrix} a_e(\mathbf{p}) \\ a_x(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1(\mathbf{p}) \\ a_2(\mathbf{p}) \end{pmatrix}$$

- Introduce mass-basis isospin ops

$$J^+_{\mathbf{p}} = a^\dagger_1(\mathbf{p}) a_2(\mathbf{p}), \qquad \text{mass } 1 : |\nu_1\rangle \longleftrightarrow |\uparrow\rangle$$
$$J^-_{\mathbf{p}} = a^\dagger_2(\mathbf{p}) a_1(\mathbf{p}), \qquad \text{mass } 2 : |\nu_2\rangle \longleftrightarrow |\downarrow\rangle$$
$$J^z_{\mathbf{p}} = \frac{1}{2}[a^\dagger_1(\mathbf{p}) a_1(\mathbf{p}) - a^\dagger_2(\mathbf{p}) a_2(\mathbf{p})]$$

obeying the usual SU(2) commutation relations

$$[J^+_{\mathbf{p}}, J^-_{\mathbf{p}}] = 2\delta_{\mathbf{pq}} J^z_{\mathbf{p}}, \quad [J^z_{\mathbf{p}}, J^\pm_{\mathbf{q}}] = \pm\delta_{\mathbf{pq}} J^\pm_{\mathbf{p}}.$$

Neutrino Many-body Hamiltonian

In Terms of Isospin

- Vacuum Oscillations

$$H_{\text{vac}} = \sum_{\mathbf{p}} \sum_{i=1}^{2} \frac{m_i^2}{2|\mathbf{p}|} a_i^\dagger(\mathbf{p}) a_i(\mathbf{p})$$

$$= \sum_{\omega} \omega \vec{B} \cdot \vec{J}_\omega + \text{const}, \quad \vec{J}_\omega = \sum_{|\mathbf{p}|=\frac{\delta m^2}{2\omega}} \vec{J}_{\mathbf{p}}$$

with $\omega = \dfrac{\delta m^2}{2|\mathbf{p}|}$ and $\vec{B} = (0, 0, -1)_m = (\sin 2\theta, 0, -\cos 2\theta)_f$

- Neutrino-Neutrino Interaction

$$H_{\nu\nu} = \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \sum_{g,h=e,x} a_g^\dagger(\mathbf{p}) a_h(\mathbf{p}) a_h^\dagger(\mathbf{q}) a_g(\mathbf{q})$$

$$= \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} + \text{const}$$

Neutrino Hamiltonian: Single-angle Approximation

In Terms of Isospin

- Make the problem more tractable by averaging over $\theta_{\mathbf{pq}}$;

$$H_{\nu\nu} \approx \frac{\sqrt{2}G_F}{V} \langle 1 - \cos\vartheta_{\mathbf{pq}} \rangle \sum_{\mathbf{pq}} \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$
$$= \mu(r) \vec{J} \cdot \vec{J}, \quad \text{where} \quad \vec{J} = \sum_{\omega} \vec{J}_{\omega}$$

- In summary, Hamiltonian for (vacuum + collective) oscillations

$$H = H_{\text{vac}} + H_{\nu\nu} \approx \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu(r) \vec{J} \cdot \vec{J}$$

Mean-field Treatment
Random-phase Approximation

- Obtain effective one-body Hamiltonian via RPA:

$$H \sim H^{\mathrm{RPA}} = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \vec{P} \cdot \vec{J}$$

  where $\vec{P} = \sum_{\omega} \vec{P}_{\omega}$, and $\vec{P}_{\omega} = 2 \langle \vec{J}_{\omega} \rangle$ is the "polarization" vector
- Self-consistency requirement of mean-field theory implies

$$\frac{d}{dt} \vec{P}_{\omega} = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_{\omega}$$

  for each ω.
- Additional interpretation: $\vec{P}_{\omega}$ is the Bloch vector of that neutrino's density matrix in mass/flavor basis. Here, $|\vec{P}_{\omega}| = 1$.

Many-body Eigenstates, Limiting Cases

$\mu \to 0, \infty$

General: Many-body neutrino Hamiltonian (two-flavors, single-angle)

$$H = \sum_{p=1}^{M} \omega_p \vec{B} \cdot \vec{J}_p + \mu \vec{J} \cdot \vec{J}$$

where p is an index for the M different ω values

- $\mu \to 0$: Eigenstates are simple tensor products of single-particle vacuum mass eigenstates $|\nu_{i_1} \cdots \nu_{i_N}\rangle$, $i_j = 1, 2$, with eigenvalues $-\frac{1}{2}\sum_{p=1}^{M} \omega_p (n_{1,p} - n_{2,p})$

$$\text{e.g.)} \quad N = 2: \quad |\nu_1\nu_1\rangle\,, \quad |\nu_1\nu_2\rangle\,, \quad |\nu_2\nu_1\rangle\,, \quad |\nu_2\nu_2\rangle$$

- $\mu \to \infty$: Eigenstates are the total isospin states $|j, m\rangle_{f/m}$ with eigenvalues $\mu j(j+1)$

Many-body Eigenstates, Generic μ

The Richardson-Gaudin Approach

- For $0 \leq \mu \leq \infty$, the two extremal states $|\nu_1 \cdots \nu_1\rangle = |\frac{N}{2}, +\frac{N}{2}\rangle$ and $|\nu_2 \cdots \nu_2\rangle = |\frac{N}{2}, -\frac{N}{2}\rangle$ are eigenstates with eigenvalues

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{n_p}{2} + \mu \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

  where n_p is the number of neutrinos at ω_p and $N = \sum_p n_p$

- Construct the remaining $2^N - 2$ eigenstates from the extremal states by hitting with the Gaudin operators

$$S^{\pm}(\zeta_\alpha) = \sum_{p=1}^{M} \frac{J_p^{\pm}}{\omega_p - \zeta_\alpha}$$

  where $\{\zeta_\alpha\}$ is a set of parameters yet to be determined.

Bethe Ansatz Equations

Determining those parameters

- For one parameter, $S^-(\zeta)\,|\frac{N}{2}, +\frac{N}{2}\rangle$ is an eigenstate with eigenvalue $E_{+N/2} + \zeta - \mu N$

$$\text{if} \quad \frac{1}{2\mu} + \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta} = 0,$$

  where $j_p = n_p/2$.

- For multiple parameters, $S^-(\zeta_1)\cdots S^-(\zeta_\kappa)\,|\frac{N}{2}, +\frac{N}{2}\rangle$ is an eigenstate with eigenvalue $E_{+N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \mu\kappa(N - \kappa + 1)$

$$\text{if} \quad \frac{1}{2\mu} + \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = -\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}.$$

Solving the Bethe Ansatz Equations

Some remarks...

$$\frac{1}{2\mu} + \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = -\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}.$$

- For κ parameters, solve a system of κ coupled algebraic equations in κ *complex* variables.
- There are many singularities, which can be avoided by converting to κ coupled, deg-$(M + \kappa - 2)$ polynomial equations
- For complete set of eigenstates, need to solve a different set of equations for each $0 < \kappa < M$.

We can avoid these complexities of the problem...

The 'Lambda' Approach

An Alternative/Reduction of Bethe Ansatz

- Let $P(\lambda) = \prod_\alpha (\lambda - \zeta_\alpha)$, the characteristic polynomial of BA eq. Define unitless log derivative, $\tilde{\Lambda}(\lambda) = \mu \frac{d}{d\lambda}(\log P(\lambda)) = \sum_\alpha \frac{\mu}{\lambda - \zeta_\alpha}$, to convert κ algebraic equations into *one* ODE for Λ:

$$\tilde{\Lambda}^2(\lambda) + \tilde{\Lambda}(\lambda) + \mu\tilde{\Lambda}'(\lambda) = \mu \sum_{q=1}^{M} 2j_q \frac{\tilde{\Lambda}(\lambda) - \tilde{\Lambda}(\omega_q)}{\lambda - \omega_q}$$

Boundary condition is quite complicated...

- Taking $\lambda \to \omega_p$ and $j_p = 1/2$ (interpretation: bin with ω_ps such that $n_p = 1$), we can look for values of $\tilde{\Lambda}_p \equiv \tilde{\Lambda}(\omega_p)$:

$$\tilde{\Lambda}_p^2 + \tilde{\Lambda}_p = \mu \sum_{\substack{q=1 \\ q \neq p}}^{M} \frac{\tilde{\Lambda}_p - \tilde{\Lambda}_q}{\omega_p - \omega_q}$$

Solving Lambda Equations
Homotopy Continuation

$$\tilde{\Lambda}_p^2 + \tilde{\Lambda}_p = \mu \sum_{\substack{q=1 \\ q \neq p}}^{M} \frac{\tilde{\Lambda}_p - \tilde{\Lambda}_q}{\omega_p - \omega_q}$$

- $\mu \to 0$: each $\tilde{\Lambda}_p = 0, -1$. Thus, 2^N in solutions total. In each soln, the number of $\tilde{\Lambda}_p$ that are -1 is the κ of the solution.
  - Even when $\mu > 0$, $\sum_p \tilde{\Lambda}_p = -\kappa$.
- Homotopy Method: each of 2^N solns for $\mu > 0$ can be constructed incrementally, starting from $\mu = 0$.
  - Step 1: Use solns for each $\tilde{\Lambda}_p|_{\mu=\mu_n}$ as starting guess for solns at $\mu = \mu_n + \delta\mu$.
  - Step 2: Apply iterative numerical methods (e.g., Newton-Raphson) to improve the guess.
- Guarantee any pair of solutions, $\{\tilde{\Lambda}_p\}_1$ and $\{\tilde{\Lambda}_p\}_2$, is different at every μ.

E.g., Ten-neutrino System

A $\{\tilde{\Lambda}_p\}$ Soln

Def: $\Lambda_p \equiv \Lambda(\omega_p) = \mu^{-1}\tilde{\Lambda}_p$.

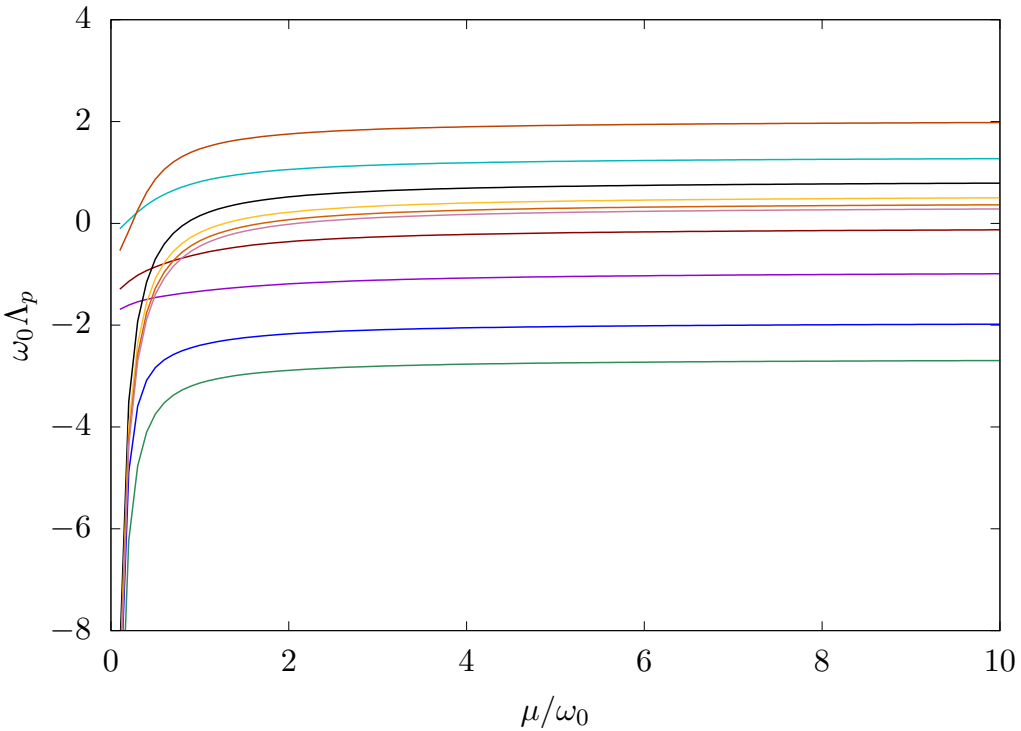


Figure: A $\kappa = 6$ soln $\{\Lambda_1, \ldots, \Lambda_{10}\}$ as a function of μ, for 10 neutrinos with frequencies $\omega_p = p\,\omega_0$. Shown: one sample soln out of $1{,}024$ for this system.

[Patwardhan, Cervia, Balantekin, Phys. Rev. D **99**, 123013 (2019).]

A Comparison of Techniques

BA v. Lambda

$$\frac{1}{2\mu} + \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}$$

$$\tilde{\Lambda}_p^2 + \tilde{\Lambda}_p = \mu \sum_{\substack{q=1 \\ q \neq p}}^{M} \frac{\tilde{\Lambda}_p - \tilde{\Lambda}_q}{\omega_p - \omega_q}$$

- For κ parameters, solve a system of κ coupled algebraic eqs. in κ *complex* variables.
- Many singularities, avoided with κ deg-$(M+\kappa-2)$ polynomial eqs.
- Solve a different set of equations for each $0 < \kappa < M$.

- For N neutrinos, solve a system of N coupled algebraic eqs. in N *real* variables.
- No singularities, already have N deg-2 polynomial eqs.
- Solve this set of equations once for all κ.

NB: Both techniques require fixing j_p for solutions

Energy Eigenvalues from Lambdas

Removing BA variables altogether, Part 1/2

- Energy eigenvalues may be rewritten in terms of $\tilde{\Lambda}_p = \sum_\alpha \frac{\mu}{\omega_p - \zeta_\alpha}$:

$$\text{Recall}: \quad E(\zeta_1, \ldots, \zeta_\kappa) = E_{N/2} - \kappa\mu(N - \kappa + 1) + \sum_\alpha \zeta_\alpha$$

$$\iff E(\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_N) = -\sum_p \frac{\omega_p}{2} + \mu\frac{N}{2}\left(\frac{N}{2} + 1\right) - \sum_p \omega_p \tilde{\Lambda}_p$$

- Instructive to categorize eigenstates/values by $\kappa = -\sum_p \tilde{\Lambda}_p$: solns are eigenstates of $J^z = \sum_p J^z_p$ with eigenvalue $m = N/2 - \kappa$. Within κ, eigenvalues split into branches for $\mu \gg \omega_0$, each associated with particular $|j, m\rangle$ with $j = |m|, |m| + 1, \ldots, N/2$.

Eigenvalues $N = 10$

$\kappa = 2$ solns

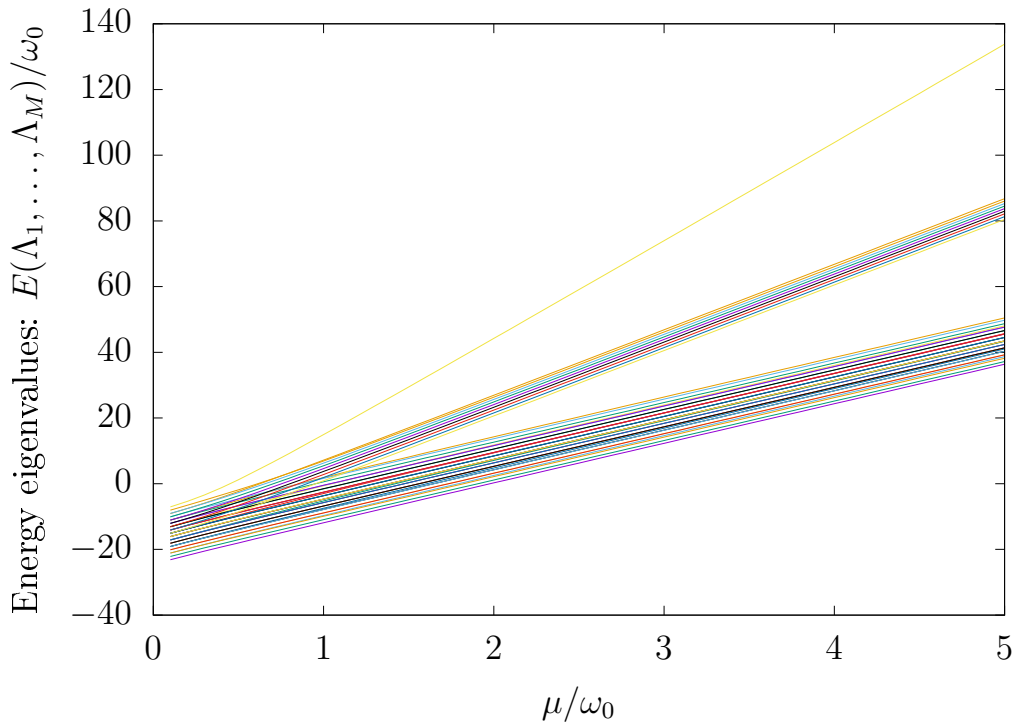


Figure: Energy eigenvalues for all $\kappa = 2$ $(m = +3)$ solns of Lambda eqs., as a function of μ, for $N = 10$.

[Patwardhan, Cervia, Balantekin, Phys. Rev. D **99** 123013 (2019).]

Eigenvalues $N = 10$

$\kappa = 5$ solns

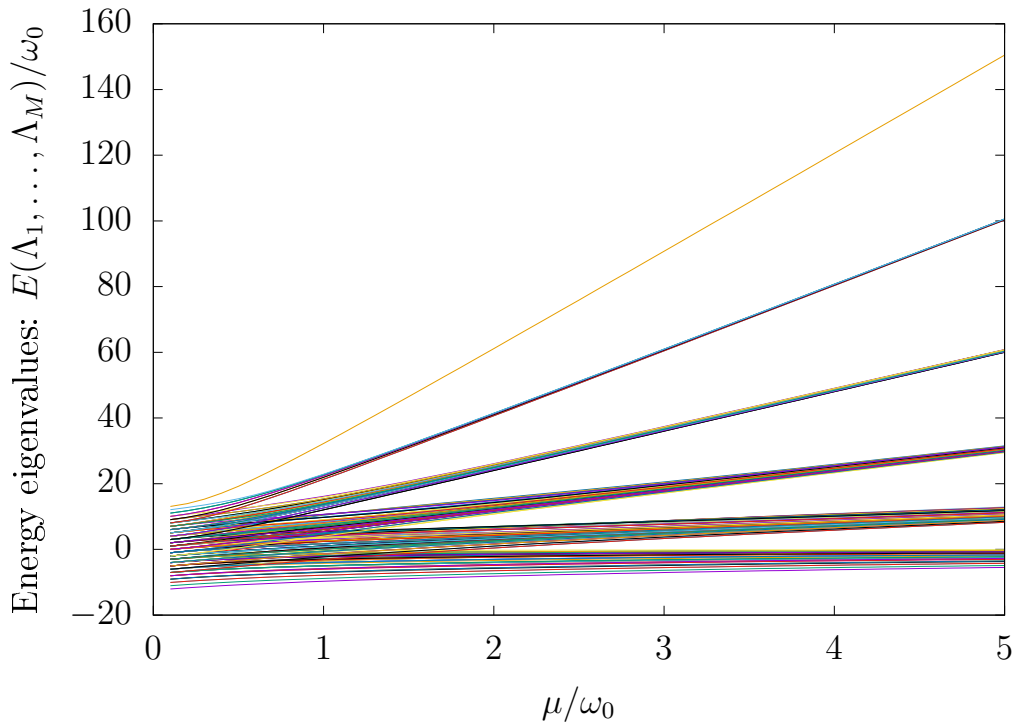


Figure: Energy eigenvalues for all $\kappa = 5$ $(m = 0)$ solns of Lambda eqs., as a function of μ, for $N = 10$.

[Patwardhan, Cervia, Balantekin, Phys. Rev. D **99** 123013 (2019).]

Energy Eigenstates from Lambdas

Removing BA variables altogether, Part 2/2

- Recognize eigenstates in terms of elementary symmetric polynomials, e_n, and power sums, P_n, of Gaudin ops $S^-_\alpha \equiv S^-(\zeta_\alpha)$:

$$\text{Recall}: \quad |\zeta_1, \ldots, \zeta_\kappa\rangle = e_\kappa(S^-_1, \ldots, S^-_\kappa)\,|\tfrac{N}{2}, +\tfrac{N}{2}\rangle$$

$$\iff |\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_N\rangle = \left[\frac{1}{\kappa}\sum_{i=1}^{\kappa}(-1)^{i-1}e_{\kappa-i}P_i\right]|\tfrac{N}{2}, +\tfrac{N}{2}\rangle$$

Via Newton's identities, calculate e_κ from power sums, given by:

$$P_n = \sum_\alpha (S^-_\alpha)^n = \sum_{i=1}^{\kappa}\sum_{p_1}\cdots\sum_{p_n}\frac{J^-_{p_1}\cdots J^-_{p_n}}{(\omega_{p_1}-\zeta_\alpha)\cdots(\omega_{p_n}-\zeta_\alpha)}$$

$$= \sum_{p_1}\cdots\sum_{p_n} J^-_{p_1}\cdots J^-_{p_n}\sum_{m=1}^{n}\Lambda_{p_m}\prod_{\substack{l=1\\ l\neq m}}^{n}(\omega_{p_l}-\omega_{p_m})^{-1}$$

Eigenstates Results
Two solns

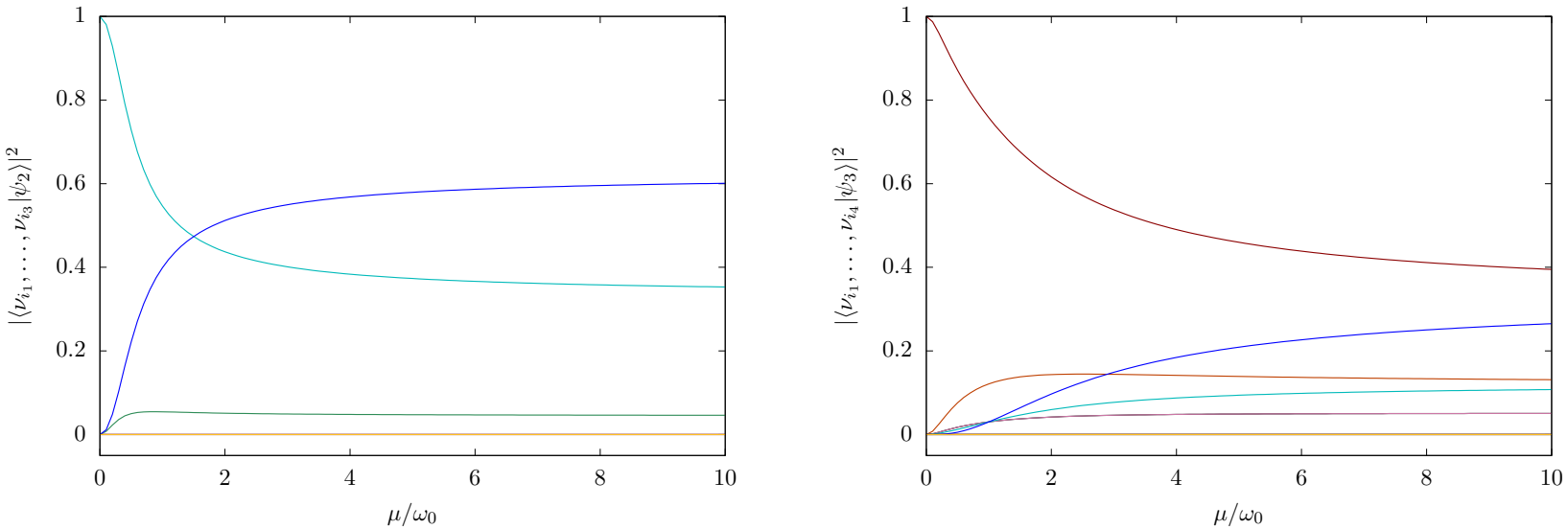


Figure: Overlaps of particular energy states ψ_n with msas basis states (with $i_1,\ldots,i_N = 1,2$), $|\langle\nu_{i_1}\cdots\nu_{i_N}|\psi_n\rangle|^2$, as functions of μ. Left: particular eigenstate with $\kappa = 1$ from $N = 3$ system. Right: particular eigenstate with $\kappa = 2$ from $N = 4$ system.

Note: for $\mu > 0$, a state with κ has ${}^{N}C_{\kappa}$ nontrivial components.

[Patwardhan, Cervia, Balantekin, Phys. Rev. D **99** 123013 (2019).]

Conclusions

- Calculations of collective neutrino flavor evolution typically employ a mean field
- Many-body calculations permit test of efficacy and limitations of mean field
- Evolution in MB can be studied by characterizing the whole spectrum of the Hamiltonian, as we have
- For certain simple systems, qualitative differences in flavor evolution can be observed between MF and MB treatments, resulting from entangled states–talk by Amol Patwardhan to follow!

Entanglement in Energy Eigenstates

A Preview, $N = 4$

Density matrix for neutrino 4 from energy state ψ_n: $\rho_4 = \mathrm{Tr}_{1,2,3}[|\psi_n\rangle\langle\psi_n|]$
Entanglement entropy between neutrino 4 and $\{1, 2, 3\}$: $S = -\mathrm{Tr}[\rho_4 \ln(\rho_4)]$

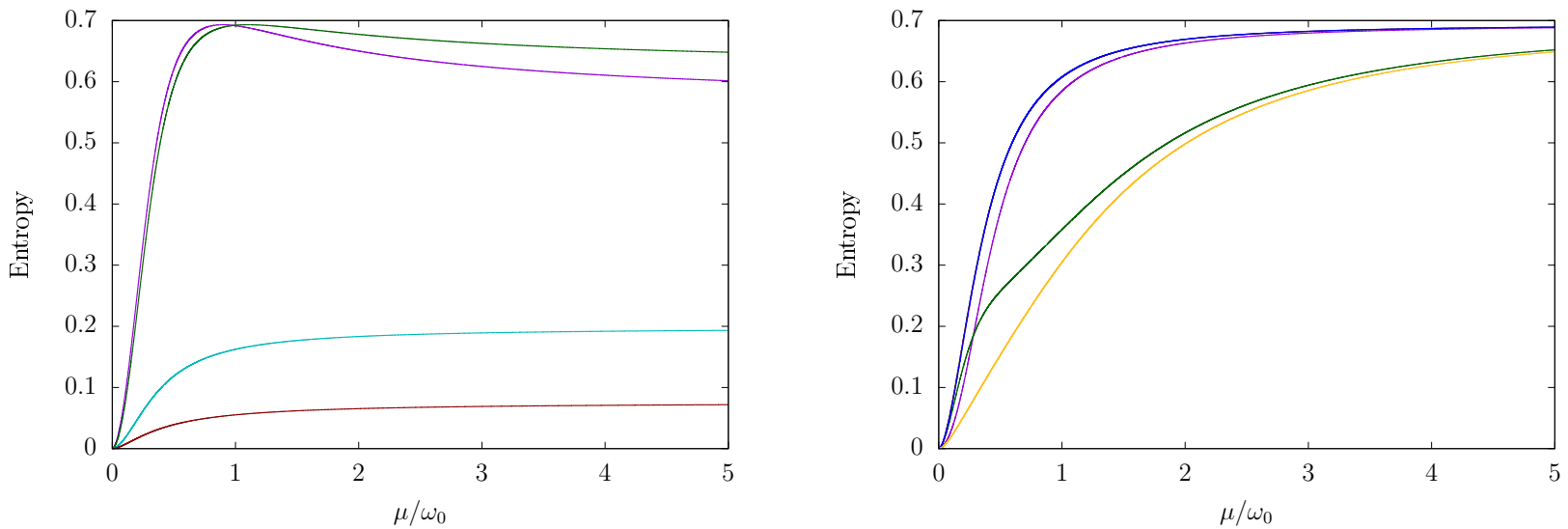


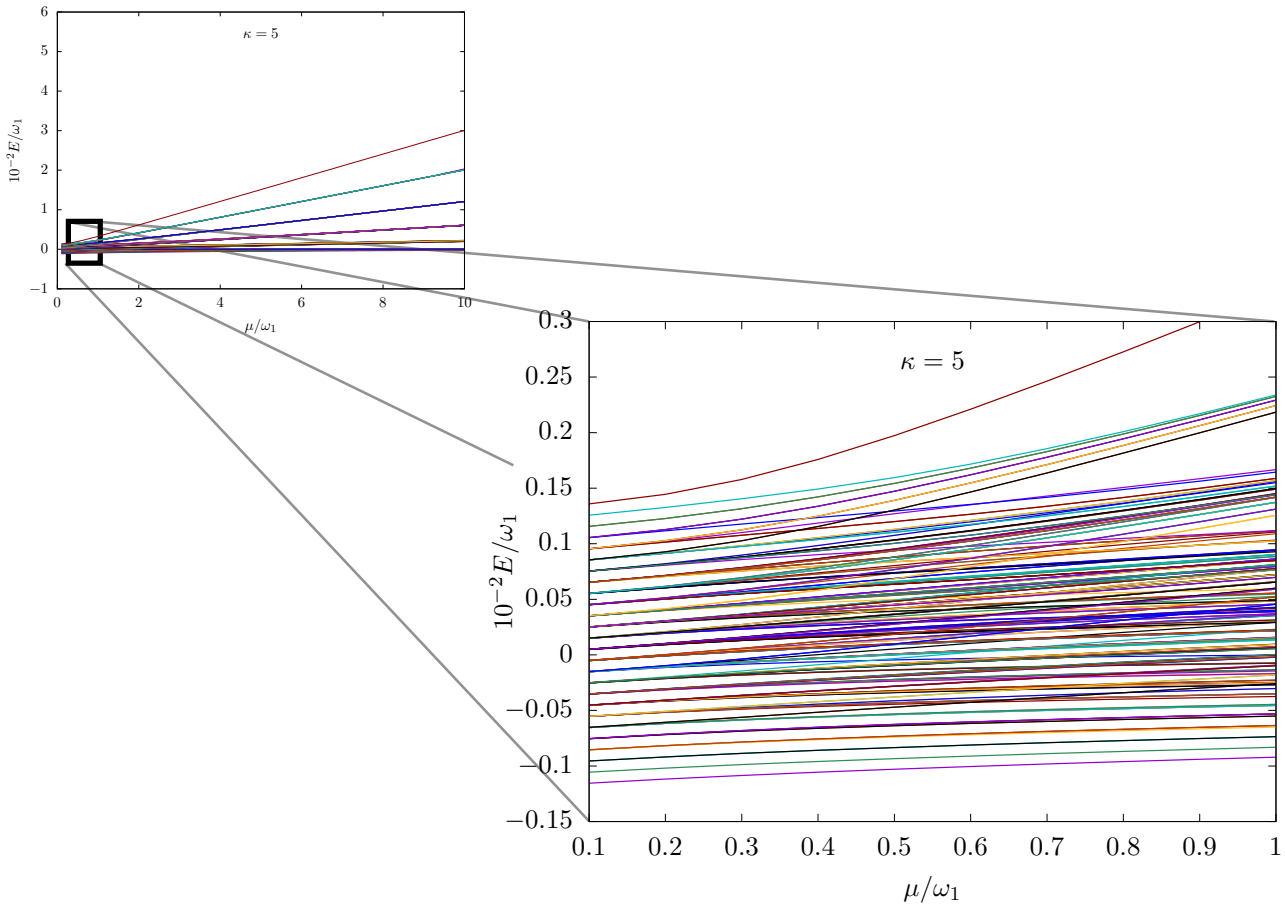
Figure: Entropy of entanglement between neutrino at frequency ω_4 and the rest of ensemble, for all $N = 4$ eigenstates with $\kappa = 1$ (left) and 2 (right).

[Cervia et al., Phys. Rev. D **100** 083001 (2019).]

Results: for $M = 10$

Energy Level Crossings

- Many level crossings, including between states of the same m
- *However*, $\Lambda_p \leftrightarrow$ eigenvalue of operator $h_p(\mu)$, $[h(\mu), H(\mu)] = 0$; the non-degeneracy of $\{\Lambda_p\}$ sets breaks these crossings.



Neutrino flavor evolution: matter effects

- Matter backgrounds (electrons, nucleons, etc.) modify flavor evolution: neutrinos acquire "effective mass" through forward scattering (like photons in medium, but via weak interactions)

- In typical environments ($T \lesssim 10$ MeV), ν_e experience charged- and neutral-current interactions, unlike ν_μ and ν_τ (only NC)

- In such a medium, ν_e acquires additional effective mass compared to ν_μ, ν_τ

$$i\frac{d}{dt}\begin{pmatrix}\psi_{\nu_e}\\ \psi_{\nu_x}\end{pmatrix} = \left[U\frac{1}{2E}\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}U^\dagger + \begin{pmatrix}V_{\rm CC} & 0\\ 0 & 0\end{pmatrix}\right]\begin{pmatrix}\psi_{\nu_e}\\ \psi_{\nu_x}\end{pmatrix}$$

where $V_{\rm CC} = \sqrt{2}G_F n_B Y_e$.

Mean-field (random phase) approximation

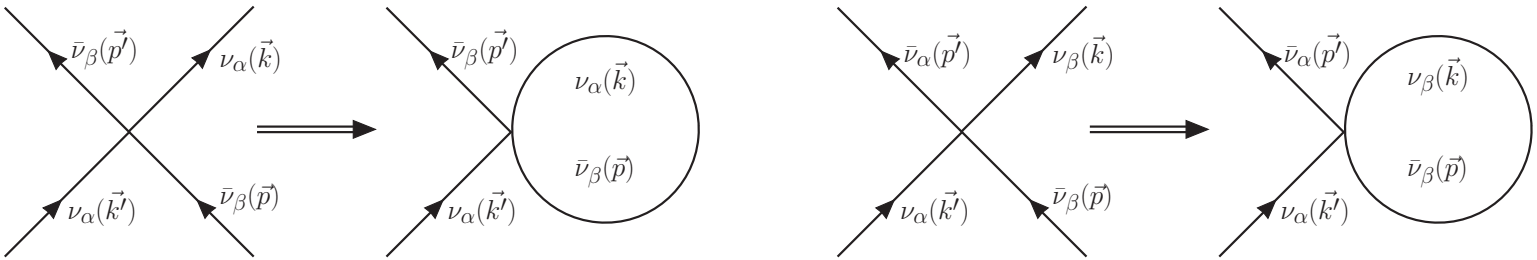


Figure: Volpe *et al.*, 2013

- In an effective one-particle approximation, a single neutrino is described as interacting with an average potential created by all other particles in the medium (including neutrinos)

- Operator product $\mathcal{O}_1\mathcal{O}_2$ approximated as

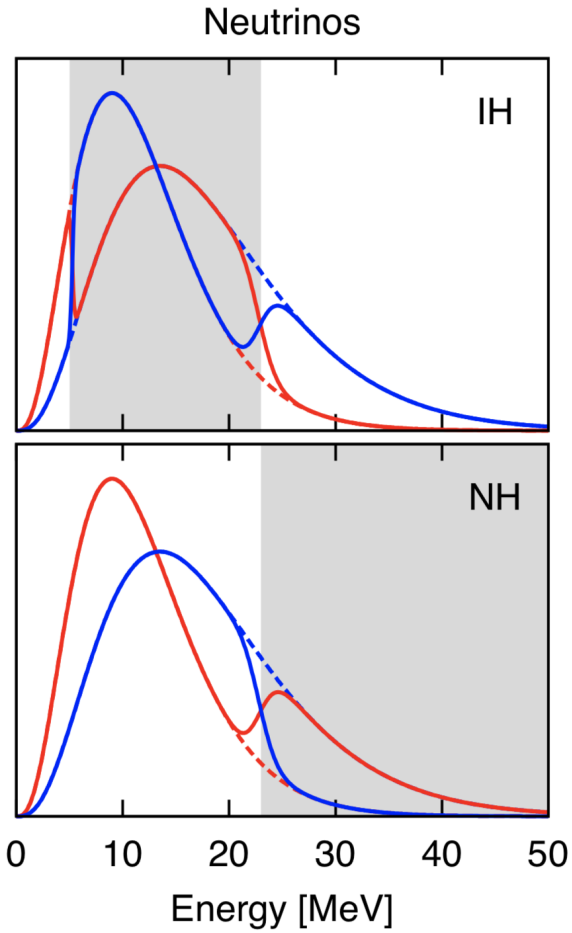
$$\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle.$$

  Above expectation values are calculated w.r.t state $|\Psi\rangle$ which satisfies $\langle\mathcal{O}_1\mathcal{O}_2\rangle = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$

Neutrino Flavor Evolution: Collective Phenomena with a Mean Field

Red: Electron flavor
Blue: Electron flavor

Top: spectral swap in inverted hierarchy. Bottom: spectral swap in normal hierarchy. (Dasgupta *et al.*, 2009).



Obtaining the roots ζ_i from the solutions $\Lambda(\omega_p)$

- One can derive a set of constraint equations involving $\{\tilde{\Lambda}_p\}$ and $\{\zeta_i\}$:

$$\sum_p j_p\,\omega_p^k\,\tilde{\Lambda}_p = -\frac{1}{2\mu}Q_k + \sum_{l=1}^{k} Q_{l-1}\left(\sum_p j_p\,\omega_p^{k-l}\right) - \frac{1}{2}\left[\sum_{l=1}^{k} Q_{k-l}\,Q_{l-1} - k\,Q_{k-1}\right],$$

  where $Q_k = \sum_i \zeta_i^k$

- In particular, the $k = 0$ constraint:

$$\sum_p j_p\,\tilde{\Lambda}_p = -\frac{\kappa}{2\mu},$$

  is useful for classifying solution sets $\{\tilde{\Lambda}_p\}$ according to κ

Obtaining the roots ζ_i from the solutions $\Lambda(\omega_p)$

- Using the previous recursion relation, all Q_k's can be calculated one-by-one, once all the $\tilde{\Lambda}_p$ are known
- The power sums Q_k are related to the elementary symmetric polynomials e_i via Newton's identities

$$k\, e_k(\zeta_1, \ldots, \zeta_\kappa) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}\, Q_i,$$

which allows us to construct coefficients of the polynomial

$$P(\lambda) \equiv \prod_{i=1}^{\kappa} (\lambda - \zeta_i) = \sum_{k=1}^{\kappa} (-1)^k\, e_k\, \lambda^{\kappa-k},$$

and the roots ζ_α can then be calculated using standard polynomial rootfinding methods

Ten neutrino system: roots $\{\zeta_\alpha\}$

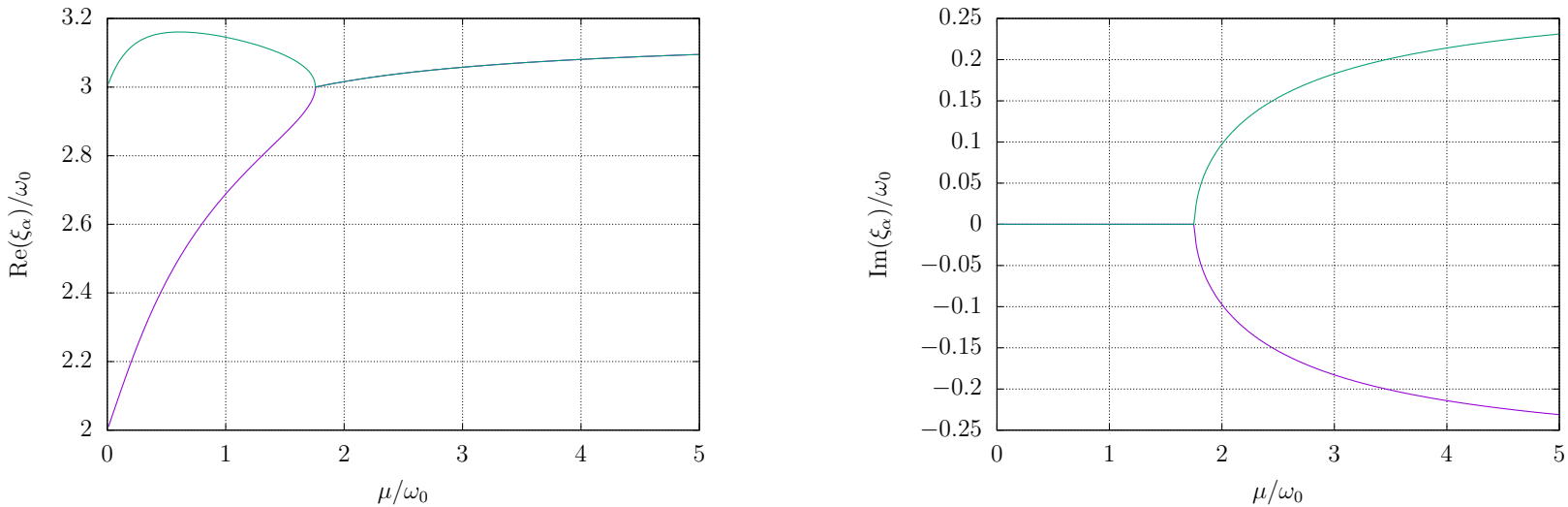


Figure: Roots of the Bethe Ansatz equations: real and imaginary parts of $\{\zeta_\alpha : \alpha = 1, \ldots, \kappa\}$, as functions of μ. Sample solution for $\kappa = 2$.

Ten neutrino system: roots $\{\zeta_\alpha\}$

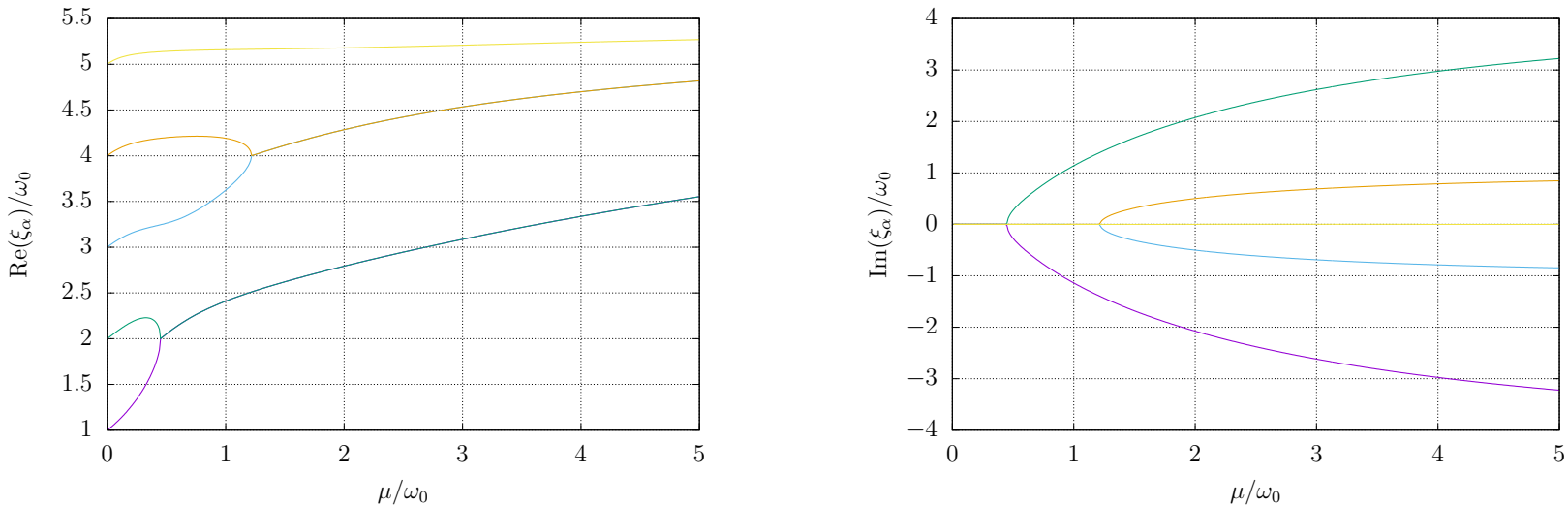


Figure: Roots of the Bethe Ansatz equations: real and imaginary parts of $\{\zeta_\alpha : \alpha = 1, \ldots, \kappa\}$, as functions of μ. Sample solution for $\kappa = 5$.