CeC Theory and 3D Simulations

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EIC Hadron Cooling Workshop
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70 YEARS OF **DISCOVERY**
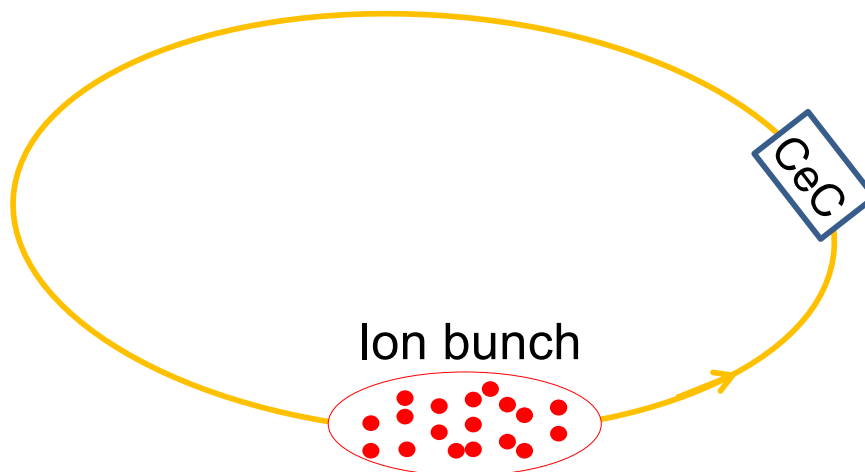A CENTURY OF SERVICE

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Overall Structure of CeC Prediction

A. Prediction of the single pass kicks received by an ion in the cooling section

| Ion's initial condition						Kicks due to CeC		
x,	x',	y,	y',	t,	E	dx',	dy',	dE
…	…	…	…	…	…	…	…	… …
…	…	…	…	…	…	…	…	… …

Interpolation

CeC

B. Long term prediction for circulating ions

Ion bunch

Outline

- Prediction of cooling force in a single pass
  - FEL-based CeC system
    - Theoretical model
    - Simulations
  - PCA-based CeC system
    - Theoretical model
    - Simulations
    - Preliminary estimates of cooling eRHIC proton beam (275 GeV) with PCA-based CeC
  - Chicane-based CeC system (time-domain analysis)
- Prediction of hadron beam evolution in the presence of cooling
  - Analytical approach: solving 1-D Fokker-Planck equation
  - Macro-particle tracking
- Transverse cooling
- Summary

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FEL-based CeC
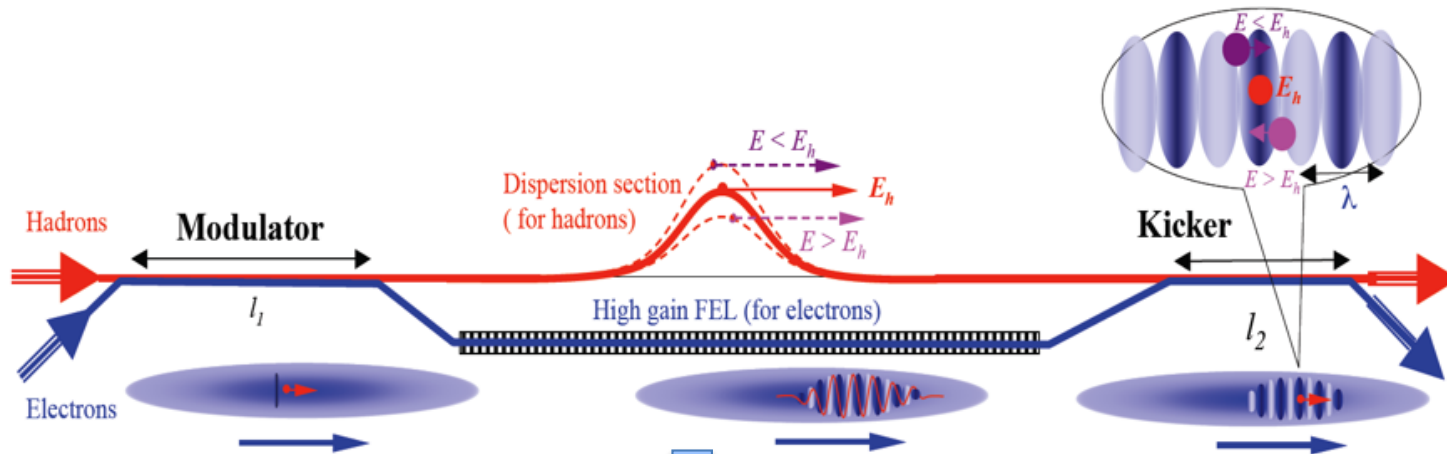
The economical version is made possible by:
1. Use relatively weak undulator field to reduce the delay of electrons;
2. Wave-packet moves faster than electrons.

Our Proof-of-Principle is an economic version of CeC, where electrons and hadrons are co-propagating along the entire CeC system
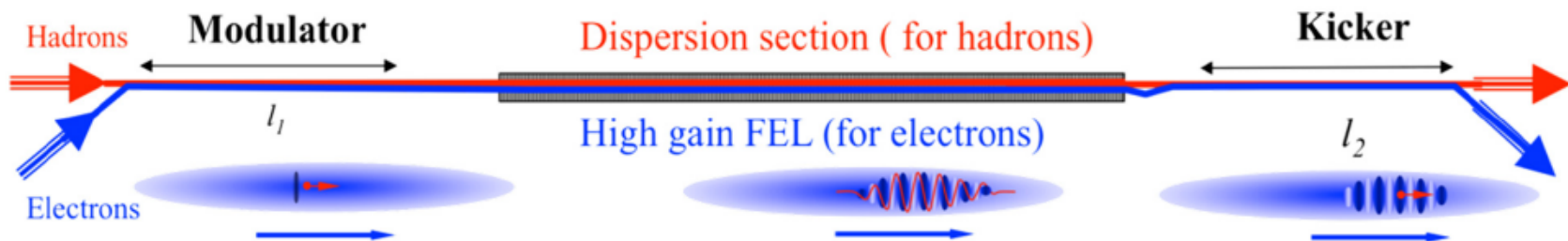
Parameters for the CeC PoP experiment

Electron beam parameters* (γ=28.6)	
Peak current*	40 A
Bunch length, (full length with uniform profile)	25~30 ps
RMS emittance, normalized	3~5 µm
RMS beam width at modulator/kicker	700 µm
RMS beam width at FEL amplifier	235 µm
RMS energy spread	1e-3

System parameters*	
Length of modulator/kicker	3 m
Undulator period	4 cm
Total number of undulator period (3 sections)	188
Undulator parameter, a_w	0.5
FEL optical wavelength	30.5 µm
Pierce parameter, ρ	0.012

* The designed peak current was changed to 75A to compensate FEL gain reduction due to longitudinal space charge

Analytical Tools for the Modulation Process I

- Cold uniform electron beam (© V.N. Litvinenko)
  - Density modulation: $q = -Ze \cdot (1 - \cos\varphi_1)$ $\qquad \varphi_1 = \omega_p l_1 / c\gamma_0$

  - Energy modulation ($\varphi_1 << 1$): $\left\langle \dfrac{\delta E}{E} \right\rangle \cong -2Z \dfrac{r_e}{a^2} \cdot \dfrac{L_{pol}}{\gamma} \cdot \left(\dfrac{z}{|z|} - \dfrac{z}{\sqrt{a^2/\gamma^2 + z^2}} \right)$
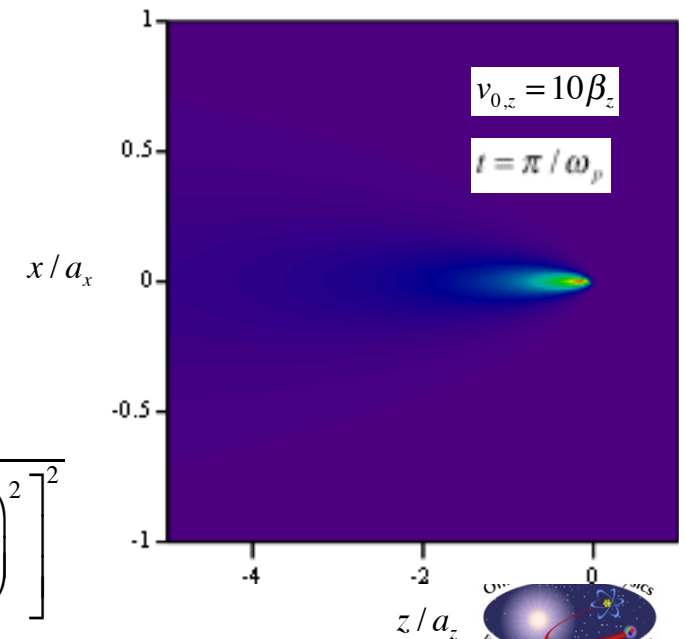
- Warm uniform electron beam with k-2 velocity distribution

$$f_0(\vec{v}) = \frac{1}{\pi^2 \beta_x \beta_y \beta_z} \left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2} \right)^{-2}$$

G. Wang and M. Blaskiewicz, Phys Rev E 78, 026413 (2008)

  - Density modulation:

$$\tilde{n}_1(\vec{x}, t) = \frac{Z_i}{\pi^2 a_x a_y a_z} \int_0^{\omega_p t} \frac{\tau \sin\tau \cdot d\tau}{\left[\tau^2 + \left(\dfrac{x}{a_x} + \dfrac{v_{0,x}}{\beta_x}\tau \right)^2 + \left(\dfrac{y}{a_y} + \dfrac{v_{0,y}}{\beta_y}\tau \right)^2 + \left(\dfrac{z}{a_z} + \dfrac{v_{0,z}}{\beta_z}\tau \right)^2 \right]^2}$$

$v_{0,z} = 10\beta_z$

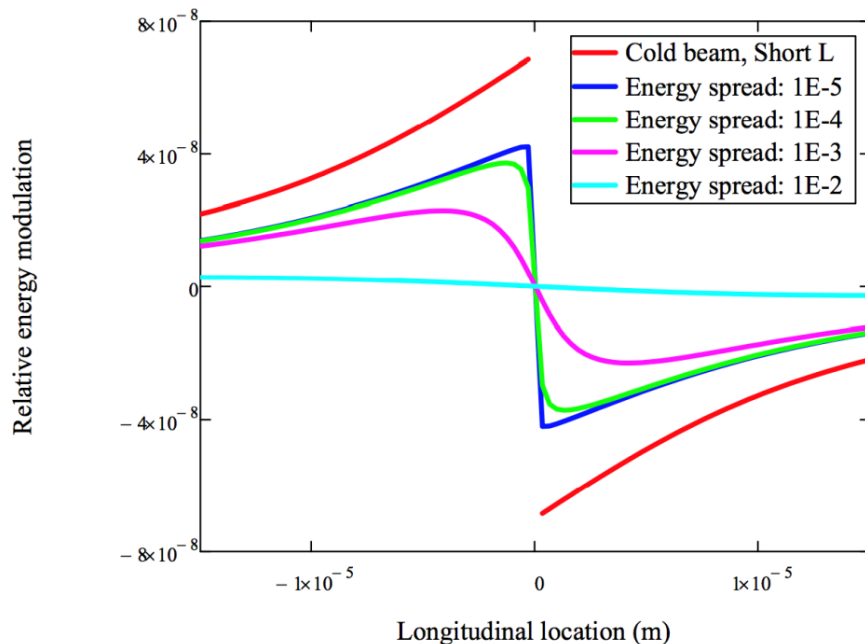$t = \pi / \omega_p$

x / a_x

z / a_z

Analytical Tools for the Modulation Process II

– Energy modulation:

$$\left\langle \frac{\delta E}{E_0} \right\rangle = \frac{\langle v_z \rangle}{c} = -\frac{1}{e n_0 \pi a^2 c} I_d\left(\gamma_0 z_l, \frac{L_{\text{mod}}}{\beta_0 \gamma_0 c} \right)$$

where $I_d(z,t)$ is an 1-D integral with finite integration limits (see backup slides).

It reduces to the previously derived cold beam result at the corresponding limits:

$$\bar{\beta} = 0 \qquad v_{0,z} = 0 \qquad L_{\text{mod}} << \beta_0 \gamma_0 c / \omega_p$$

$$\left\langle \frac{\delta E}{E} \right\rangle \approx -2 Z_i \frac{r_e}{a^2} \frac{L_{\text{mod}}}{\gamma} \cdot \left[\frac{z_l}{|z_l|} - \frac{z_l}{\sqrt{z_l^2 + a^2/\gamma^2}} \right].$$

Validating Numerical Simulations with Theoretical Prediction

- Simulations based on perturbative trajectory approach (© J. Ma, with code SPACE)
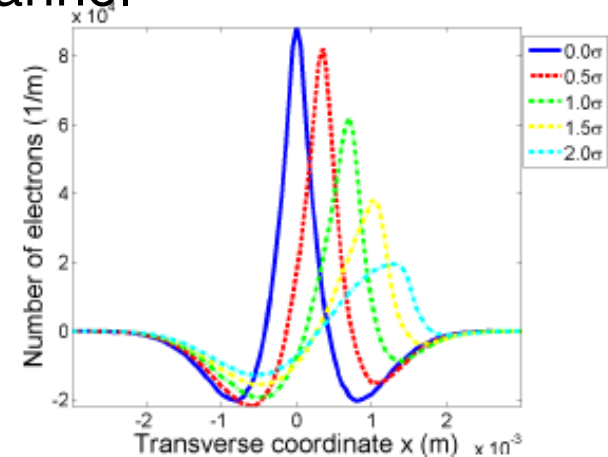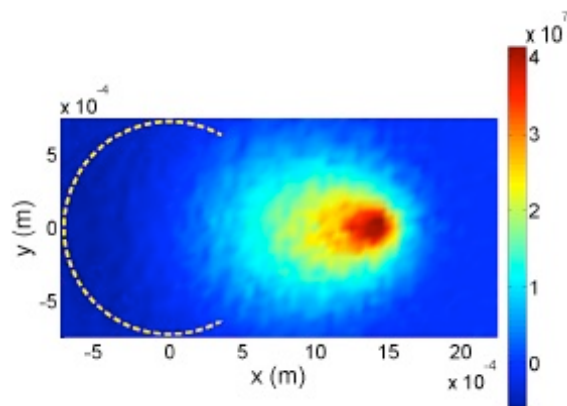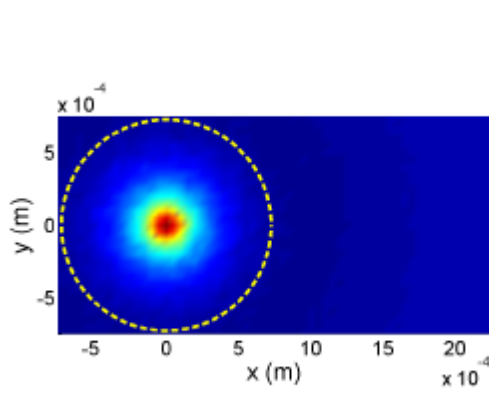 - Benchmarked with theory for uniform warm beam
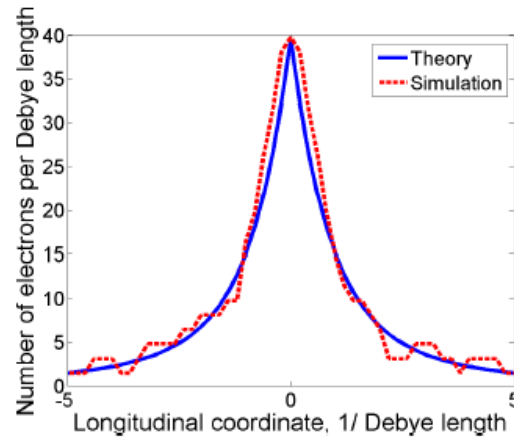
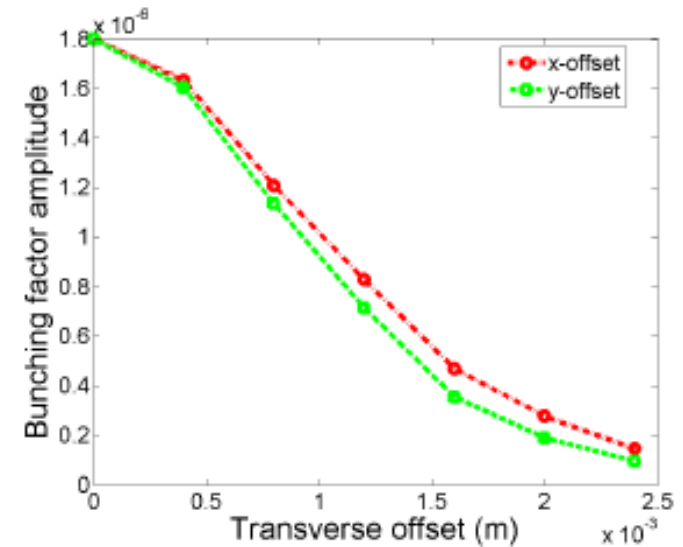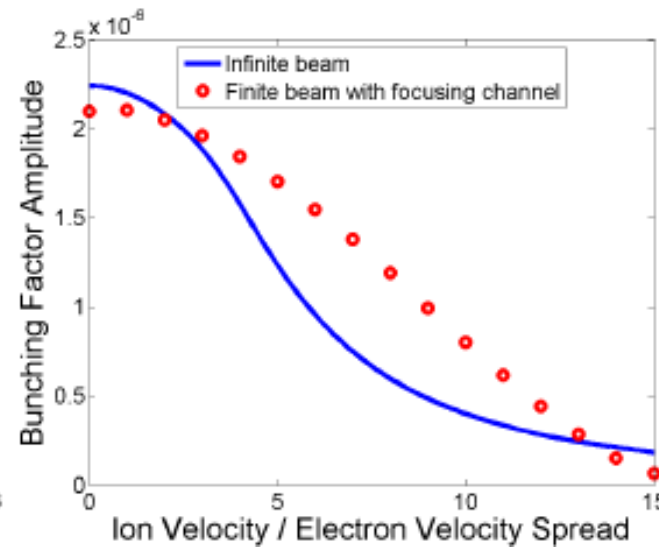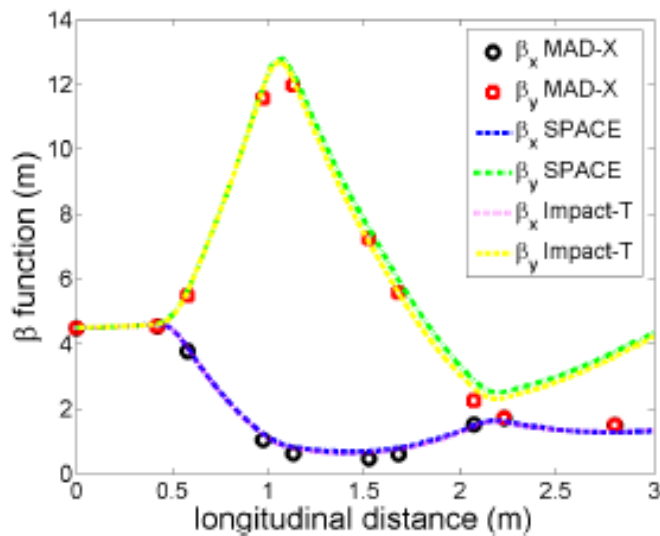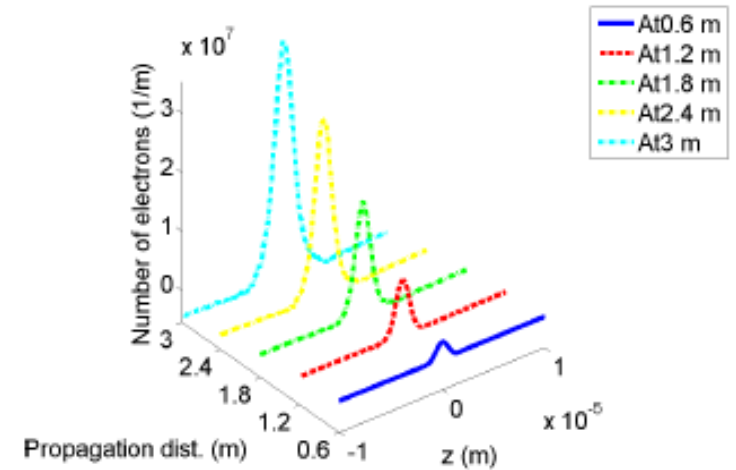(a) Longitudinal density
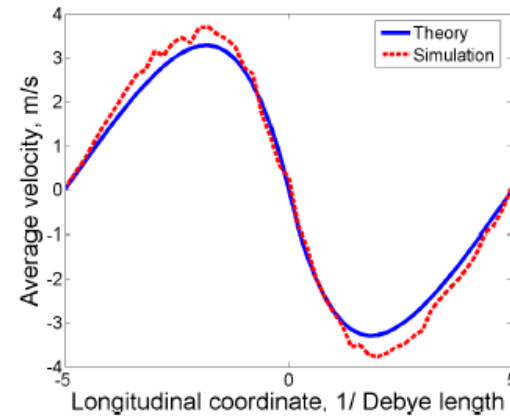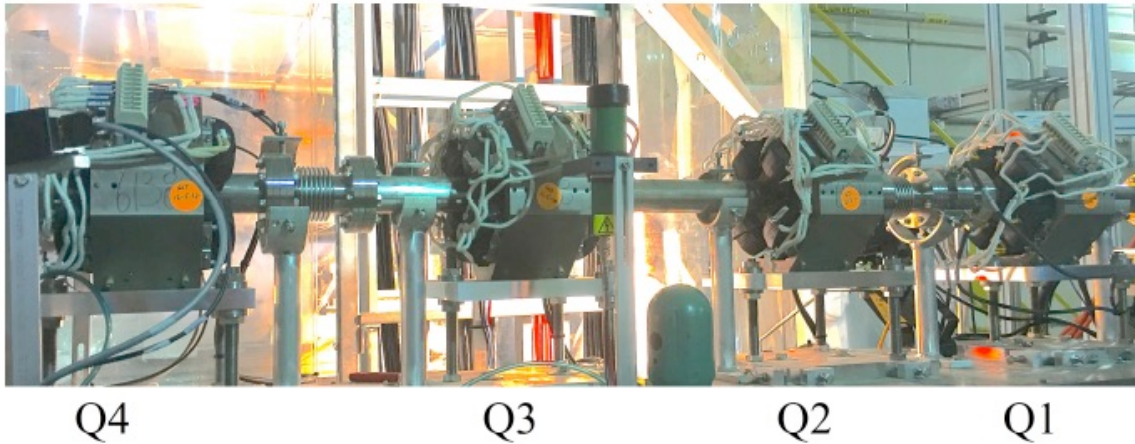
(b) Longitudinal velocity

- Gaussian beam in continuous focusing channel

*SPACE is a PIC code, *X. Wang et al., "Adaptive Particle-in-Cloud method for optimal solutions to Vlasov-Poisson equation," J. Comput. Phys., 316 (2016), 682 - 699.*

3D Modulator simulation in Quadrupole Focusing Channel

© J. Ma, SPACE simulation

$$b \equiv \frac{1}{N_\lambda} \sum_{k=1}^{N_\lambda} e^{i\frac{2\pi}{\lambda_{opt}}z_k}, \quad -\frac{\lambda_{opt}}{2} \leq z_k \leq \frac{\lambda_{opt}}{2},$$

Analytical Prediction for FEL Amplifier

- For the CeC PoP parameters, the FEL amplifier works in the diffraction dominated regime

$$\eta_d = \frac{l_{1D}}{2k_{opt}\sigma_x^2} = 6.6 \gg 1 \qquad l_{1D} = \frac{1}{2\sqrt{3}\rho k_u} = 0.15m \qquad \sigma_{x,FEL} = 0.235mm$$

  and hence we can't rely on 1-D theory to predict the performance of the amplifier.

- Using Ming Xie's empirical formula, we obtain $l_{3D} = l_{1D}(1+\Lambda_{3D}) = 0.363m$

- Estimate of the maximal gain before FEL saturation

  Using the saturating criteria that radiation power at saturation is $\rho\gamma mc^2 I / e$
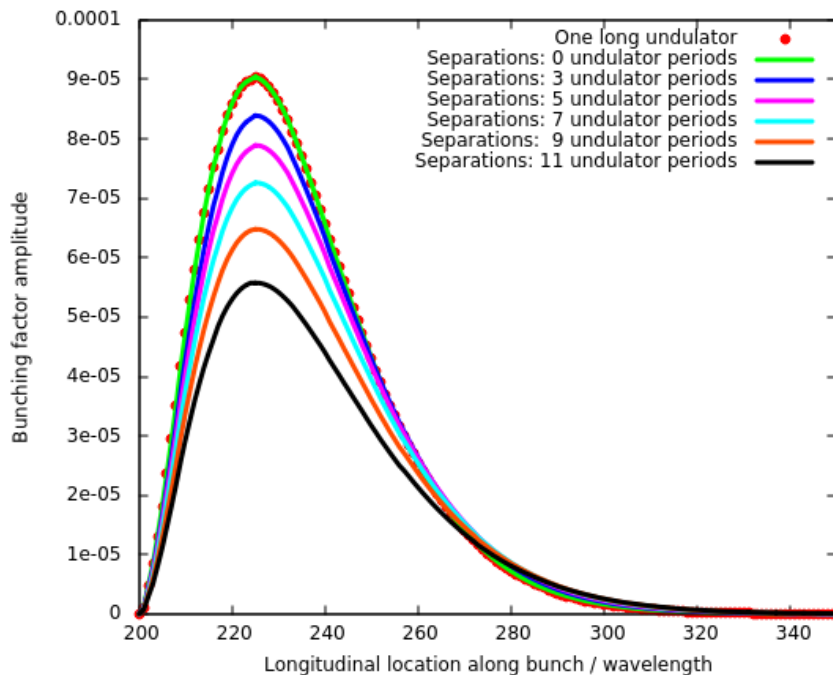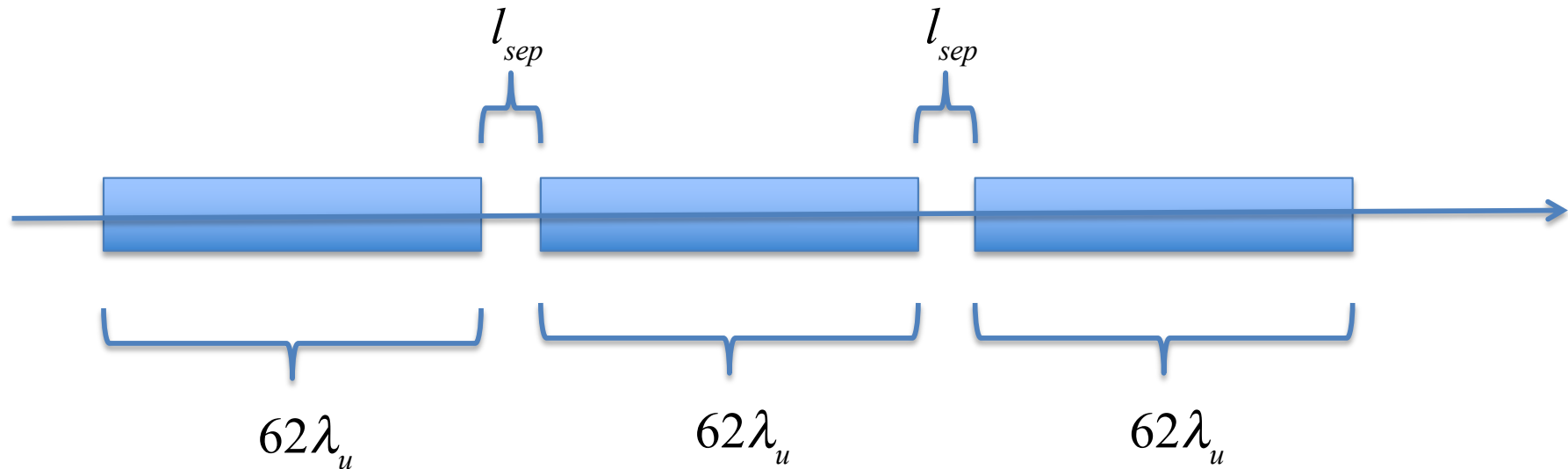
$$l_{sat} = 22.8 l_{3D} = 8.3m \qquad G_{sat} = \frac{1}{3}\delta_{\omega,sat}\exp\left(\frac{l_{sat}}{2l_{3D}}\right) = 330$$

Using the criteria $\left.\left|\delta\hat{n}/n_0\right|\right._{max} < 1 \Rightarrow$ $\quad G_{sat} = \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}} = \sqrt{N_{e\lambda}\delta_{\omega,sat}} = 523$

* There is a factor of $(2\pi)^{\frac{-1}{4}} \approx 0.632$ between the two estimates.

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Effects due to multi-subsections

$$l_{sep}$$ $$l_{sep}$$

$$62\lambda_u$$ $$62\lambda_u$$ $$62\lambda_u$$
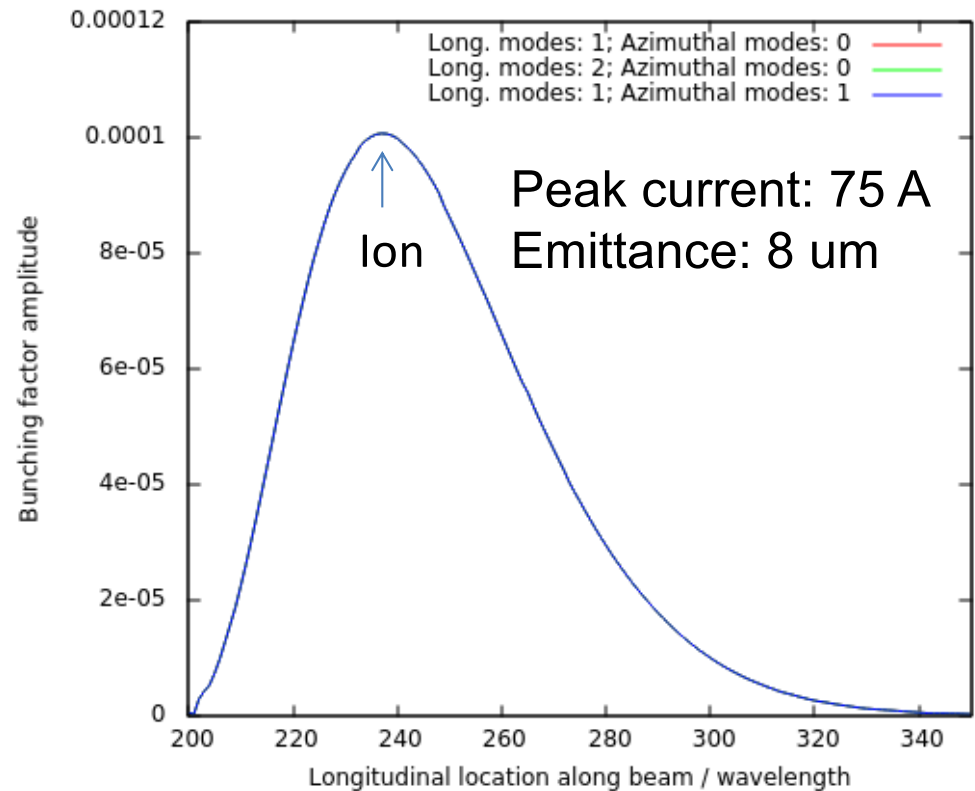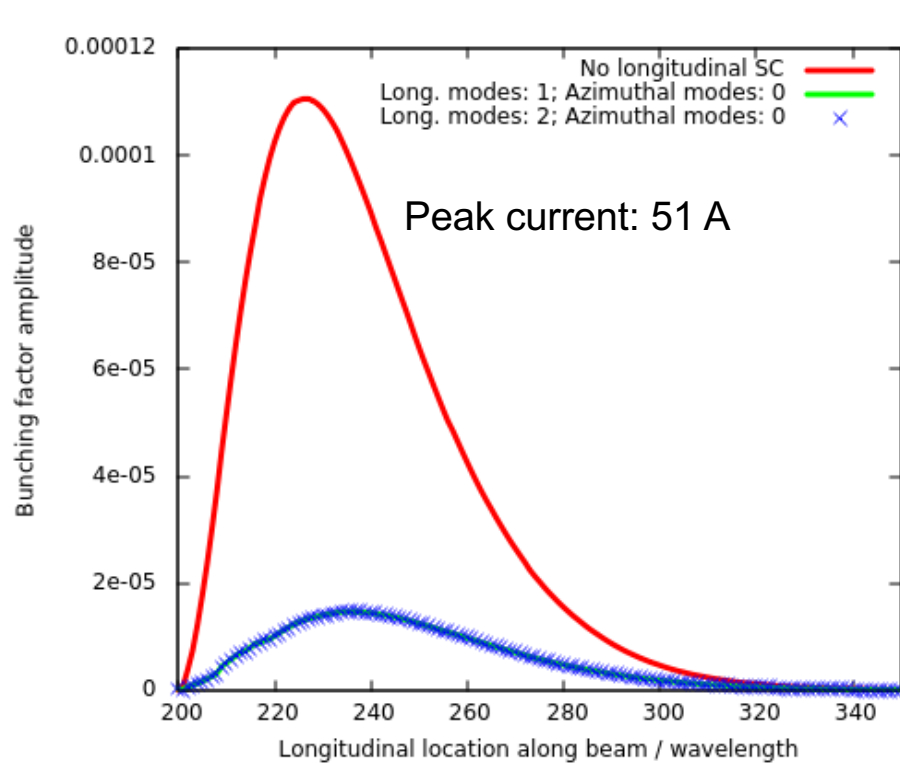

$$\sigma_r = \sqrt{\frac{2\lambda_1 L_G}{4\pi}} = \sqrt{\frac{30\times10^{-6}\times0.4}{2\pi}} = 1.38mm$$ The radiation width

$$L_R = \frac{\pi\sigma_r^2}{\lambda_1} = \frac{\pi\times1.38^2\times10^{-6}m^2}{30\times10^{-6}m} = 20cm$$ Rayleigh length

$$R = \frac{A(s=0)}{A(s=44cm)} = \frac{1}{1+\frac{44^2}{20^2}} = 17\%$$ Only 17% of the radiation power propagates to the next subsection

- Our undulator consists of 3 sub-sections and more than 80% of radiation power is lost in the 44cm gap between two successive subsections.

Effects due to Longitudinal Space Charge

- Longitudinal space charge has more pronounced effects on the FEL gain than what to be expected from theory.

Kicker

- Dynamic equation in Kicker is very similar to that in the modulator except the initial modulation in 6D phase space dominates the process. For 1D FEL output with the certain assumptions for the transverse distribution, the following analytical formula for density modulation can be derived (IPAC'10 proceedings, pp. 873)

$$\tilde{n}_1(k_z,t) = -Z_i \tilde{\Lambda}_{drive}(k_z) e^{ik_z v_{0z}t} e^{-|k_z|\sigma_{vz}t} \left[\cos(\omega_p t) + \frac{(\lambda_1 + i\hat{C})\beta_z c\gamma_z \Gamma - ik_z v_{0z} + |k_z|\sigma_{vz}}{\omega_p} \sin(\omega_p t) \right]$$

- The macro-particles from GENESIS simulation are imported into SPACE for the kicker simulation © J. Ma, SPACE simulation
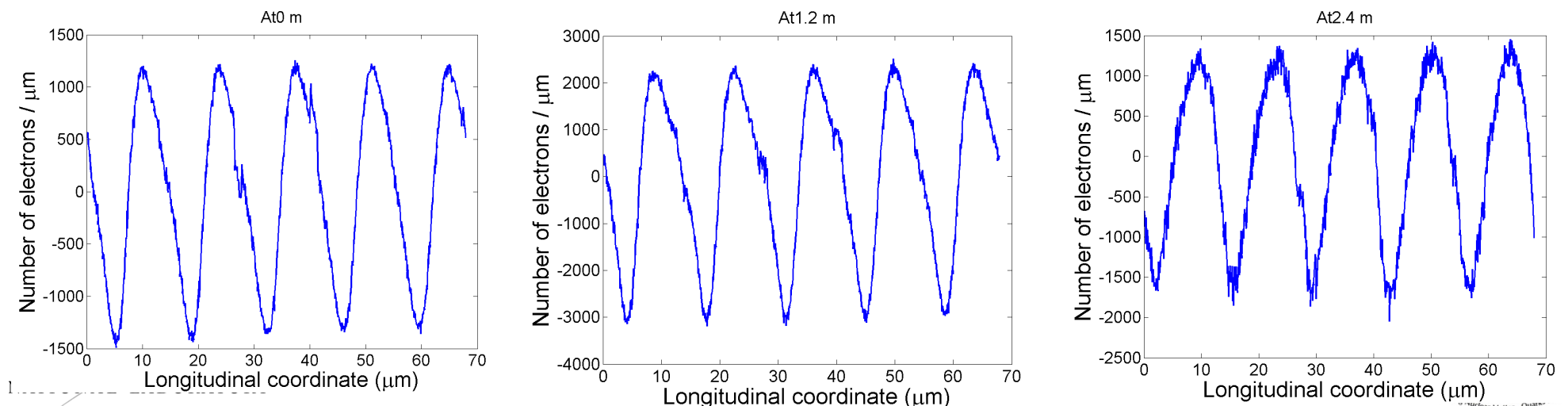
Field Reduction due to Finite Transverse Size

To estimate the reduction of electric field due to finite transverse size, we solve Poisson equation for charge distribution of the form:

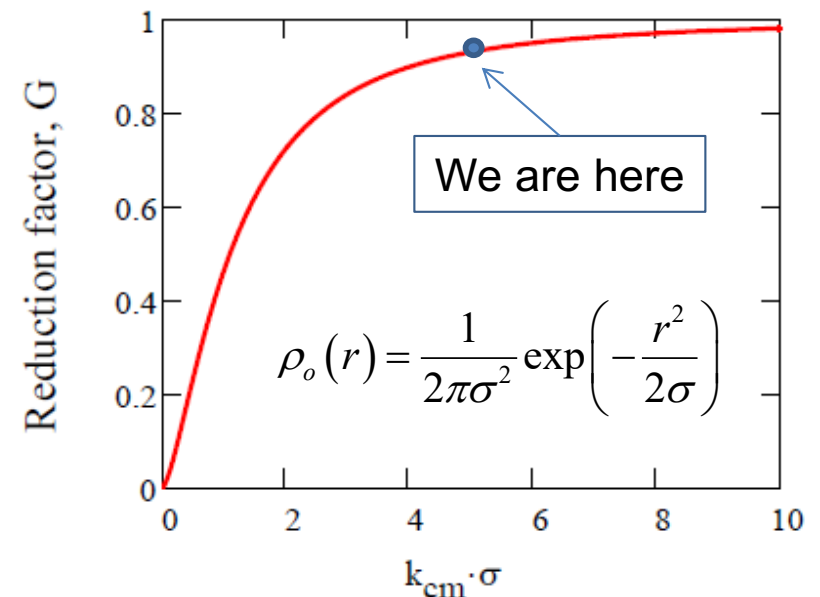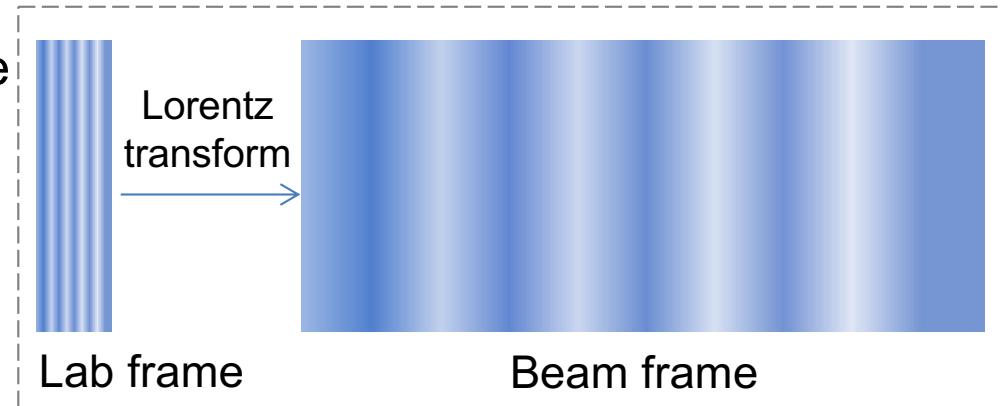$$\rho(\vec{r}) = \rho_o(r) \cdot \cos(k_{cm}z)$$

and get the on-axis longitudinal electric field

Lorentz transform

Lab frame   Beam frame

$$E_z(r=0) = -\frac{1}{k_{cm}\varepsilon_0}\sin(k_{cm}z)\int_0^\infty \eta K_0(\eta)\cdot\rho_o\left(\frac{\eta}{k_{cm}}\right)d\eta$$

For Gaussian transverse density distribution, we obtain

$$\boxed{E_z(r=0) = -\frac{\sin(k_{cm}z)}{2\pi k_{cm}\varepsilon_0\sigma^2}G(k_{cm}\sigma)}$$

Reduction factor, G

We are here

$$\rho_o(r) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma}\right)$$

$k_{cm}\cdot\sigma$

$$G(q) = \int_0^\infty \eta K_0(\eta)\cdot\exp\left(-\frac{\eta^2}{2q^2}\right)d\eta = \frac{1}{q^2}\int_0^\infty \eta\left[1-\eta K_1(\eta)\right]\cdot\exp\left(-\frac{\eta^2}{2q^2}\right)d\eta$$
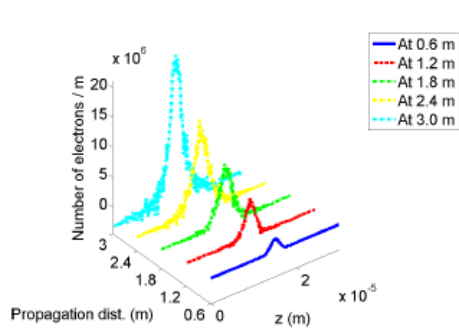
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NATIONAL LABORATORY   *The results reproduces what previously derived by G. Stupakov.

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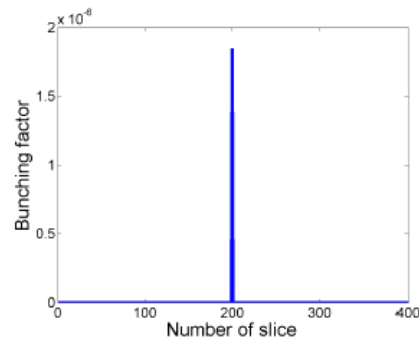# Start-to-end Simulations of Cooling Section

Electron beam parameters

© J. Ma, SPACE simulation

- Energy $\gamma = 28.5$
- R.M.S. energy spread 1e-3
- Peak current 75 A
- Normalized R.M.S. emittance 8 mm mrad

(a) Modulator

(b) Entrance of FEL
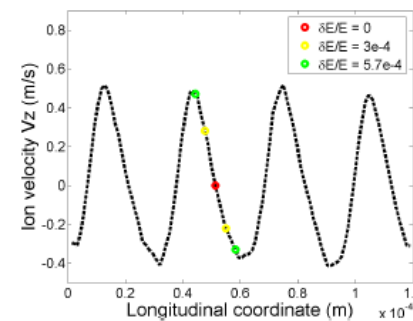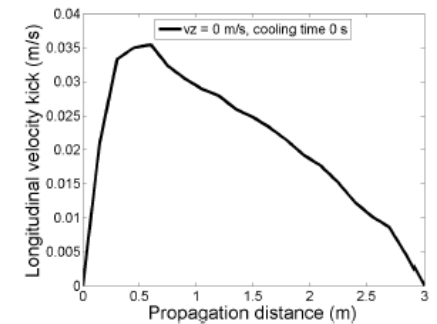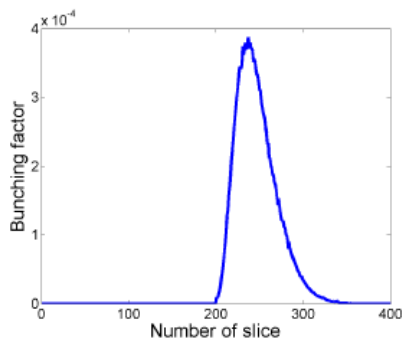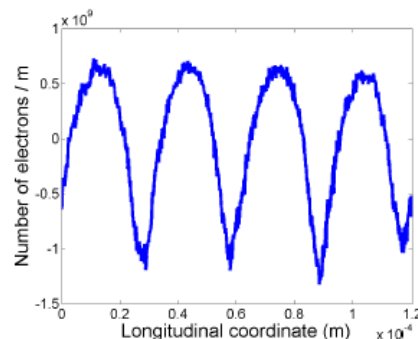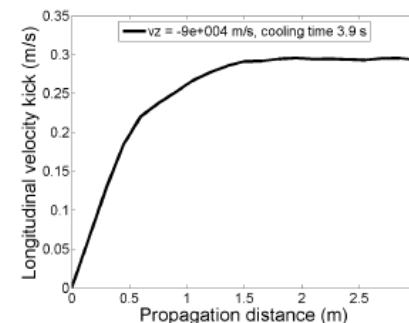
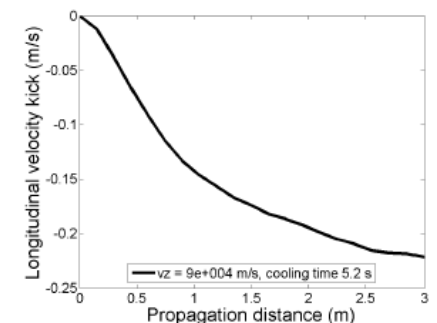(a) Ions at different locations (b) Ion with reference energy

(c) Exit of FEL

(d) Kicker

(c) Ion with lower energy

(d) Ion with higher energy

# Applying Single-pass Kick to Predict Ion Beam Evolution with Cooling

1. Macro-particle tracking
2. Solving Fockker-Planck equation

Electron beam parameters		Ion beam parameters	
Peak current, A	75	Charge number, Z_{ion}	79
Full bunch length, ps	25	Bunch intensity	10^8
Norm. emittance, RMS, μm	5	Bunch length, RMS, ns	3.06
Relative energy spread, RMS	10^{-3}	Relative energy spread, RMS	3.35×10^{-4}
Beam energy, γ	28.66	RF frequency, MHz	28

Table 1: Beam parameters for the proof of CeC principle experiment

Gain of FEL amplification	80	FEL wavelength, μm	30.5
Peak correcting field, V/m	36	R_{56}, cm	1.2
Kicker length, m	3	Coherent length, σ_w, mm	0.54
Coherent kick amplitude, g_γ	4.657×10^{-8}	Local cooling time (T_0), s	3.185
CeC diffusive kick from neighbor ions, d_{ion}	1.163×10^{-5}	CeC diffusive kick from electrons, d_e	2.038×10^{-5}
IBS diffusive kick at bunch center, $d_{IBS}(0)$	1.886×10^{-6}		

Table 2: CeC system parameters of the proof of CeC principle experiment

40 minutes of cooling

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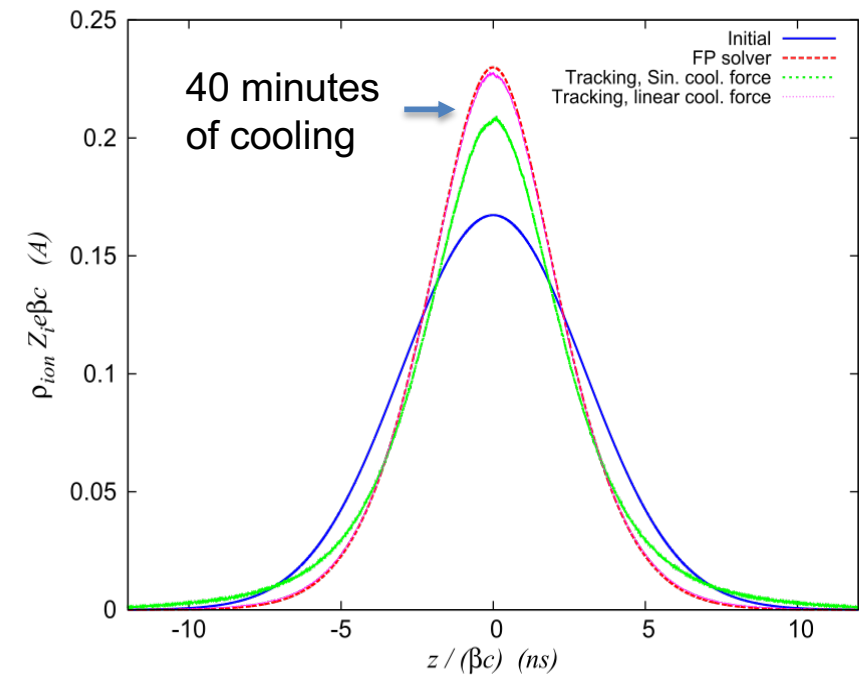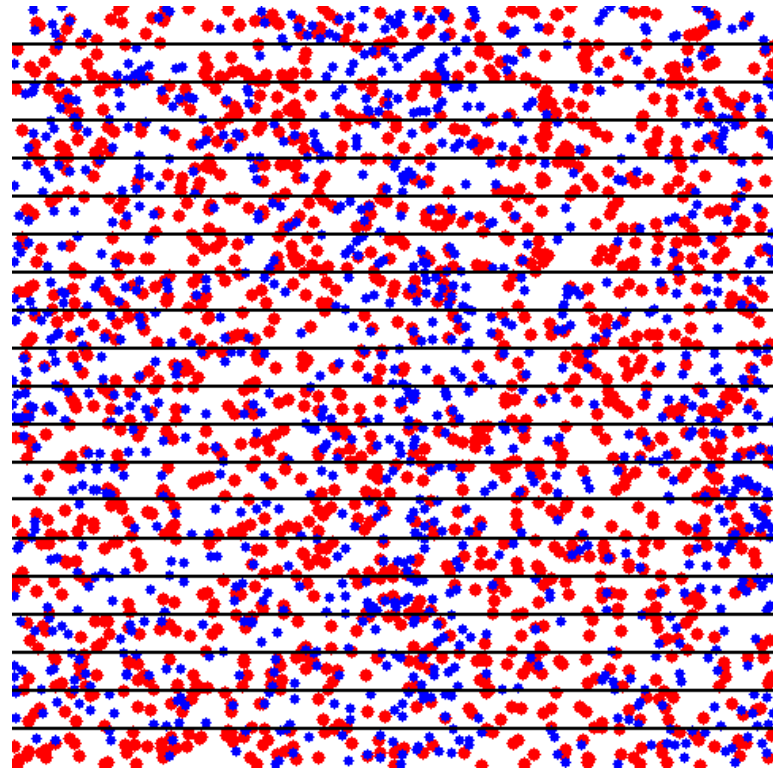
PCA-based CeC

- ## Plasma-Cascade Instability
  - ### Longitudinal plasma oscillation

Longitudinal plasma oscillation in a neutral plasma (from the internet, only for illustration)

$$\frac{d^2\tilde{n}}{dt^2} + \omega_p^2\tilde{n} = 0;$$

$$\omega_p^2 = \frac{4\pi n_o e^2}{m};$$
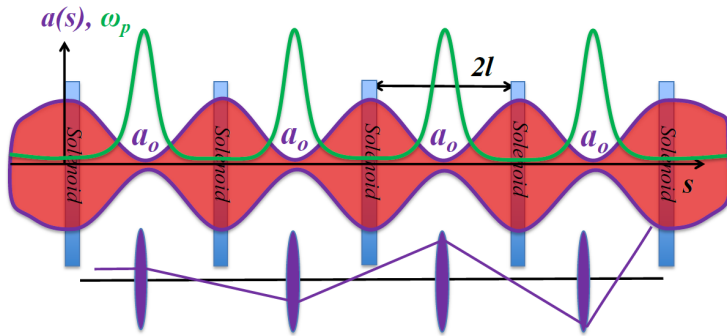
Plasma oscillation also exist in single-species plasma like electron beam.

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Plasma-Cascade Instability

Longitudinal plasma oscillation with periodically varying plasma frequency

$$\frac{d^2\tilde{n}}{dt^2} + \omega_p^2(t)\tilde{n} = 0;$$

$$\hat{n}'' + 2k_{sc}^2 \hat{a}(\hat{s})^{-2}\hat{n} = 0$$

$$\hat{a}'' = k_{sc}^2 \hat{a}^{-1} + k_\beta^2 \hat{a}^{-3}$$

$$\hat{a} = \frac{a}{a_o}; \hat{s} = \frac{s}{l} \in \{-1, 1\};$$

$$k_{sc} = \sqrt{\frac{2}{\beta_o^3 \gamma_o^3}\frac{I_o}{I_A}\frac{l^2}{a_o^2}}; \quad k_\beta = \frac{\varepsilon l}{a_o^2}$$

Stability condition

$$\begin{pmatrix} \hat{n} \\ \hat{n}' \end{pmatrix}_{s=-l} = M_{total} \begin{pmatrix} \hat{n} \\ \hat{n}' \end{pmatrix}_{s=l}$$
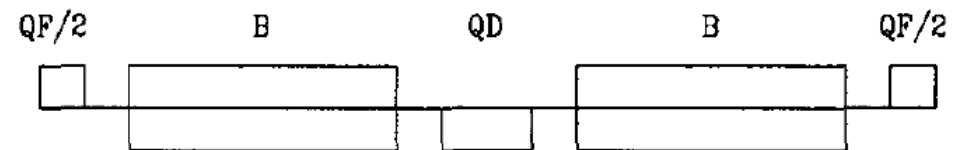
$$|\lambda| = \left| (M_{total})_{1,1} \pm \sqrt{(M_{total})_{1,1}^2 - 1} \right| \leq 1$$

$$(M_{total})_{1,1} = 2m_{11}m_{22}|_{\hat{s}=1} - 1 \qquad \begin{pmatrix} \hat{n}(\hat{s}) \\ \hat{n}'(\hat{s}) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \hat{n}(0) \\ \hat{n}'(0) \end{pmatrix}$$
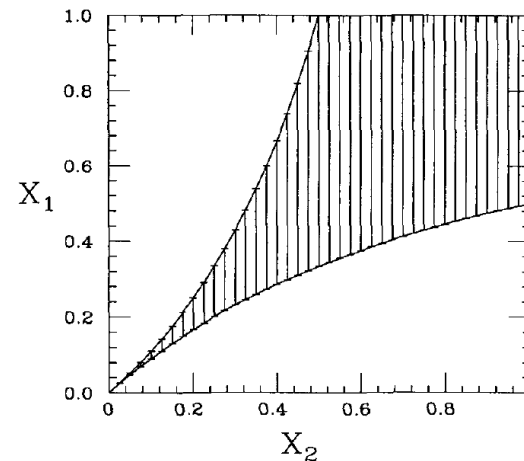
Betatron motion in a FODO cell

$$y'' + K_y(s)y = 0,$$

FODO CELL

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{L_1}{f_2} - \frac{L_1}{f_1} - \frac{L_1^2}{2f_1f_2} & 2L_1(1 + \frac{L_1}{2f_2}) \\ \frac{1}{f_2} - \frac{1}{f_1} - \frac{L_1}{f_1f_2} + \frac{L_1}{2f_2^2} + \frac{L_1^2}{4f_1^2f_2} & 1 + \frac{L_1}{f_2} - \frac{L_1}{f_1} - \frac{L_1^2}{2f_2^2} \end{pmatrix},$$

$$X_1 = L_1/2f_1$$

$$X_2 = L_1/2f_2$$

Analytical solution for emittance dominated beam

If we assume $k_\beta >> k_{sc}$, the envelope equation become $\hat{a}'' = k_\beta^2 \hat{a}^{-3}$ and its solution for initial condition of $\hat{a}(0) = 1$ and $\hat{a}'(0) = 0$ is

$$\hat{a}^2 = k_\beta^2 \hat{s}^2 + 1$$

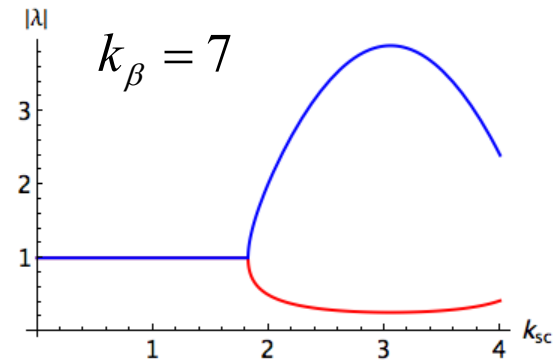and the equation of longitudinal density perturbation becomes

$$\hat{n}'' + \frac{2k_{sc}^2}{k_\beta^2 \hat{s}^2 + 1} \hat{n} = 0$$

which can be solved in terms of Hypergeometric function:

$$\hat{n}(\hat{s}) = c_1 \cdot F(\alpha_1, \beta_1; \gamma_1; x_1) + c_2 \hat{s} \cdot F\left(\alpha_1 + \frac{1}{2}, \beta_1 + \frac{1}{2}; \gamma_1 + 1; x_1\right)$$

The eigenvalues for the transfer matrix of one cell is

$$\lambda = (M_{total})_{1,1} \pm \sqrt{(M_{total})_{1,1}^2 - 1}$$

$|\lambda|$

$k_\beta = 7$

k_{sc}

$$(M_{total})_{1,1} = 2F\left(\alpha_1, \beta_1; \frac{1}{2}; -k_\beta^2\right)\left\{F\left(\alpha_1 + \frac{1}{2}, \beta_1 + \frac{1}{2}; \frac{3}{2}; -k_\beta^2\right) - \frac{2}{3}k_{sc}^2 F\left(\alpha_1 + \frac{3}{2}, \beta_1 + \frac{3}{2}; \frac{5}{2}; -k_\beta^2\right)\right\} - 1$$
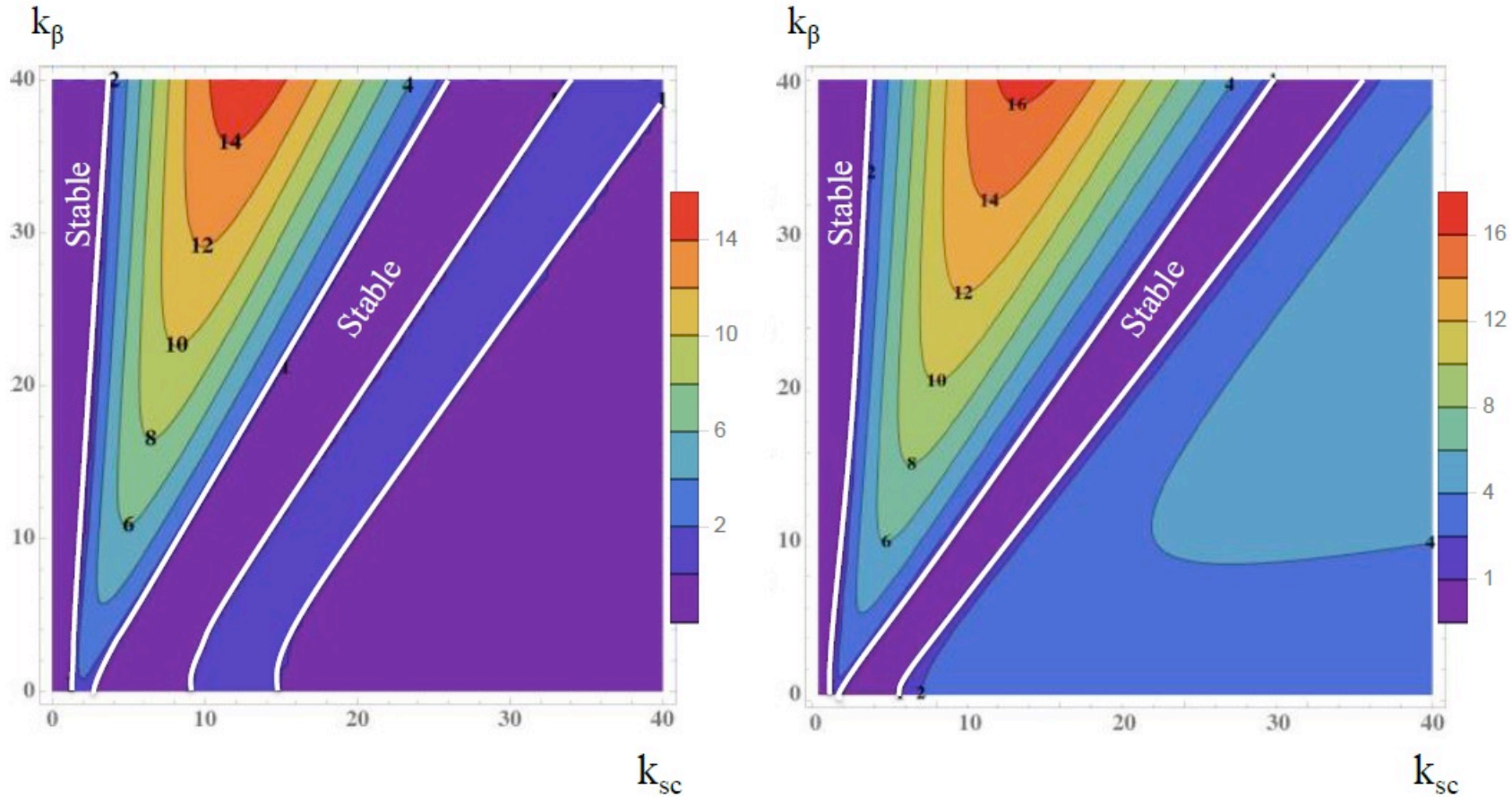
Gain of Plasma-Cascade Instability

Figure 7. Contour plots of $\lambda = \max\left(\left|\mathrm{Re}\,\lambda_1\right|, \left|\mathrm{Re}\,\lambda_2\right|\right)$, the absolute value of maximum growth rate per cell, using (a) an analytical solution (67) for an approximate beam envelope $\hat{a}(\hat{s})$ (as in eq. (33), and (b) exact numerical solution of the problem using code described in Appendix A. Purple area highlighted by white lines indicates areas of stable oscillation $\left|\lambda_{1,2}\right| = 1$. Outside these areas oscillations are growing exponentially.

Estimate Cooling Force for PCA-based CeC: Parameters

Energy, γ	28.5
Electron beam peak current, A	100
Bunch length, ns	0.015
Bunch charge, nC	1.5
Modulator length, m	3
Amplifier length, m	8 (4 sections)
Beam width at modulator, mm	0.94
Amplifier gain (Cold, infinitely wide), g_{amp}	200
RMS energy spread	1e-4
KV envelope norm. emittance, μm	8
Minimal beam width at PCA, mm	0.2

$$k_{sc} = 3.6 \qquad k_\beta = 7$$
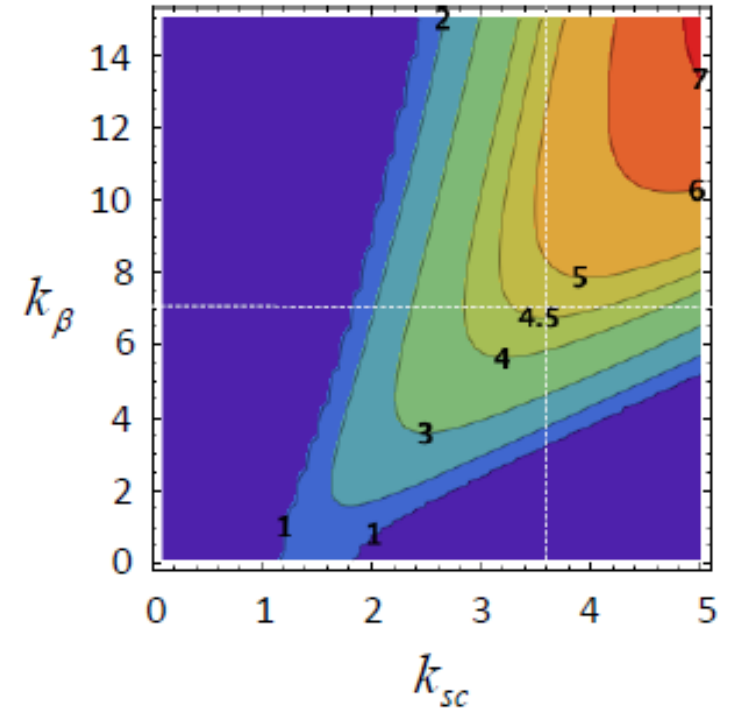
$$|\lambda| \approx 4.5 \qquad g_{amp} \equiv \frac{1}{2}|\lambda|^4 \approx 200$$
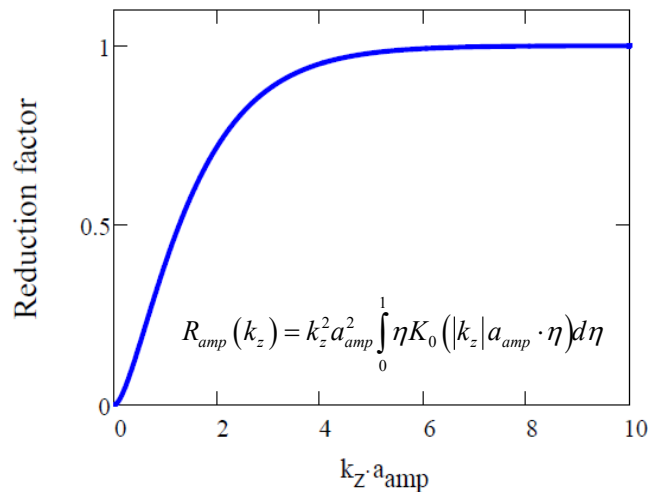
Estimate Cooling Force for PCA-based CeC: Line density perturbation

Line density perturbation at the exit of the modulator:

$$\tilde{\rho}_1(k_z) = \frac{Z_i e}{1 + \bar{\lambda}_z(k_z)^2}\left[1 - e^{\bar{\lambda}_z(k_z)\psi_m}\left(\cos\psi_m - \bar{\lambda}_z(k_z)\sin\psi_m\right)\right] \qquad \rho_{1z}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_z z}\tilde{\rho}_1(k_z)\,dk_z = \frac{Z_i e}{\pi a_z}\int_0^{\psi_m} \frac{\tau\sin\tau}{\left(\dfrac{z}{a_z} + \dfrac{v_{0z}}{\beta_z}\tau\right)^2 + \tau^2}$$

Line density perturbation at the exit of the Plasma-Cascade Amplifier:

$$\tilde{\rho}_2(k_z) = g_{amp} R_{amp}(k_z)\exp\left(-|k_z|\beta_z \frac{L_{amp}}{\gamma c}\right)\tilde{\rho}_1(k_z)$$
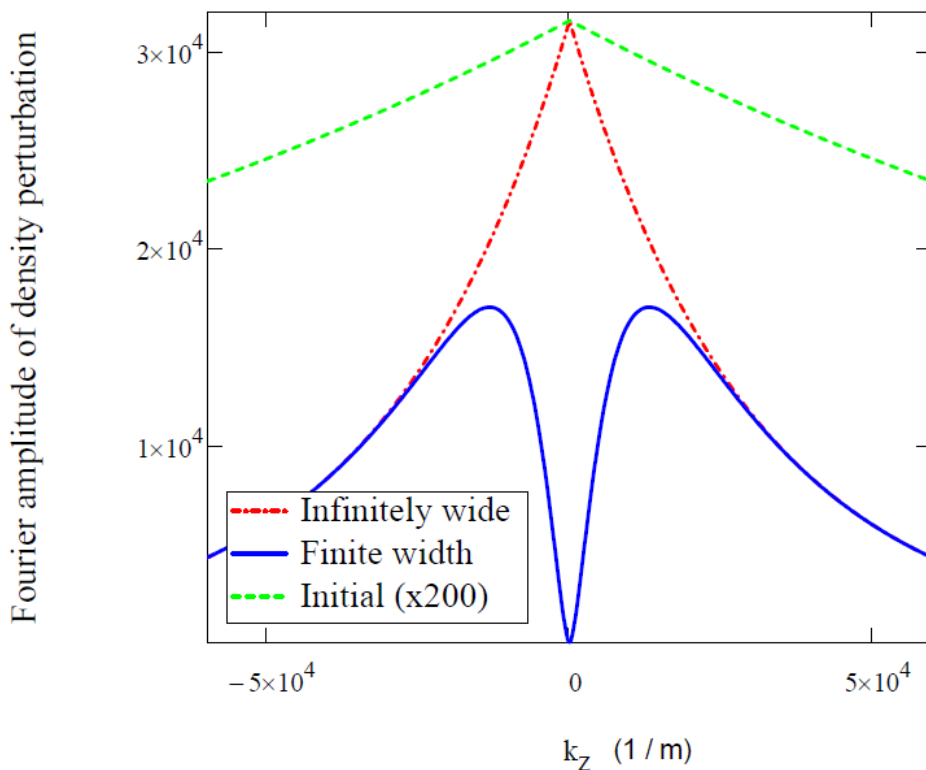

$$R_{amp}(k_z) = k_z^2 a_{amp}^2 \int_0^1 \eta K_0\left(|k_z|a_{amp}\cdot\eta\right)d\eta$$

Gain reduction factor due to finite transverse beam size

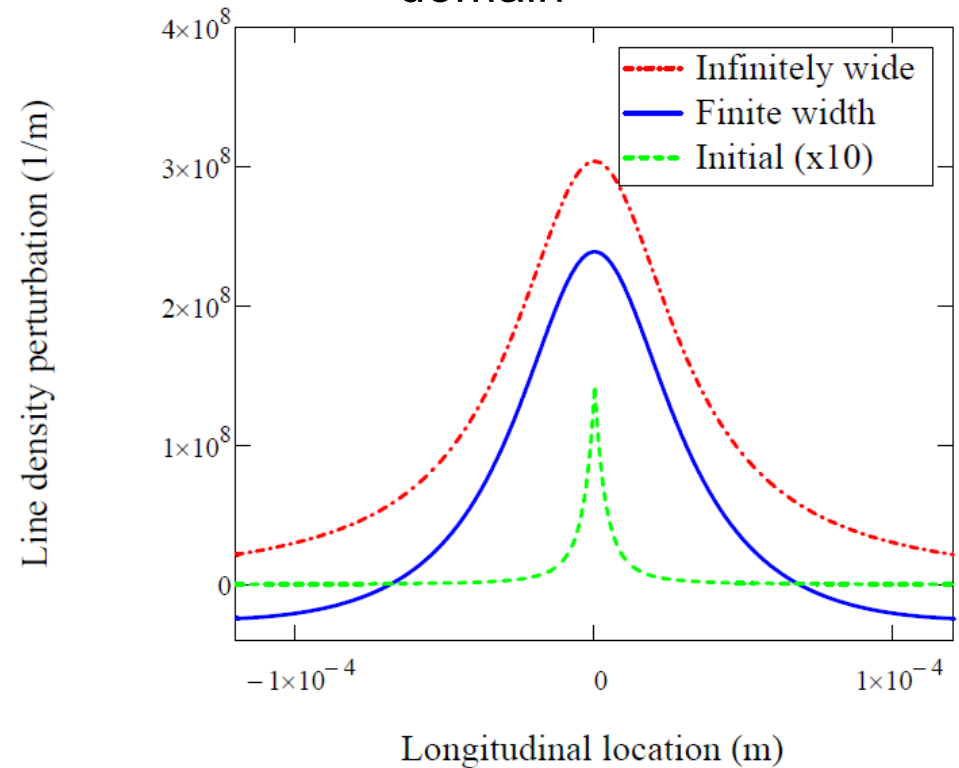Landau damping factor for Lorentzian energy distribution

Estimate Cooling Force for PCA-based CeC:
Line density perturbation

Line density perturbation in wave-number domain

$$\tilde{\rho}_2(k_z) = g_{amp} R_{amp}(k_z) \exp\left(-|k_z|\beta_z \frac{L_{amp}}{\gamma c}\right) \tilde{\rho}_1(k_z)$$

Line density perturbation in spatial domain

$$\rho_2(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\rho}_2(k_z) e^{ik_z z} dk_z$$

Estimate Cooling Force for PCA-based CeC: Longitudinal Electric Field in the Kicker Section

The electric potential induced by the line density perturbation is determined by the following equations

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\varphi(r,z)\right)\right]+\frac{\partial^2}{\partial z^2}\varphi(r,z)=\frac{1}{\varepsilon_0}\rho_2(z)f_\perp(r)$$
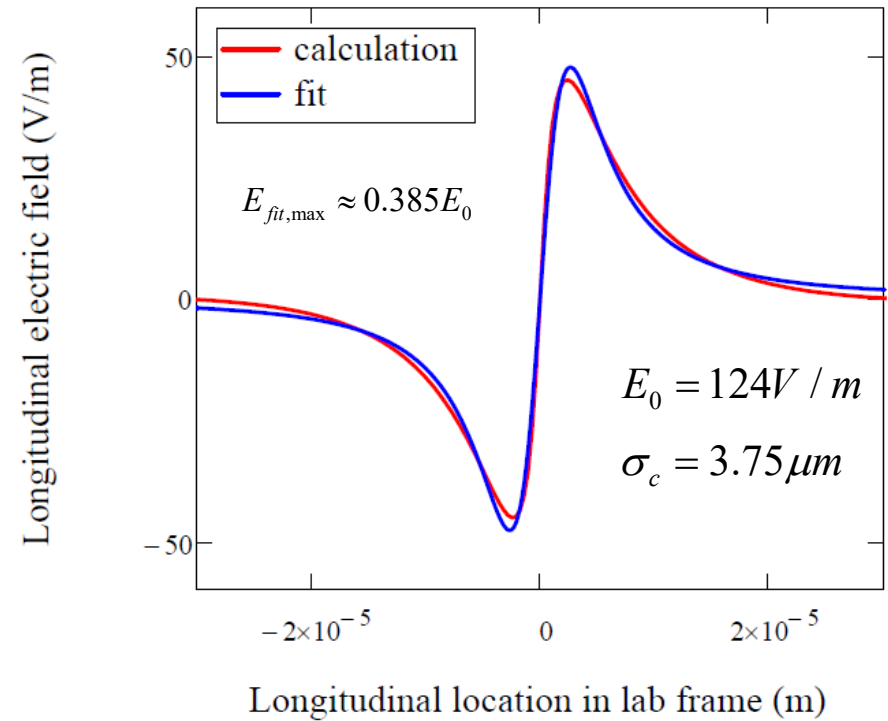
If we take the transverse distribution of the electrons as

$$f_\perp(r)=\frac{1}{\pi a^2}H(a-r)$$

The electric field can be solved as

$$E_z(r,z)=-\frac{\partial\varphi}{\partial z}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_z(r,k_z)e^{ik_zz}dk_z$$

$$\tilde{E}_z(r)=-ik_z\frac{\tilde{\rho}_2(k_z)}{\pi\varepsilon_0}$$

$$\times\left[I_0(k_zr)\int_{r/a}^{1}\eta K_0(k_za\cdot\eta)d\eta+K_0(k_zr)\int_{0}^{r/a}\eta I_0(k_za\cdot\eta)d\eta\right]$$

$$E_{fit,\max}\approx 0.385E_0$$

$$E_0=124V/m$$

$$\sigma_c=3.75\mu m$$

For easy implementation into ion tracking code, we use the fitting formula:

$$E_{fit}(z)=E_0\cdot\frac{z}{\sigma_c}\left[1+\frac{z^2}{\sigma_c^2}\right]^{-3/2}$$

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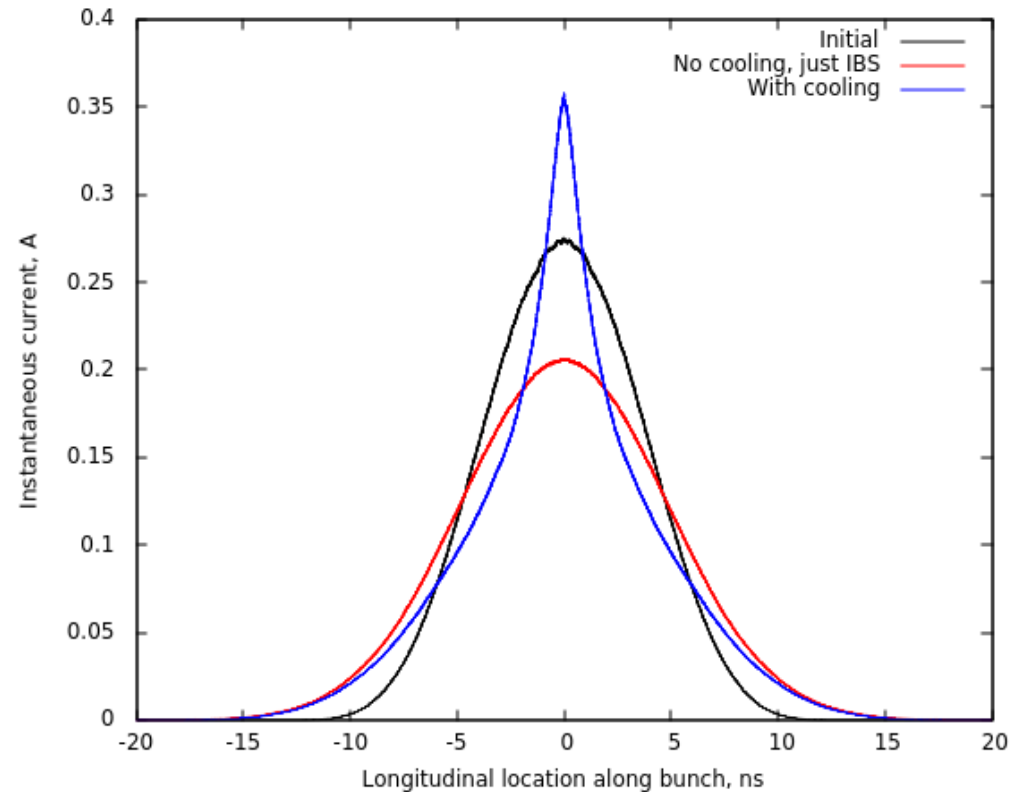
Single-pass Kick and Tracking Results for PCA-based CeC

$$\Delta\gamma_{j,N} = -g_\gamma \frac{(D \cdot \delta_{j,N})}{\sigma_c}\left[1 + \frac{(D \cdot \delta_{j,N})^2}{\sigma_c^2}\right]^{-3/2} + g_\gamma\sqrt{\frac{3\pi}{8}}\rho_{ion}(z_{j,N})\sigma_c \cdot X_{j,N} + \frac{g_\gamma}{Z_i}\sqrt{\frac{3\pi}{8}}\rho_e(z_{j,N})\sigma_c \cdot Y_{j,N}$$

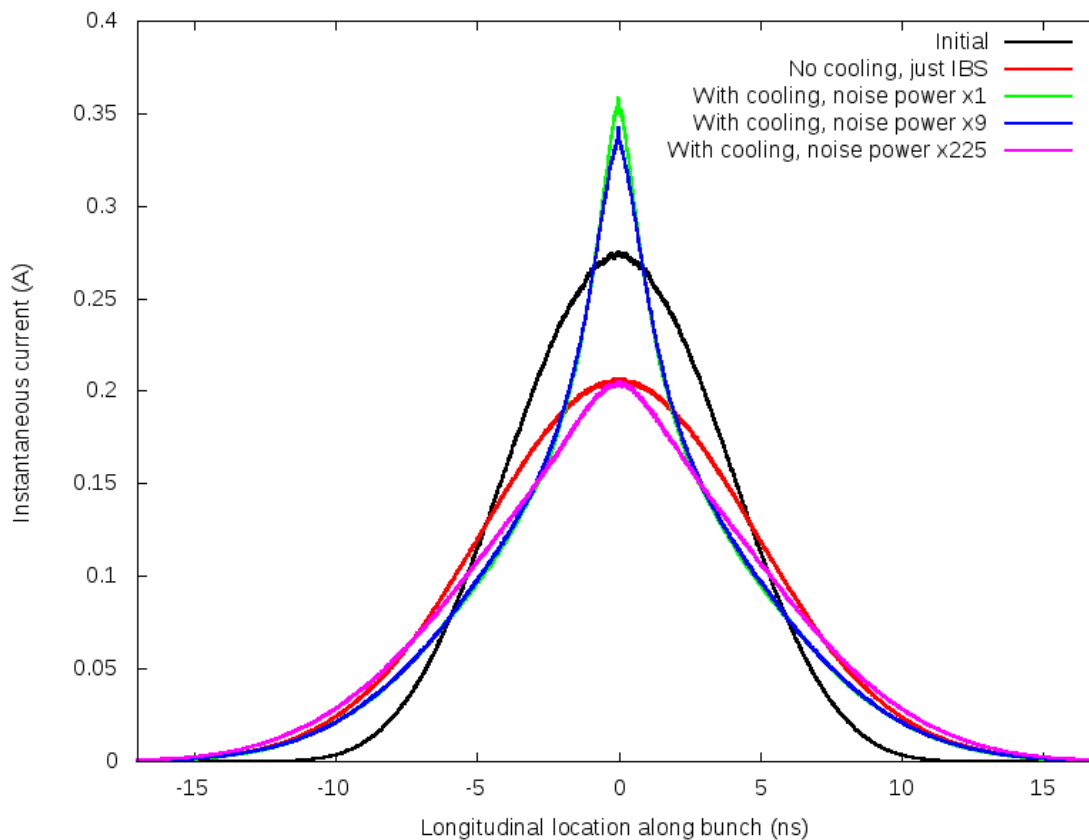$$g_\gamma = Z_i e E_0 L_k / \left(A_i m_u c^2\right)$$

Parameters used in ion tracking	
$E_0{}^*$	62 V/m
σ_c	3.75 μm
Z_{ion}	79
Ion bunch intensity	2E8
D (R_{56})	1.2 cm

*E_0 is reduced by a factor of 2 to account for reduced cooling for ions with large betatron amplitude

Tolerance of PCA-based CeC on the noise of electron beam

$$\Delta\gamma_{j,N} = -g_\gamma \frac{(D\cdot\delta_{j,N})}{\sigma_c}\left[1+\frac{(D\cdot\delta_{j,N})^2}{\sigma_c^2}\right]^{-3/2} + g_\gamma\sqrt{\frac{3\pi}{8}}\rho_{ion}(z_{j,N})\sigma_c\cdot X_{j,N} + \sqrt{R_{NL}}\cdot\frac{g_\gamma}{Z_i}\sqrt{\frac{3\pi}{8}}\rho_e(z_{j,N})\sigma_c\cdot Y_{j,N}$$


$$R_{NL} = \frac{\text{Actual noise power in the electron beam}}{\text{Shot noise power for uncorrelated electrons}}$$

- According to the simulation, the noise power in the electron beam should not exceed 200 times of the shot noise power.

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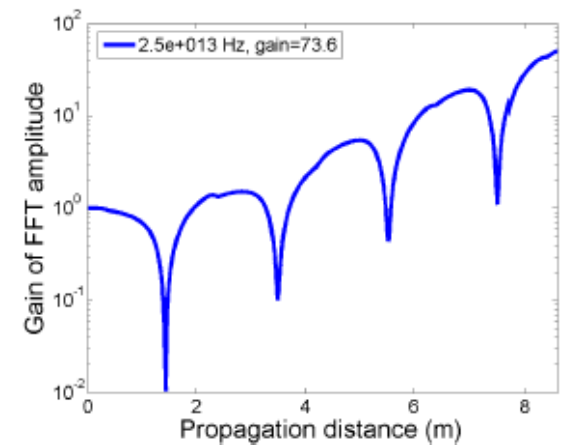Exploring Nuclear Matter - Quarks to Stars

3D Simulation of PCA

- KV distribution
- Energy $\gamma = 28.5$, R.M.S. energy spread $\delta E/E$ =2e-4
- Peak current 100 A
- Normalized envelope emittance 8 mm mrad, beam waist a_o =2e-4 m
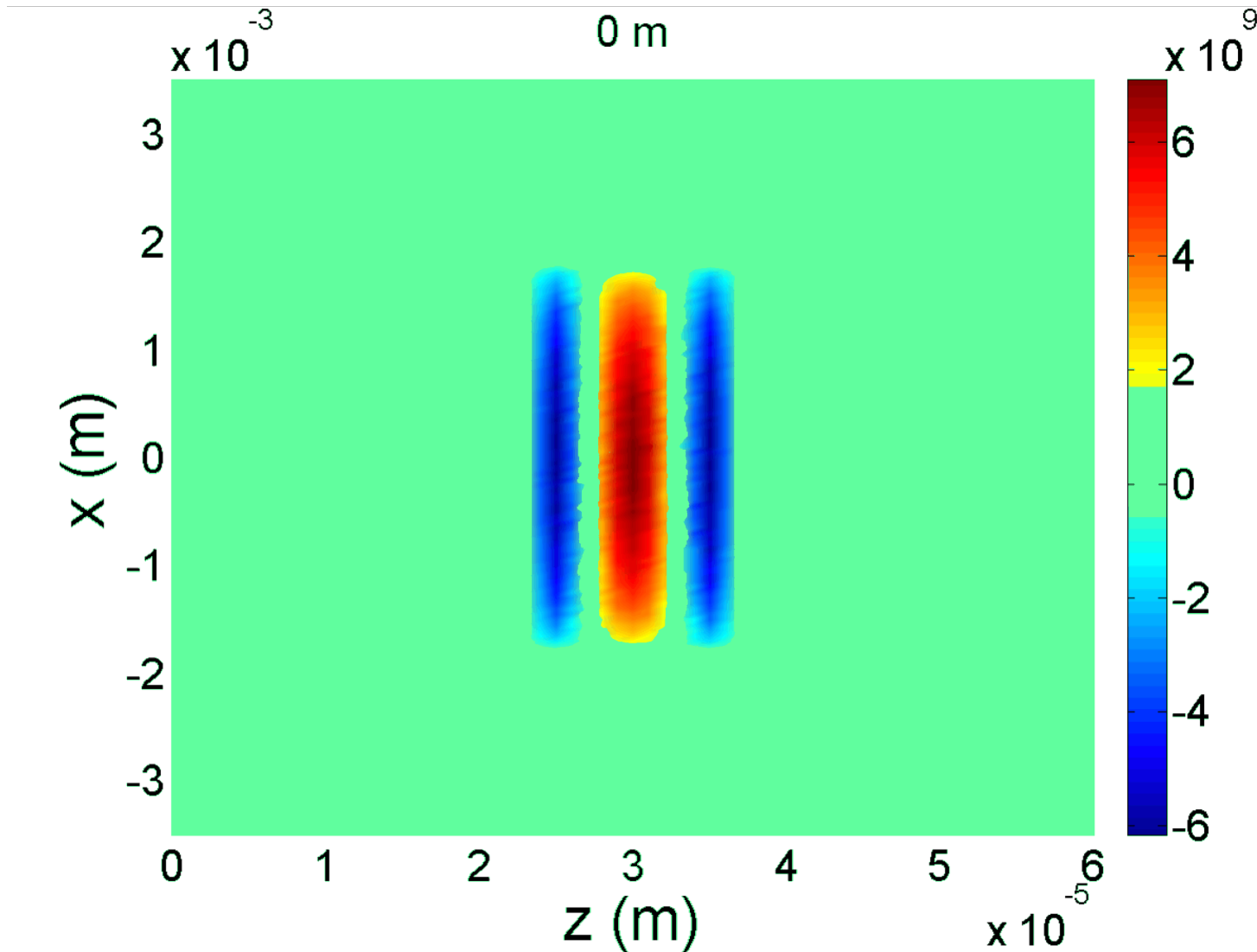- Cell length 2 m
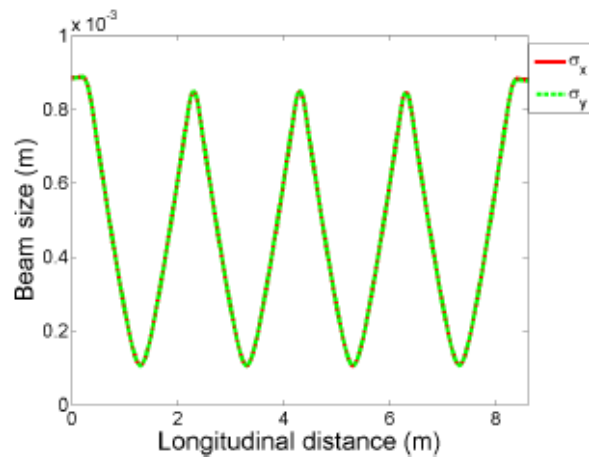
(a) Beam size
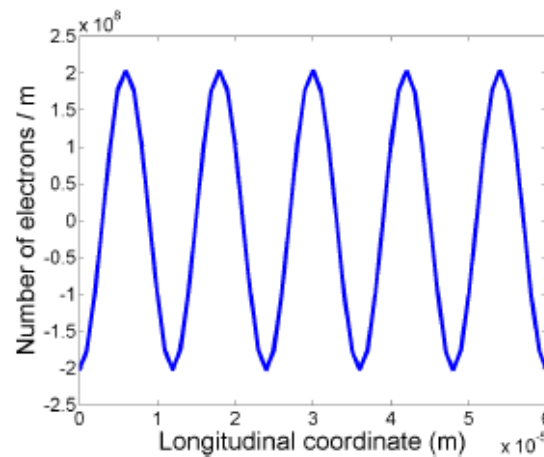
(b) Initial signal at 2.5e+13 Hz

(c) Gain of 2.5e+13 Hz signal
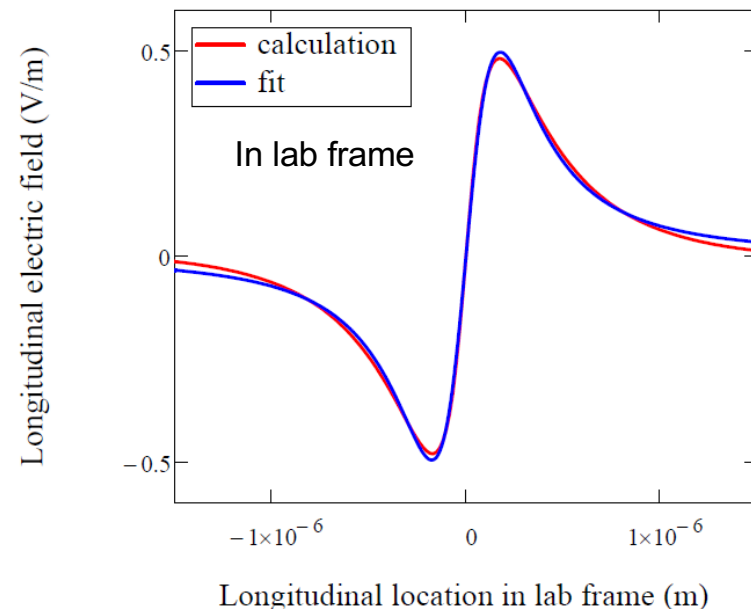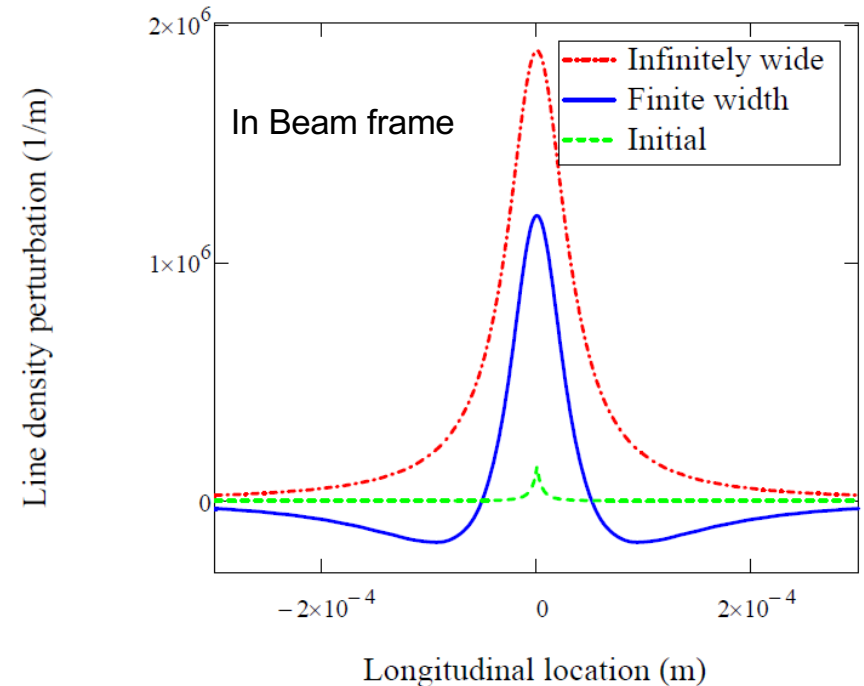
3D simulation of PCA

Preliminary Estimates for eRHIC

Electron beam parameters

Energy, γ	293.1
Bunch length, ps	50
Bunch charge, nC	12.5
RMS energy spread	1e-4
Beam width at modulator and kicker, mm	0.6
Minimal beam width at PCA, mm	0.1

Other parameters

Modulator length, m	40
PCA length, m	80
Kicker length, m	20
PCA gain	100

$$E_{fit}(z) = E_0 \frac{z}{\sigma_c} \cdot \left(1 + \frac{z^2}{\sigma_c^2}\right)^{-3/2}$$

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Preliminary Estimate for eRHIC

Ion beam parameters	
Energy, γ	293.1
RMS bunch length, ps	160
Bunch intensity	6E10
RMS energy spread	4.6e-4
R56 from modulator to kicker (drift), mm	0.93

1.5 hours of cooling

Local cooling time for ions with small synchrotron oscillation amplitude

$$E_{peak} = 0.48 V/m \qquad \sigma_{peak} = \frac{z_{peak}}{R_{56}} = 1.8 \times 10^{-4}$$

$$z_{peak} = 0.17 \mu m$$

$$\tau_{min} = \frac{T_{rev} \sigma_{peak} E_p}{e E_{peak} L_k} = 66.5 s$$
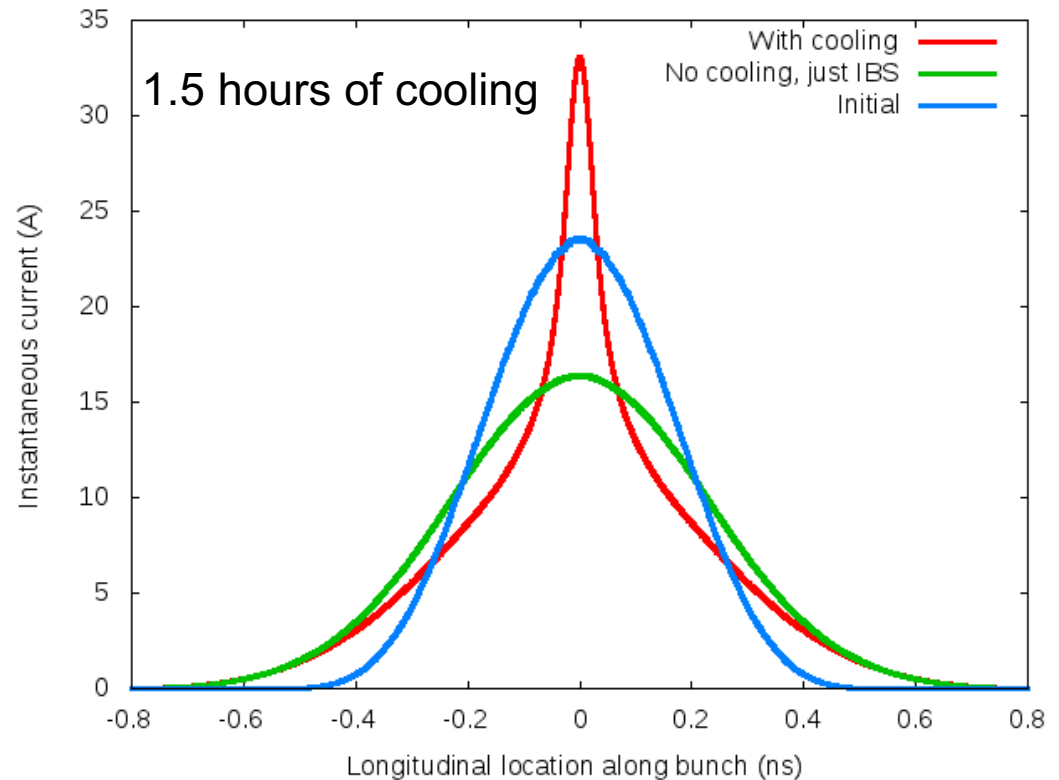
R_{56} of ions is about a factor of 3 larger than the ideal value (0.3 mm) and consequently ions with large energy deviation are not cooled efficiently. One possible way to increase the cooling range is 'electron painting'.
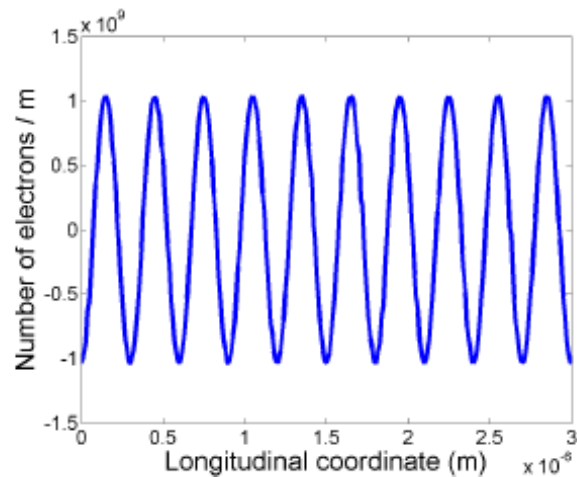
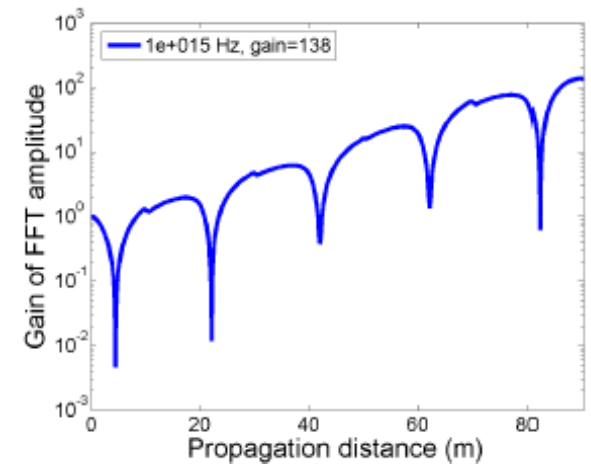# 3D Simulation of PCA Gain for eRHIC Set-up

© J. Ma

- Energy $\gamma = 275$, R.M.S. energy spread $\delta E/E = $1e-4
- Peak current 250 A
- Normalized envelope emittance 2 mm mrad, beam waist $a_o = $1e-4 m
- Cell length 20 m
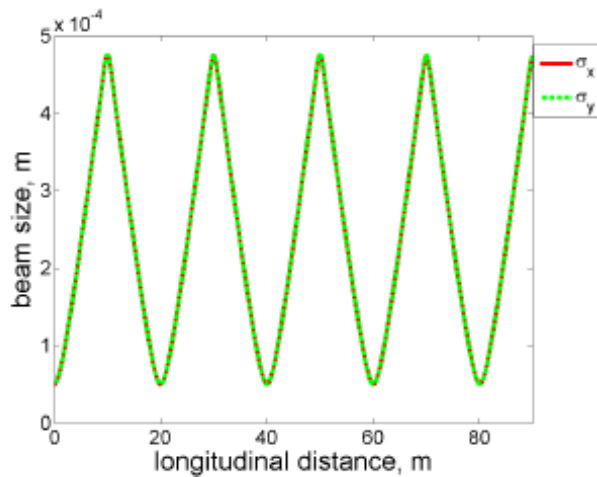
(a) Beam size

(b) Initial signal at 1e+15 Hz

(c) Gain of 1e+15 Hz signal
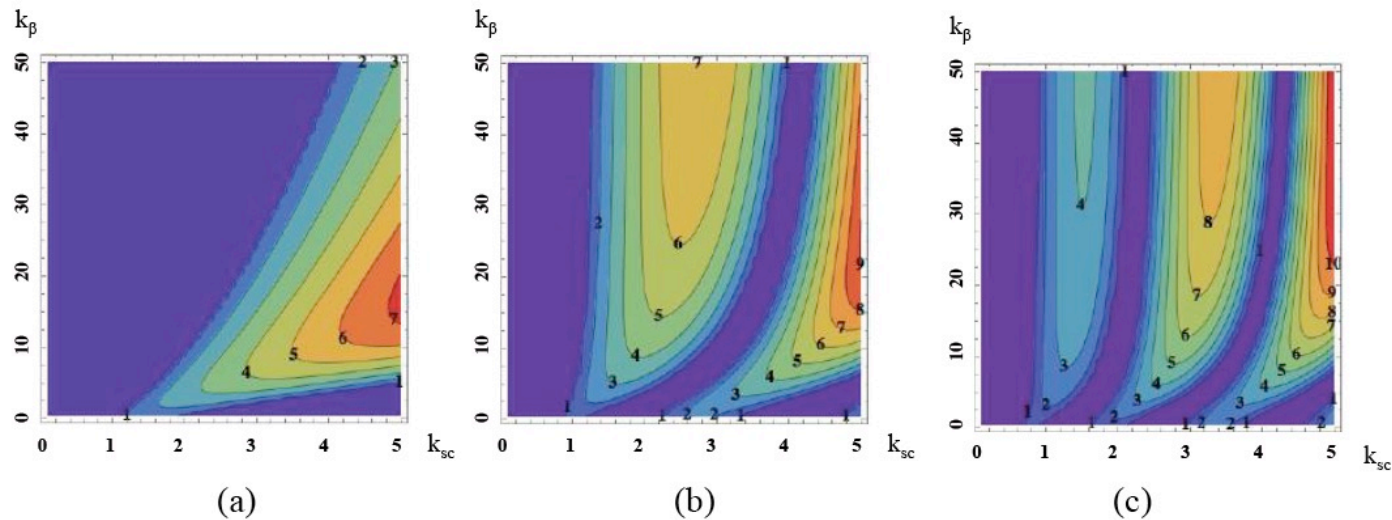
PCA with dedicated central Solenoids

Fig. 16. Comparison of PCA cell eigen values for a regular scheme (a) and scheme with dedicated central section (Fig. 4(a)) maintaining constant beam size: fig. (b) shows the case when the central section occupies $1/5^{th}$ of the cell and fig. (c) for the case of $1/3^{rd}$. We used similar program as shown in Appendix A, which included additional "focusing element".
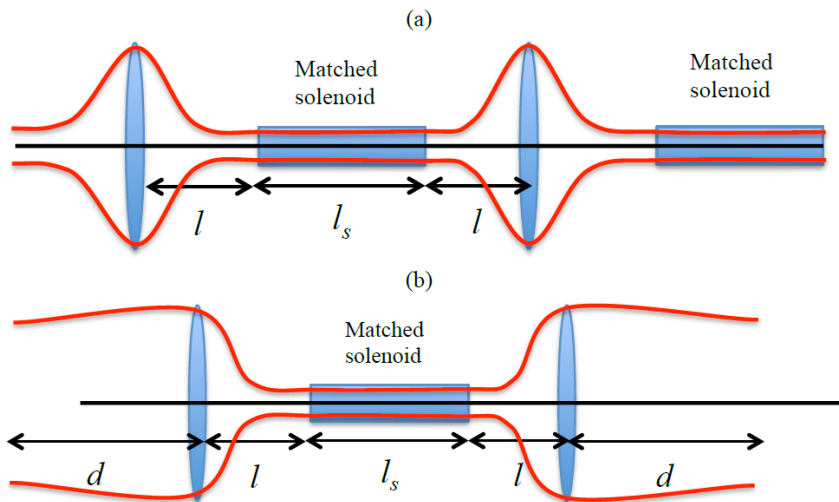

Figure 4. Alternative schemes providing for separation of the "strong focusing" in the cell center and "fast expansion" outside it (a) and also for reducing cell's chromaticity (b) by early interspersing the beam size expansion.

We also work on PCA schemes with dedicated central solenoids, which may relax the requirements on peak current and emittance of the electron beam.

Chicane-based CeC
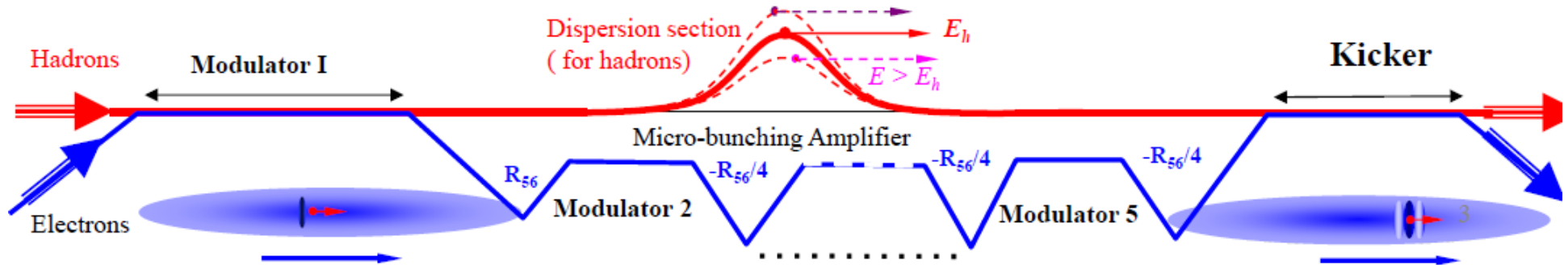
Enhanced bunching: single stage – VL, FEL2007

Micro-bunching: MB Amplifier, Single & Multi-stage, D. Ratner, PRL, 2013

Cooling rate for microbunched electron cooling without amplification, G. Stupakov, PRAB, 2018

Microbunched electron cooling with amplification cascade, G. Stupakov and P. Baxevanis, PRAB, 2019

Analysis of Chicane-based CeC: Single Ion Approach I

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Consider the following initial distribution of electrons (Spatially Beercan and Gaussian energy distribution)

$$f_1(r, \vec{p}_\perp, x_1, p_1) = n_o \cdot \theta(r-a) \cdot \theta(z-l_z) \cdot \frac{e^{-\frac{p_1^2}{2\sigma_{p_1}^2}}}{\sqrt{2\pi}\sigma_{p_1}} \cdot g(\vec{p}_\perp) \qquad p \equiv \frac{\delta\gamma}{\gamma_o}$$

After the modulation (kicked by a single ion) and the buncher, the distribution function is

$$f_2\left(\vec{r}_2, \vec{p}_2\right) = f_1(\vec{r}_1\left(\vec{r}_2, \vec{p}_2\right), \vec{p}_1\left(\vec{r}_2, \vec{p}_2\right));$$

$$p_2 = p_1 - \frac{Zr_e L_{\text{mod}}}{\gamma_o^3} \frac{x_1}{\left(r^2/\gamma_o^2 + x_1^2\right)^{3/2}}; \qquad x_2 = x_1 + D\left(p_1 - \frac{Zr_e L_{\text{mod}}}{\gamma_o^3} \frac{x_1}{\left(r^2/\gamma_o^2 + x_1^2\right)^{3/2}}\right)$$

The line density modulation at the exit of the buncher is thus

$$\rho(z) = 2\pi \int f_2(r, \vec{p}_\perp, z, p_2) r\, dr\, d\vec{p}_\perp^2 dp_2$$

$$= 2\pi \int \delta(z - x_2(r, x_1, p_1)) f_1(r, \vec{p}_\perp, x_1, p_1) r\, dr\, d\vec{p}_\perp^2 dp_1 dx_1$$

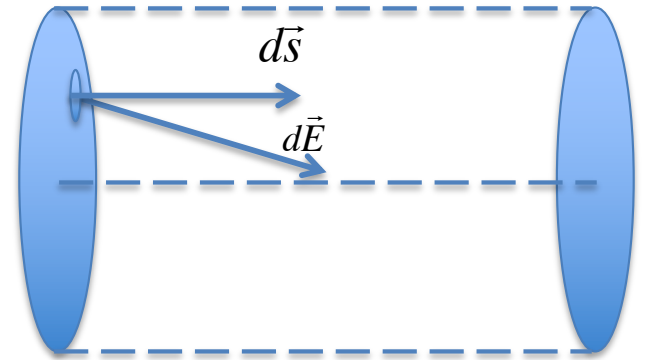# Analysis of Chicane-based CeC: Single Ion Approach II

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For the specific form of initial distribution, the integral can be reduced to

$$\rho(z) = 2\pi n_o \sigma_\gamma^2 D^2 \cdot \Psi_{u2}\left(\frac{\gamma_o z}{\sigma_\gamma |D| \sqrt{2}}, Z\frac{r_e L_{\text{mod}}}{2\sqrt{2}\sigma_\gamma^3 D|D|}, \frac{a}{\sqrt{2}\sigma_\gamma |D|}\right)$$

$$\Psi_{u2}(\zeta,\Xi,\tilde{a}) = \int_0^\infty q\, dq \left\{ \frac{Erf\left(q(1-\Xi q^{-3})+\zeta\right)+Erf\left(q(1-\Xi q^{-3})-\zeta\right)}{1-\Xi q^{-3}} - \frac{Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)+\zeta\right)+Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)-\zeta\right)}{1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}} \right\}$$

The electric field due to the density modulation can be calculated as follows (disc charge model)

$$E_{z,disc}(0,0,z_{lab}) = \frac{e\sigma z_{lab}}{2\varepsilon_0}\left[\frac{1}{\sqrt{z_{lab}^2+a^2/\gamma_0^2}} - \frac{1}{|z_{lab}|}\right]$$
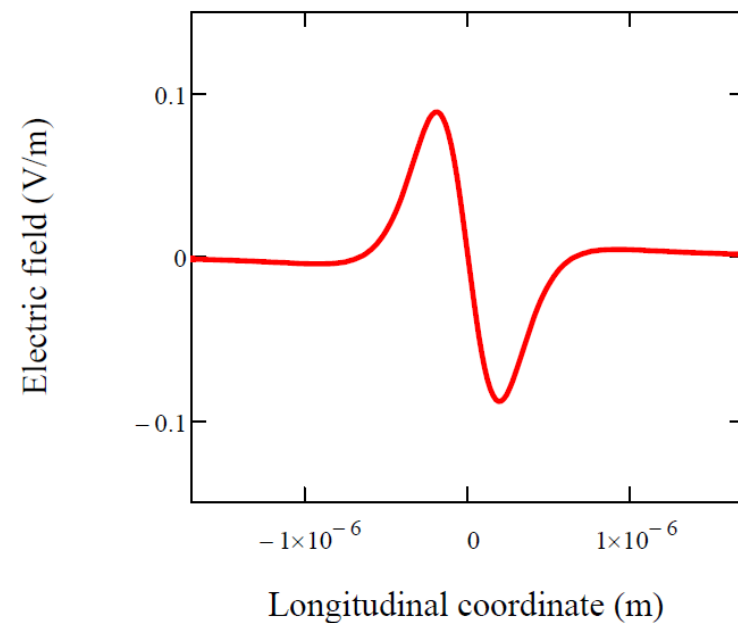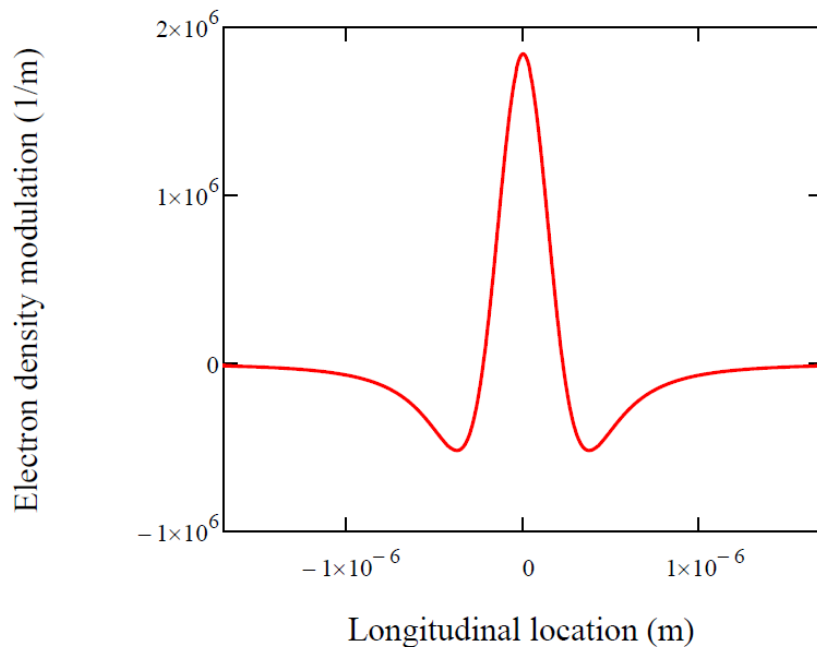
$$E_z(0,0,z) = \frac{1}{\pi a^2}\int_{-\infty}^{\infty} dz_1 \tilde{\rho}(z_1)\frac{1}{\sigma}E_{z,disc}(0,0,z-z_1) = \frac{\sqrt{2}e n_o \sigma_\gamma^3 D^2|D|}{\varepsilon_0 a^2 \gamma_o}\int_{-\infty}^{\infty}\Psi_{u2}(\zeta_1,\Xi,\tilde{a})\cdot\left[\frac{\tilde{z}-\zeta_1}{\sqrt{(\tilde{z}-\zeta_1)^2+\tilde{a}^2}} - \frac{\tilde{z}-\zeta_1}{|\tilde{z}-\zeta_1|}\right]d\zeta_1$$
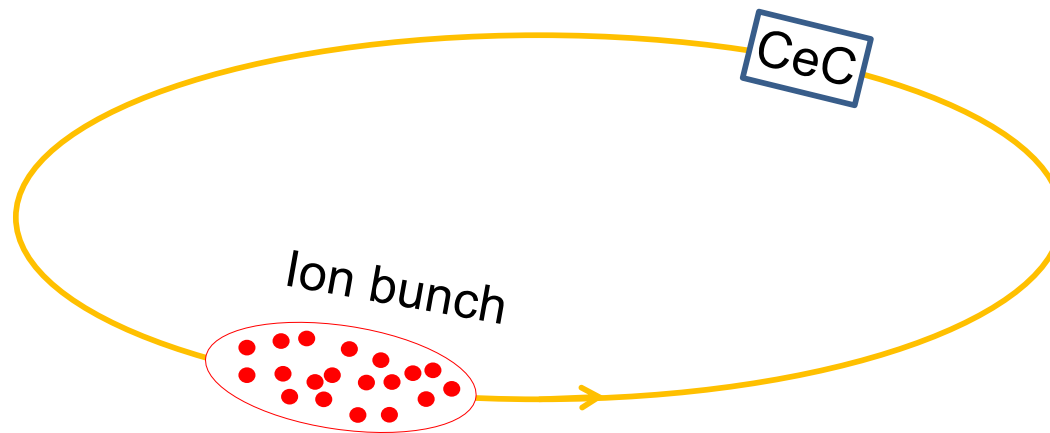
Analysis of Chicane-based CeC: Single Ion Approach III

The following shows an example of our analysis:

$$a = 0.1mm \qquad I_{peak} = 1A \qquad L_{kick} = 10m \qquad \sigma_{\delta,ion} = 6 \cdot 10^{-4}$$

$$L_{mod} = 10m \qquad \sigma_{\delta,e} = 10^{-4} \qquad D = 1.5mm \qquad E_p = 275GeV$$

Circulating Ion Beam Evolution in the Presence of CeC

We take two approach to predict the evolution of the ion bunch in the presence of CeC:

- Solving 1-D Fokker-Planck equation (analytical tool);
  - Very fast (a few minutes on a pc)
  - With limitations (currently work with linear cooling force, no beam losses from RF bucket, static diffusion coefficient…)

- Macro-particle tracking (simulation tool).
  - Time consuming (a few hours on a pc)
  - More realistic and versatile

How to evaluate CeC: the original recipe

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- Linear response of electron beam on perturbations – no saturation, superposition principle

Field induced by an ion

$$\delta \vec{\mathbf{E}}_h = Ze \cdot \vec{\mathbf{G}}_{Eh}\left(\vec{r},\vec{r}_h,\gamma_h,t,t_h\right); \delta \vec{\mathbf{B}}_h = Ze \cdot \vec{\mathbf{G}}_{Bh}\left(\vec{r},\vec{r}_h,\gamma_h,t,t_h\right);$$

Field induced by an electron

$$\delta \vec{\mathbf{E}}_e = -e \cdot \vec{\mathbf{G}}_{Ee}\left(\vec{r},\vec{r}_e,\gamma_e,t,t_e\right); \delta \vec{\mathbf{B}}_e = -e \cdot \vec{\mathbf{G}}_{Be}\left(\vec{r},\vec{r}_e,\gamma_e,t,t_e\right);$$

Field induced by all ions and electrons

$$\vec{\mathbf{E}} = Ze \cdot \sum_h \vec{\mathbf{G}}_{Eh}\left(\vec{r},\vec{r}_h,\gamma_h,t,t_h\right) - e \cdot \sum_e \vec{\mathbf{G}}_{Ee}\left(\vec{r},\vec{r}_e,\gamma_e,t,t_e\right);$$

$$\vec{\mathbf{B}} = Ze \cdot \sum_h \vec{\mathbf{G}}_{Bh}\left(\vec{r},\vec{r}_h,\gamma_h,t,t_h\right) - e \cdot \sum_e \vec{\mathbf{G}}_{Be}\left(\vec{r},\vec{r}_e,\gamma_e,t,t_e\right)$$

- Energy and momentum kick received by an ion in the cooling section

$$\delta E_i = eZ \int \vec{\mathbf{E}} \cdot d\vec{r}_i; \quad \delta \vec{p}_i = eZ \int \left(\vec{\mathbf{E}} + \frac{\left[\vec{p}_i \times \vec{\mathbf{B}}\right]}{\gamma_i m} \right) \cdot dt;$$

- Evaluation of hadron distribution function using Fokker-Plank equation with both damping and diffusion terms

$$\bar{f} = \langle \tilde{f} \rangle; \tilde{f} = \sum_h \delta\left(X - X_i(t)\right)$$

$$\frac{\partial \bar{f}(X,s)}{\partial t} + \frac{\partial}{\partial X_i}\left[\frac{dX_i(X,t)}{dt}\bar{f}(X,s) \right] - \frac{1}{2}\frac{\partial^2}{\partial X_i \partial X_k}\left[D_{ik}(X,t)\bar{f}(X,t) \right] = 0,$$

$$\left\langle \frac{dX_i(X,t)}{dt} \right\rangle = \frac{1}{\tau}\int (X_i - Z_i) \cdot W(Z,X|\tau,t)dZ = \frac{1}{T_o}\langle \delta X_i \rangle;$$

$$D_{ik}(X,t) = \frac{1}{2\tau}\int (X_i - Z_i)(X_k - Z_k) W(Z,X|\tau,t)dZ = \frac{1}{T_o}\langle \delta X_i \cdot \delta X_i \rangle.$$

Analytical tools for predicting the influences of CeC to a circulating ion beam I

- Evolution of the longitudinal phase space density of the ion bunch, after averaging over the synchrotron oscillation phase, follows the 1-D Fokker-Planck equation

$$\frac{\partial}{\partial t}F(I,t) - \frac{\partial}{\partial I}(\zeta(I)\cdot I \cdot F(I,t)) - \frac{\partial}{\partial I}\left(I \cdot D(I)\cdot \frac{\partial F(I,t)}{\partial I}\right) = 0$$

- In the limit of $D(I) = 0$, an analytical solution can be derived for the following form of cooling profile and initial condition,

$$\zeta(I) = \zeta_0 \frac{I_e}{I+I_e} \qquad F_0(I) = \exp\left(-\frac{I}{I_{ion}}\right)$$

$P_{\log}(x)$ is called product logarithm function and can be directly evaluated in Mathematica.

as

$$F(I,t) = \left(1+\frac{I_e}{I}\right)\frac{P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right)\exp\left(\frac{-I_e}{I_{ion}}P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right)\right)}{1+P_{\log}\left(\frac{I}{I_e}\exp\left(\zeta_0 t + \frac{I}{I_e}\right)\right)}$$

# Analytical tools for predicting the influences of CeC to a circulating ion beam II

- The longitudinal line density of ion bunch is given by

$$\rho_{ion}(t,z) = \int_{-\infty}^{\infty} F(z^2 + \delta^2, t)\,d\delta$$

- For $D(I) \neq 0$, the 1-D Fokker-Planck equation can be solved numerically with arbitrary form of cooling rate and initial ion distribution.

- The analytical studies reveals the fact that the central blips due to local cooling tends to be smeared out by diffusive kicks from IBS and more significantly, from incoherent kicks induced by neighbor ions.
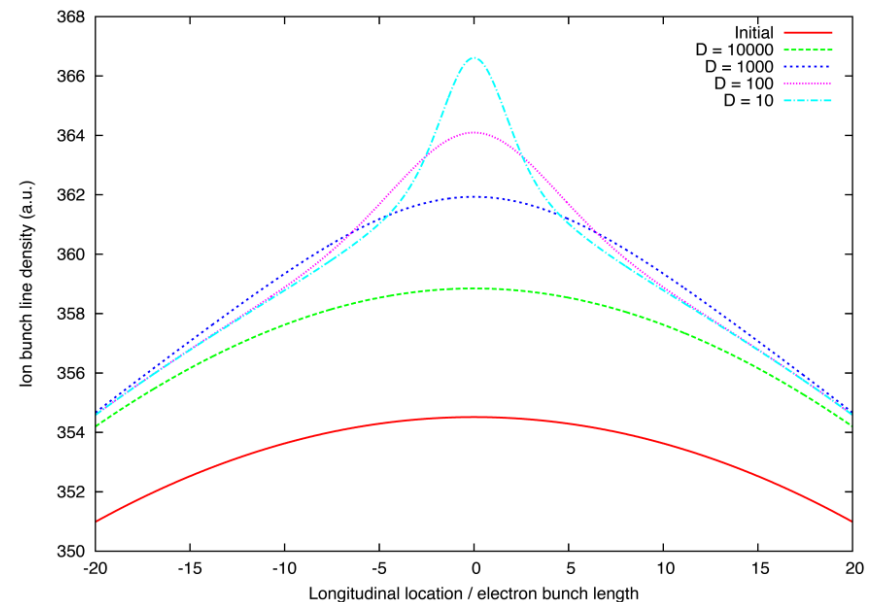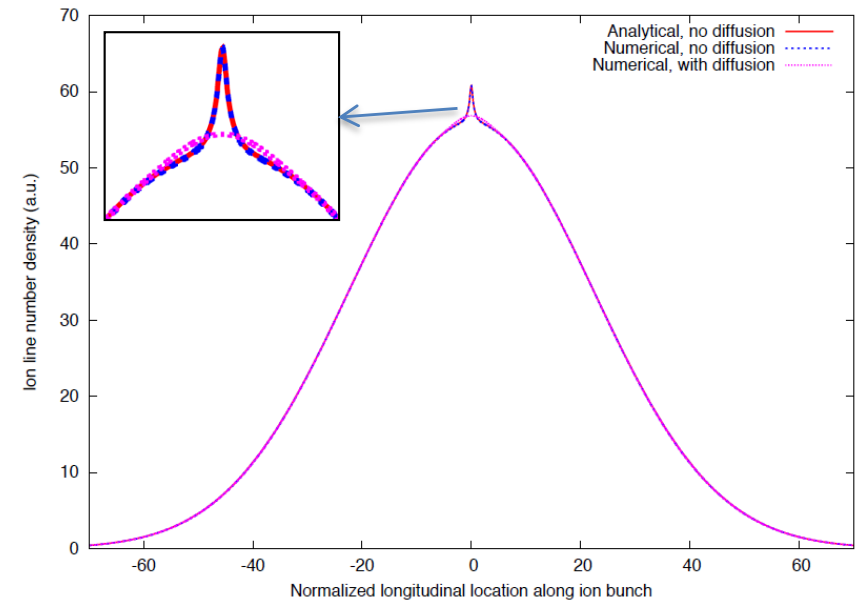
Simulation tools for predicting the influences of CeC to a circulating ion beam I
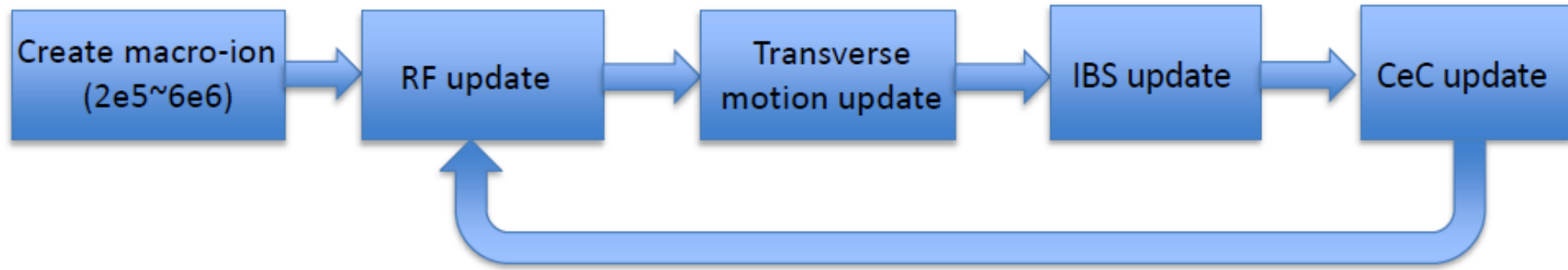
Energy kicks from CeC is $\Delta E_j = \Delta E_{coh,j} + \Delta E_{inc,j}$

Coherent kick induced by the ion itself $\quad \Delta E_{coh,j} \equiv -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right)$

Incoherent kick induced by the neighbor ions (using the Gaussian profile as obtained by quadratic expansion of FEL eigenvalues)

$$\Delta E_{inc,j} \equiv -Z_i e E_p l \sum_{i \neq j} \exp\left[-\frac{\left(z_j - z_i\right)^2}{2\sigma_{z,rms}^2}\right] \sin\left(k_0\left(D\delta_j + z_j - z_i\right) - k_2^2\left(z_j - z_i\right)^2\right)$$

z_i: longitudinal location of the ith ion; $\quad \sigma_{z,rms}$: RMS width of the wave-packet; $\quad D : R_{56}$.

Since there is no correlation between any successive incoherent kicks, one can use a random kick to represent the incoherent kicks

For a random number uniformly distributed between -1 and 1

$$\Delta E_{j,N} \approx -Z_i e E_p l_1 \sin\left(k_0 D \cdot \delta_j\right) + \sqrt{\frac{\left\langle \Delta E_{inc,j}^2 \right\rangle}{\left\langle X^2 \right\rangle}} \cdot X_{j,N}$$

$$\left\langle X^2 \right\rangle = \frac{1}{2}\int_{-1}^{1} X^2 dX = \frac{1}{3}$$

Simulation tools for predicting the influences of CeC to a circulating ion beam II

- Assuming the ion density does not vary significantly over the width of the wave-packet

$$\left\langle \Delta E_{inc,j}^2 \right\rangle = \frac{\left(Z_i e E_p l_1\right)^2}{2} \int_{-\infty}^{\infty} \rho_{ion}(z_i) e^{-\frac{(z_i - z_i)^2}{\sigma_{z,rms}^2}} dz_i \approx \frac{\left(Z_i e E_p l_1\right)^2}{2} \sqrt{\pi} \rho_{ion}(z_j) \sigma_{z,rms}$$
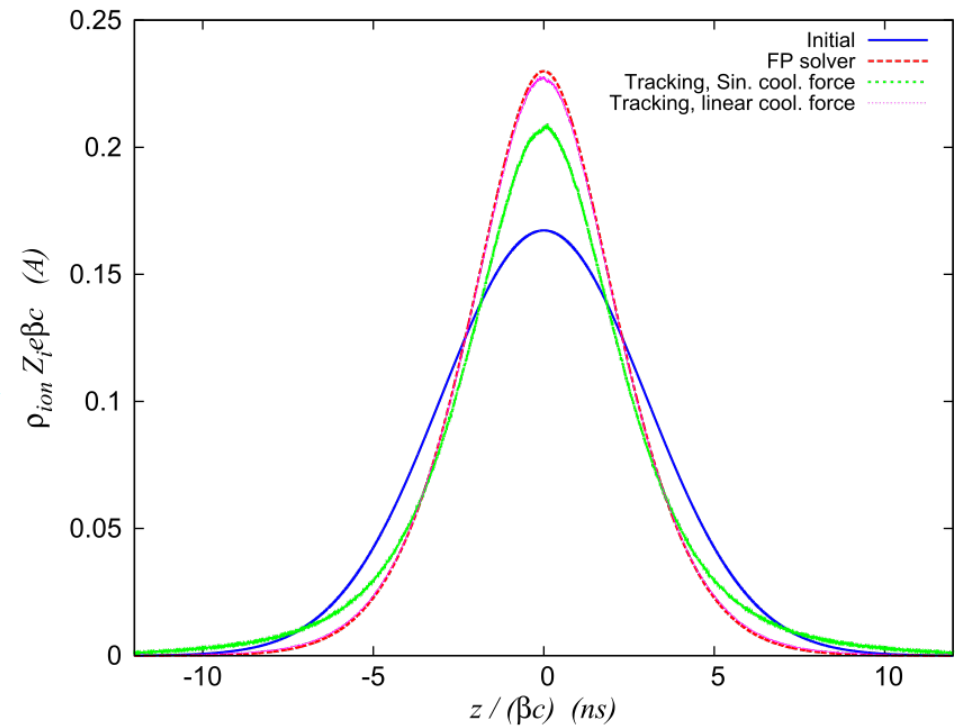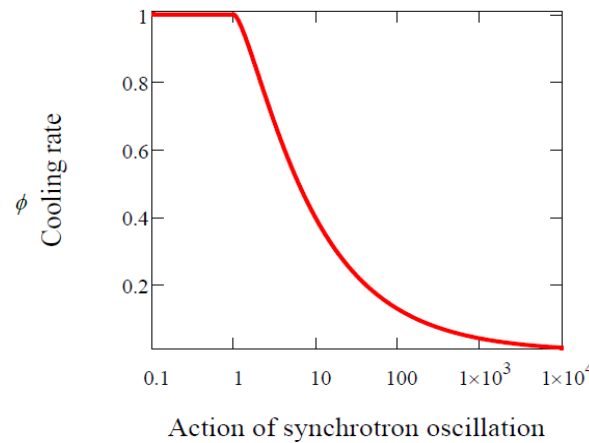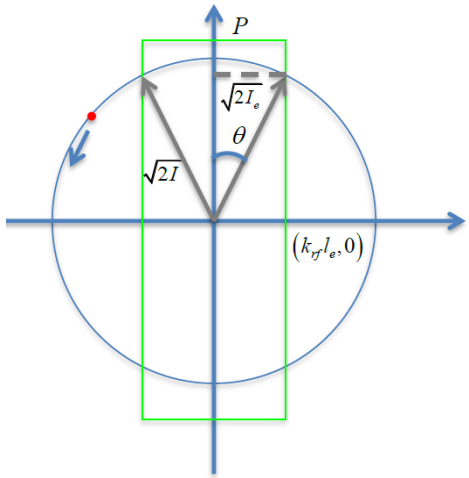
- The one-turn energy kick due to CeC is

$$\Delta E_{j,N} \approx -Z_i e E_p l \sin\left(k_0 D \cdot \delta_j\right) + Z_i e E_p l \sqrt{\frac{3}{2} \sqrt{\pi} \rho_{ion}(z_j) \sigma_{z,rms}} \cdot X_{j,N} + \Delta E_{j,N}^e$$

Diffusive kick induced by neighbor electrons, i.e. electrons' shot noise

$$\Delta E_{j,N}^e \approx e E_p l \sqrt{\frac{3}{2} \sqrt{\pi} \rho_e(z_j) \sigma_{z,rms}} \cdot Y_{j,N}$$

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Comparison with Macro-particle Tracking

$$\bar{\zeta}(r) = \begin{cases} \dfrac{2}{\pi}\arcsin\left(\dfrac{1}{r}\right) + \dfrac{2}{\pi r^2}\sqrt{r^2-1}; & \text{for } r \geq 1 \\ 1 & ; \quad \text{for } r < 1 \end{cases}$$

Some limitations of the Fokker-Planck solver:

1. The Fokker-Planck solver assumes constant cooling rate (linear cooling force) inside electron bunch while macro-particle tracking assume sinusoidal cooling force;
2. The Fokker-Planck solver does not account for particle leakage from rf bucket or other rf-related effects.
3. The diffusion coefficients applied in Fokker-Planck equation does not change with time.

How to cool transversely : a simple case

Only energy kick

$$\Delta x_6 = \frac{\delta E_h}{E_o} = const - \sum_{i=1}^{6} \zeta_i \cdot x_i$$

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$$X = \frac{1}{2}\sum_{k=1}^{3}\left(a_k Y_k(s)e^{i\psi_k} + c.c.\right); \quad \mathbf{Y}_{k=1,2} = \begin{pmatrix} Y_{k\beta} \\ -Y_{k\beta}^T SD \\ 0 \end{pmatrix}; \quad \mathbf{Y}_3 \cong \frac{1}{\sqrt{\Omega}}\begin{pmatrix} D \\ -i\Omega \\ 1 \end{pmatrix}; \quad Y_{k\beta} = \begin{bmatrix} y_{k1} \\ y_{k1} \\ y_{k2} \\ y_{k4} \end{bmatrix}; \quad D = \begin{bmatrix} D_x \\ D'_x \\ D_y \\ D'_y \end{bmatrix};$$

$$\langle \Delta a_k \rangle = -\xi_k a_k \rightarrow a_k = a_{k0}e^{-n\xi_k} \qquad \mathrm{Re}\,\xi_{(1,2)} = -\frac{i}{2}\left(Y_{(1,2)\beta}^{\ T}SD\right)^* \sum_{i=1}^{4} y_{(1,2)i}\cdot\zeta_i; \quad \xi_s = \mathrm{Re}\,\xi_3 = \frac{1}{2}\left(\zeta_6 + \sum_{i=1}^{4} D_i\cdot\zeta_i\right),$$

No x-y coupling

$$\beta_{x,y} = \mathrm{w}_{x,y}^2; \quad \alpha_{x,y} = -\mathrm{w}'_{x,y}\mathrm{w}_{x,y}$$

$$Y_{1\beta} \equiv Y_x = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} \\ 0 \\ 0 \end{bmatrix}; \quad Y_{2\beta} \equiv Y_y = \begin{bmatrix} 0 \\ 0 \\ \mathrm{w}_y \\ \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \end{bmatrix}; \quad D = \begin{bmatrix} D \\ D' \\ 0 \\ 0 \end{bmatrix};$$

$$\xi_x = \mathrm{Re}\,\xi_1 = -\left(D\zeta_1 + D'\zeta_2\right); \quad \xi_s = \xi_6 - \xi_x.$$

Qx-Qy resonance

$$Y_1 = \frac{1}{\sqrt{1+|\alpha|^2}}\left(Y_x + \alpha Y_y\right); \quad Y_2 = \frac{1}{\sqrt{1+|\alpha|^2}}\left(-\alpha^* Y_x + Y_y\right) \qquad \mathrm{Re}\,\xi_1 = -\frac{D\zeta_1 + D'\zeta_2}{1+|\alpha|^2}; \mathrm{Re}\,\xi_2 = -|\alpha|^2\frac{D\zeta_1 + D'\zeta_2}{1+|\alpha|^2}.$$

Can use a non-achromatic transport (time of flight dependence)
or transverse beam separationto couple longitudinal and transverse cooling

Distribution of cooling between longitudinal and transverse degrees of freedom – linearized kick
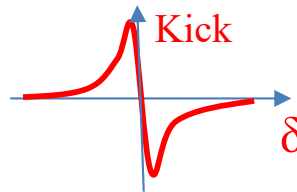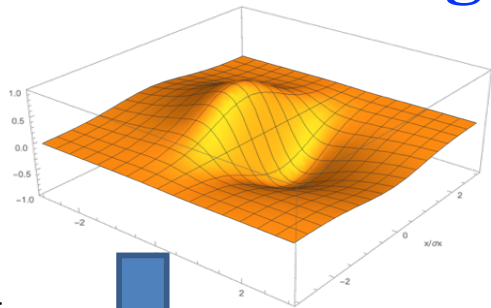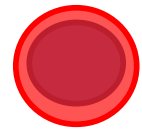
$$\frac{\delta E_h}{E_o} = const - \zeta_1 x - \zeta_6 \frac{E_h - E_o}{E_o}; \quad r = D\zeta_1 / \zeta_6$$
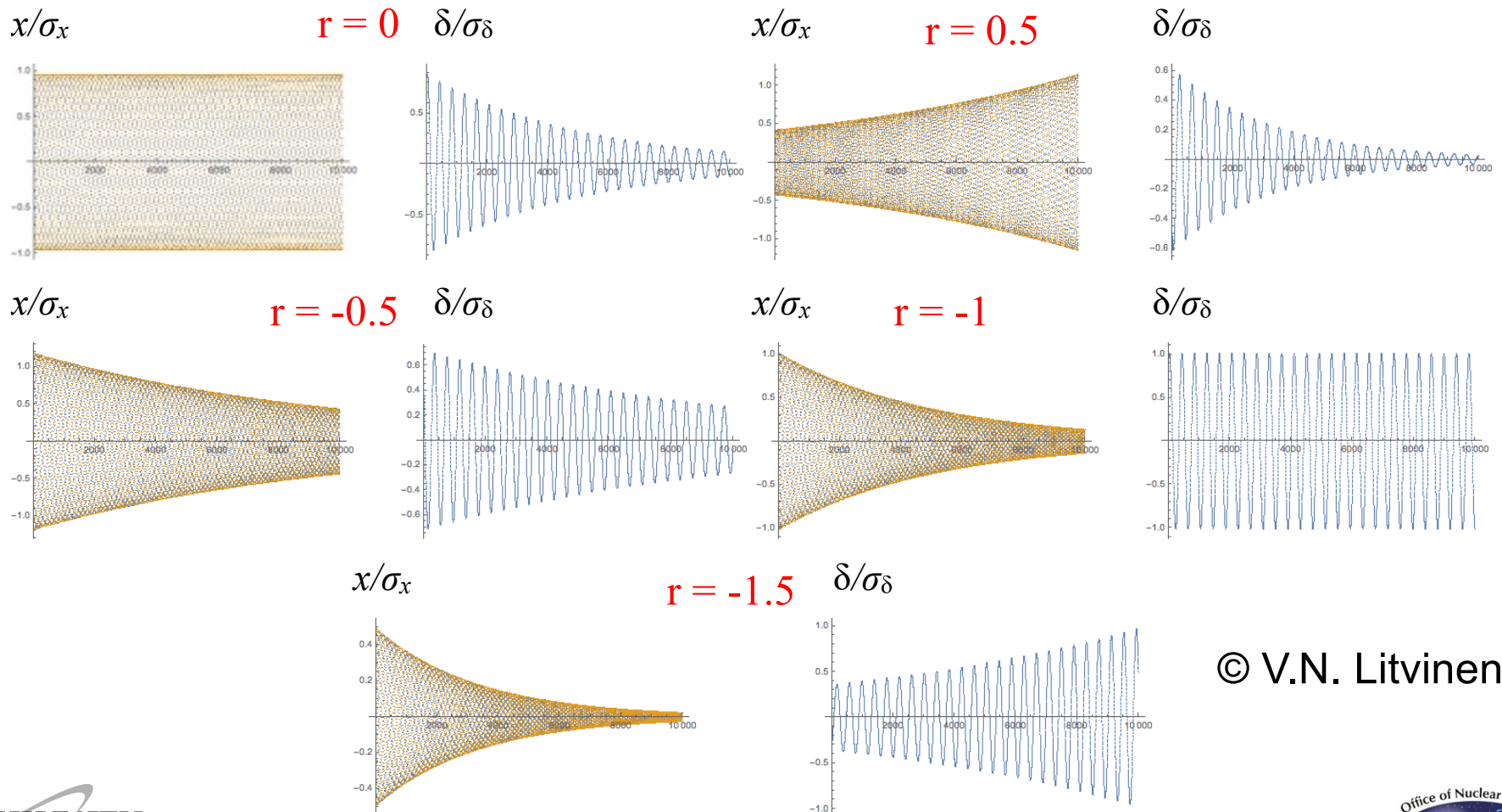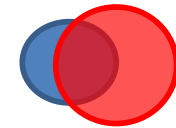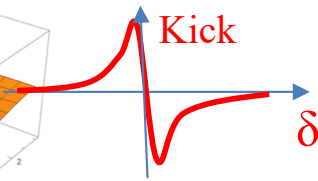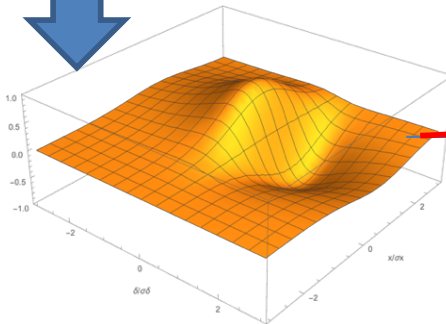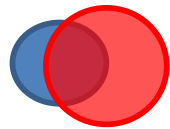

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Distribution of cooling between longitudinal and transverse degrees of freedom – real kick

$\Delta x = 0.75\sigma_x$ zero energy kick at $0.4\sigma_\delta$

Kick

δ

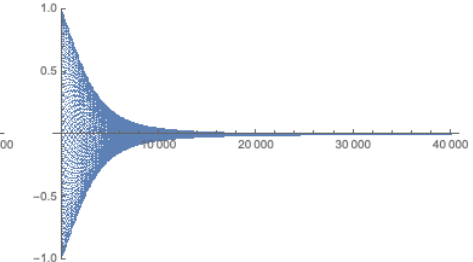x/σ_x

δ/σ_δ

Wrong sign of displacement
$\Delta x = -0.75\sigma_x$

x/σ_x

δ/σ_δ
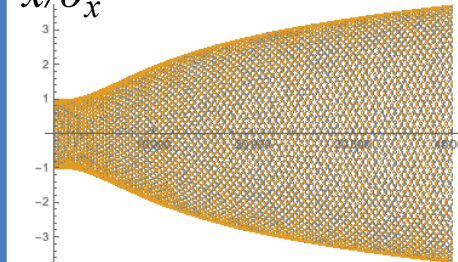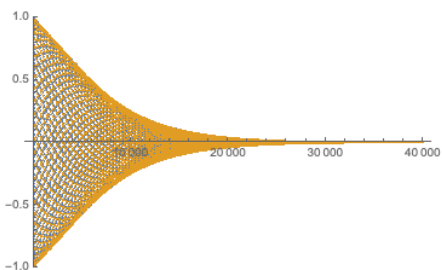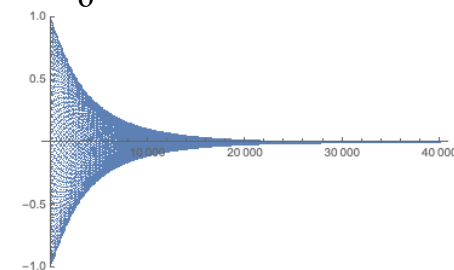
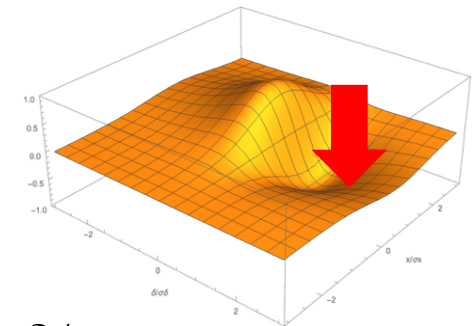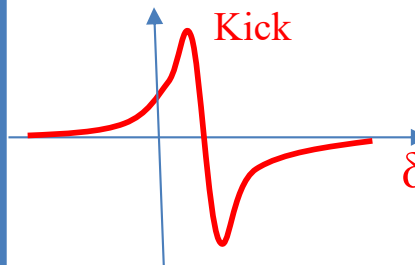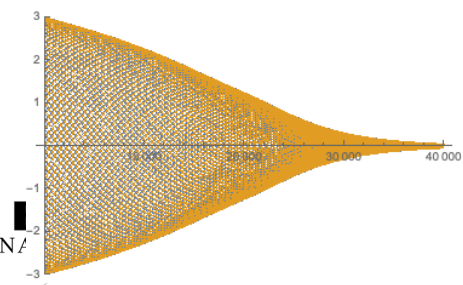Excessive shifting of zero-kick point to $\delta = 0.6\sigma_\delta$

Kick

δ

x/σ_x

δ/σ_δ

x/σ_x

δ/σ_δ
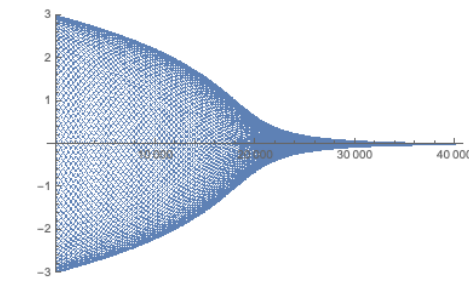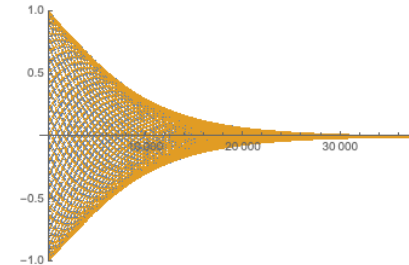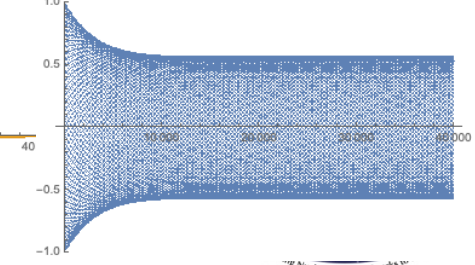
Summary

- Theoretical tools and start-to-end simulations for FEL-based CeC have been developed in the past few year;

- We are currently working on improving theoretical model for PCA-based CeC as well as conducting start-to-end 3-D simulations with code SPACE;

- Preliminary estimate of eRHIC cooling shows that 'electron painting' may be needed to cool ions with large synchrotron amplitude with PCA-based CeC;

- We are also working on PCA schemes with central solenoid which could reduce the peak current required to cool eRHIC proton beam.

- We show that transverse cooling can be achieved by displacing electron beam w.r.t the ion beam.

Backup slides

Future works

- Since PCA-based CeC is planned to be tested in the next two years, it is our main focus to complete and improve the existing model and simulations:
    - Theoretical studies
   - Develop PCA model for electron bunch with initial energy chirp and accelerator;
        - Develop 3-D model for PCA (Plasma oscillation with finite transverse size, transverse mode…);
        - Investigate how transverse-longitudinal coupling affects PCA;
        - Exploring how cooling performance scales with various beam/accelerator parameters and searching for optimal settings;
        - Continue works on evaluating transverse cooling;
        - Study PCA process with central solenoid (important for reducing requirements on electron beam for cooling EIC);
    - 3-D simulation with SPACE
        - Start-to-end simulation for PCA-based CeC test at RHIC;
        - Generating single-pass kick as a function of ion's 6-D coordinate;
        - Sensitivity studies and optimizations for the PCA-based CeC test at RHIC;
        - Incorporating bunch compression and acceleration into SPACE;
        - Simulating PCA process with central solenoids.
- For predicting the evolution of ion beam with cooling, further improvements include solving Fokker-Planck equation for non-linear cooling force and including transverse cooling into ion tracking code.

3D simulation of PCA
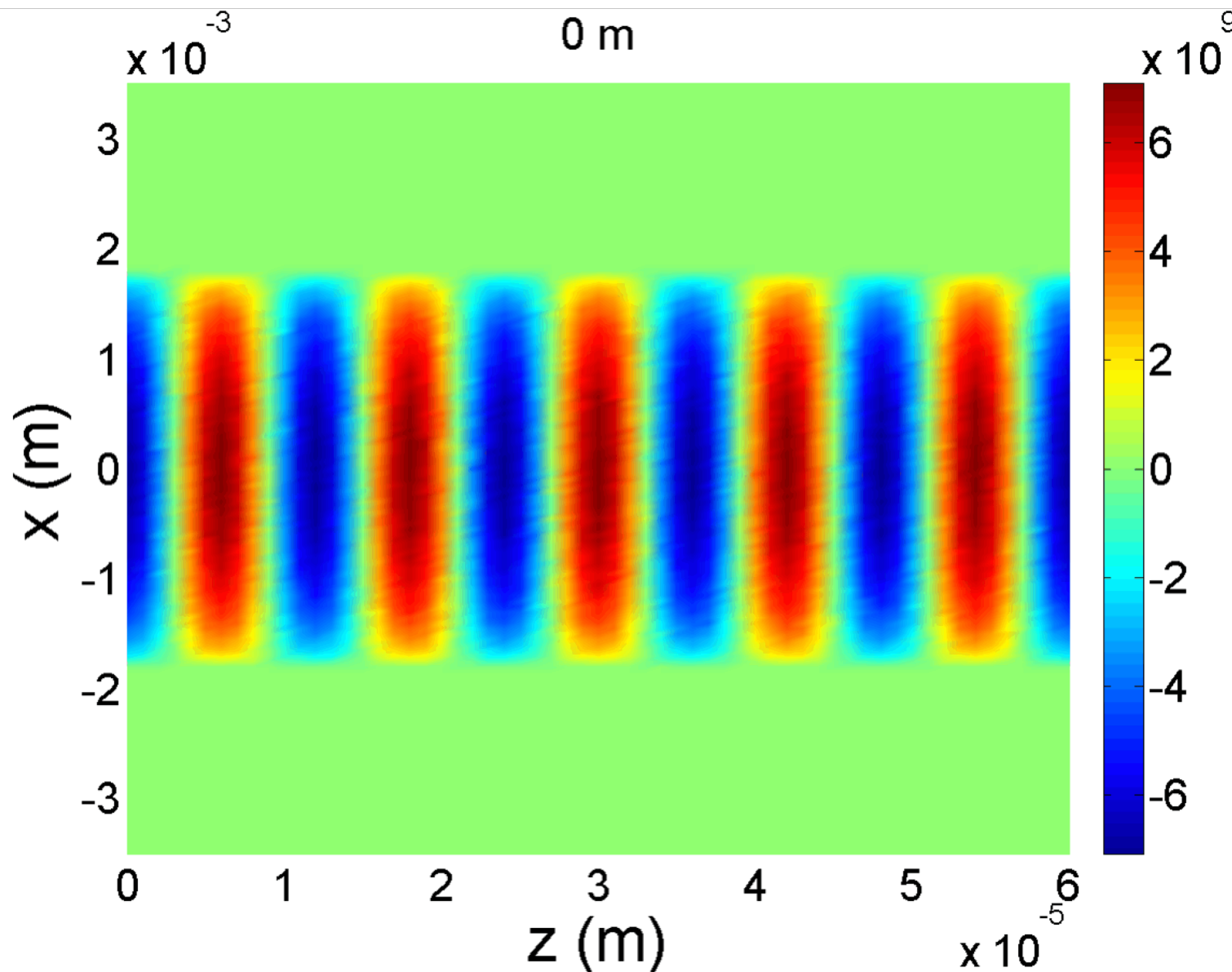
Approach Used in PRL 111, 084802 (2013) by D. Ratner

Stage 1 — Stage 2 — ... — Stage 5 — Kicker
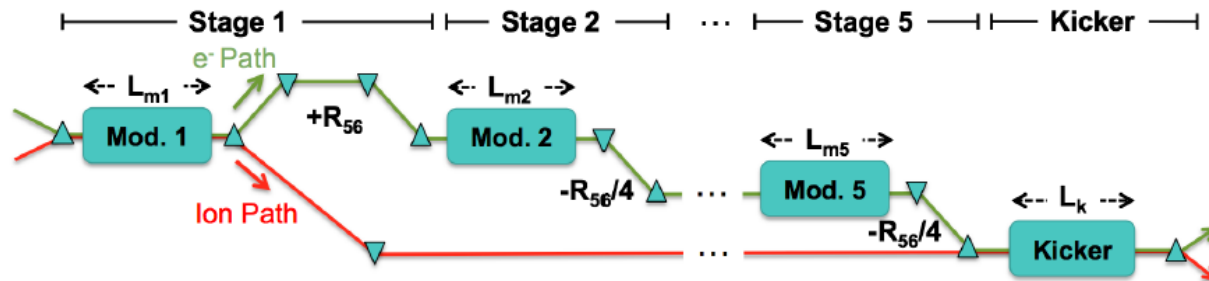
© D. Ratner
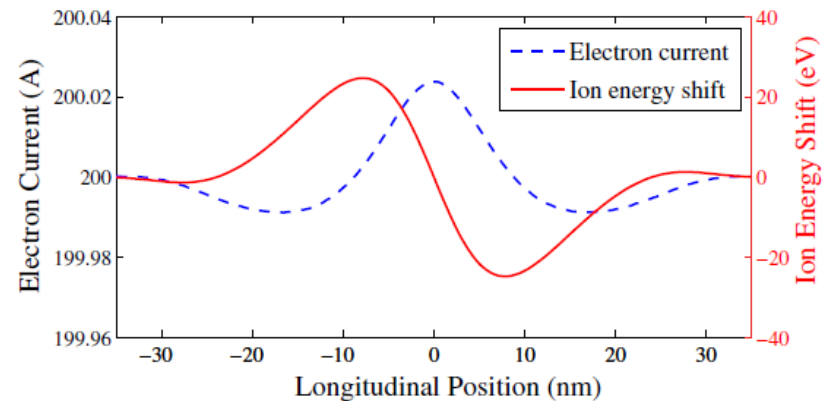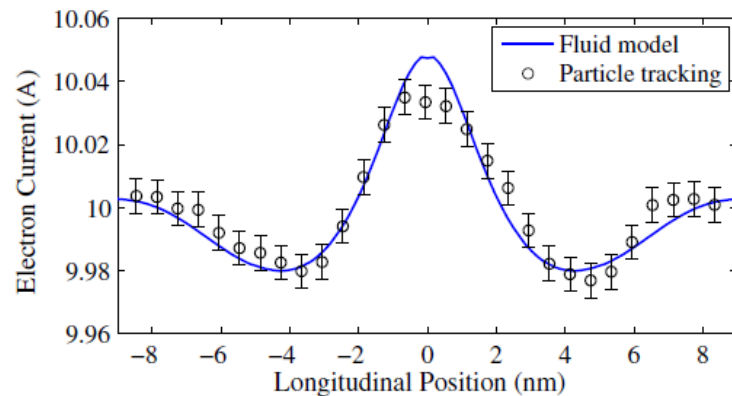
$$A_1 \equiv -2cq\dot{L}_m/a^2 I_A$$

$$A_2 \equiv -qIL_k/2\epsilon_0 c\pi a^2,$$

$$L_m \gg \beta \qquad M(z) = \frac{-2cqL_m}{\gamma a^2 I_A}\left[\frac{z}{|z|} - \frac{\gamma z}{\sqrt{a^2 + \gamma^2 z^2}}\right],$$

$$\langle W_I(z_I)\rangle = \frac{|R_{56}A_1|A_2}{\gamma}\int_{-\infty}^{\infty} d\xi \frac{e^{-\xi^2/2\sigma_\xi^2}}{\sqrt{2\pi}\sigma_\xi}\int_{-\infty}^{\infty} d\zeta$$

'Disc' model

$$\times\left(\iota(\zeta,\xi) - \frac{[\zeta_I - \tilde{\zeta}(\zeta,\xi)]}{\sqrt{\alpha^2 + [\zeta_I - \tilde{\zeta}(\zeta,\xi)]^2}}\right),$$

$$L_m \ll \beta \qquad M(r,z) = \frac{-cqL_m}{\gamma I_A}\frac{\gamma z}{[r^2 + \gamma^2 z^2]^{3/2}}.$$

Left plot: Electron Current (A) vs Longitudinal Position (nm); legend: Fluid model, Particle tracking.

Right plot: Electron Current (A) and Ion Energy Shift (eV) vs Longitudinal Position (nm); legend: Electron current, Ion energy shift.

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Debye shielding in an uniform electron plasma with anisotropic velocity distribution

The system can be described by linearized Vlasov-Maxwell equations

$$\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t) + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t) - \frac{e\vec{E}}{m_e}\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_o}\left\{Ze\delta(\vec{x}) - e\tilde{n}_1(\vec{x},t)\right\}$$

$$\tilde{n}_1(\vec{x},t) = \int f_1(\vec{x},\vec{v},t)d^3v.$$

$$f_0(\vec{v}) = \frac{1}{\pi^2\beta_x\beta_y\beta_z}\left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2}\right)^{-2}$$

In 3-D Fourier domain, the equations reduces to a non-homogeneous 2nd ODE

$$\frac{d^2}{dt^2}\tilde{H}_1(\vec{k},t) + \omega_p^2\tilde{H}_1(\vec{k},t) = \omega_p^2 Z_i e^{-\lambda(\vec{k})t}$$

$$\tilde{H}_1(\vec{k},t) = \tilde{n}_1(\vec{k},t)e^{-\lambda(\vec{k})t}$$

The solution for zero initial density and velocity modulation in Fourier domain can be found

$$\dot{\tilde{n}}_1(\vec{k},t) = Z_i\omega_p \sin(\omega_p t)e^{\lambda(\vec{k})t}$$

$$\lambda(\vec{k}) = i\vec{k}\cdot\vec{v}_0 - \sqrt{(k_x\beta_x)^2 + (k_y\beta_y)^2 + (k_z\beta_z)^2}$$

By inverse Fourier transformation, we obtain the density modulation in space domain

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 a_x a_y a_z}\int_0^{\omega_p t} \frac{\tau\sin\tau\cdot d\tau}{\left[\tau^2 + \left(\frac{x}{a_x} + \frac{v_{0,x}}{\beta_x}\tau\right)^2 + \left(\frac{y}{a_y} + \frac{v_{0,y}}{\beta_y}\tau\right)^2 + \left(\frac{z}{a_z} + \frac{v_{0,z}}{\beta_z}\tau\right)^2\right]^2}$$

1-D integral for energy modulation

$$I_d(z,t) = -\frac{Z_i e \omega_p^2}{\pi} \int_0^t d\tau (z + v_{0,z}\tau) \left\{ \frac{a_z \sin(\omega_p \tau)}{\left[\bar{\beta}^2\tau^2 + (z + v_{0,z}\tau)^2\right]\left[1 + \bar{\beta}^2\tau^2 + (z + v_{0,z}\tau)^2 / a^2\right]} \right.$$

$$\left. - \cos(\omega_p \tau) \left[\frac{\arctan\left(|z + v_{0,z}\tau| / (\bar{\beta}\tau)\right)}{|z + v_{0,z}\tau|} - \frac{\arctan\left(\sqrt{(z + v_{0,z}\tau)^2 + a^2} / (\bar{\beta}\tau)\right)}{\sqrt{(z + v_{0,z}\tau)^2 + a^2}} \right] \right\}$$

Start-to-end simulation for the single pass I

Steps for single pass start-to-end simulation:

1. At the entrance of the FEL, create macro-particles for the whole electron beam with proper shot noise. The 6-D distribution of the particles is determined by the beam dynamic simulation.
2. At the entrance of modulator, create one slice of macro-particles (with duration of one optical wavelength) with proper shot noise and 6-D distribution.
3. Run modulator simulation with the slice created in step 2. Due to periodic condition, the shot noise of the slice will stay correct.
4. Replace the corresponding slice created from step 1 with that output from step 3.
5. Run Genesis simulation. (Need to add macro-particles with negative energy to make it work as Genesis require each slice has the same number of macro-particles).
6. Take a proper portion of macro-particles output from GENESIS and import them into SPACE for kicker simulation.
7. Repeat step 1-6 but without the ion. The difference of step 6 and step 7 provides the single-pass coherent kick solely due to the ion.
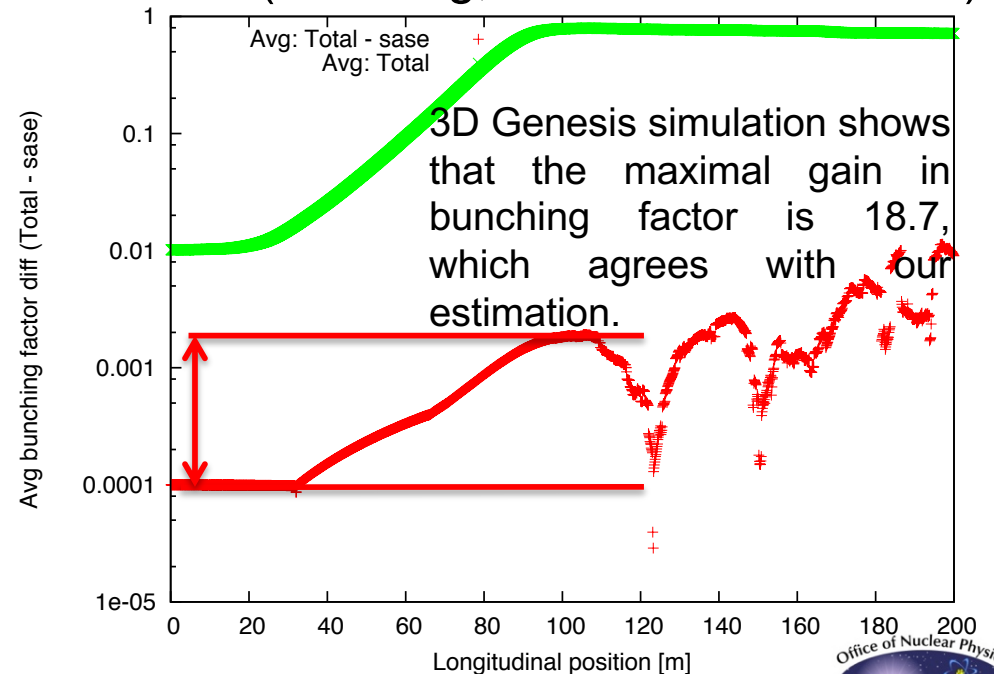
Analytical tools for FEL amplifier IV

- By requiring the relative density variation is smaller than one, we derived the upper limit of the FEL gain for the amplifier to work in the linear regime
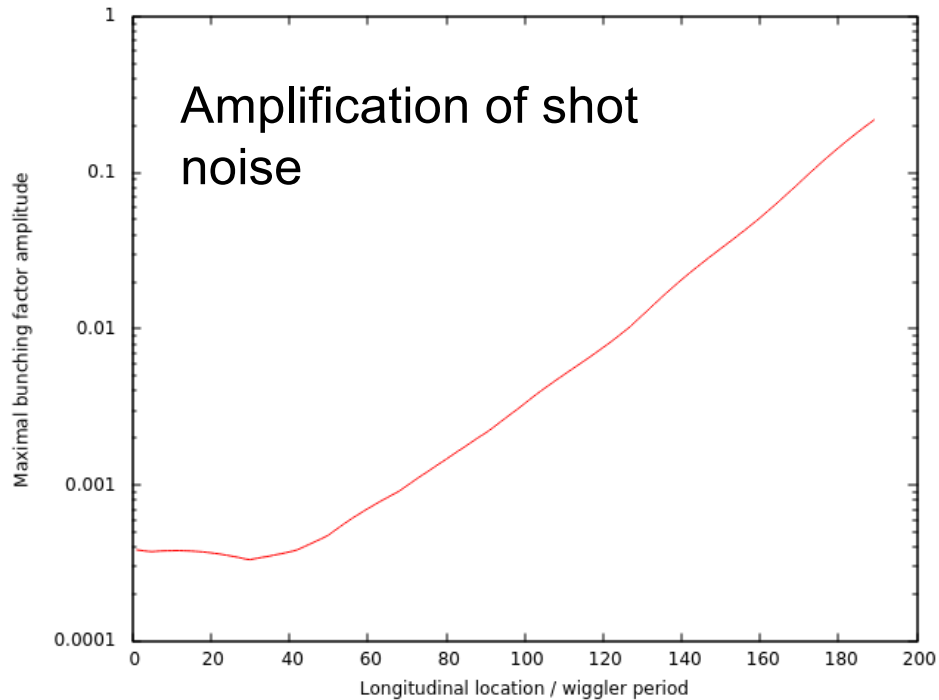
$$\left|\delta\hat{n}/n_0\right|_{max} < 1 \Rightarrow \quad \left|g\right|_{max} < \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}} \Rightarrow \quad g_{max} \sim 72 \cdot \sqrt{\frac{I_e[A] \cdot \lambda_o[\mu m]}{M_c}} = 14.1$$

- γ=7460.52
- Peak current: 30 A
- Norm emittance 1 mm mrad
- RMS energy spread 2.5e-5
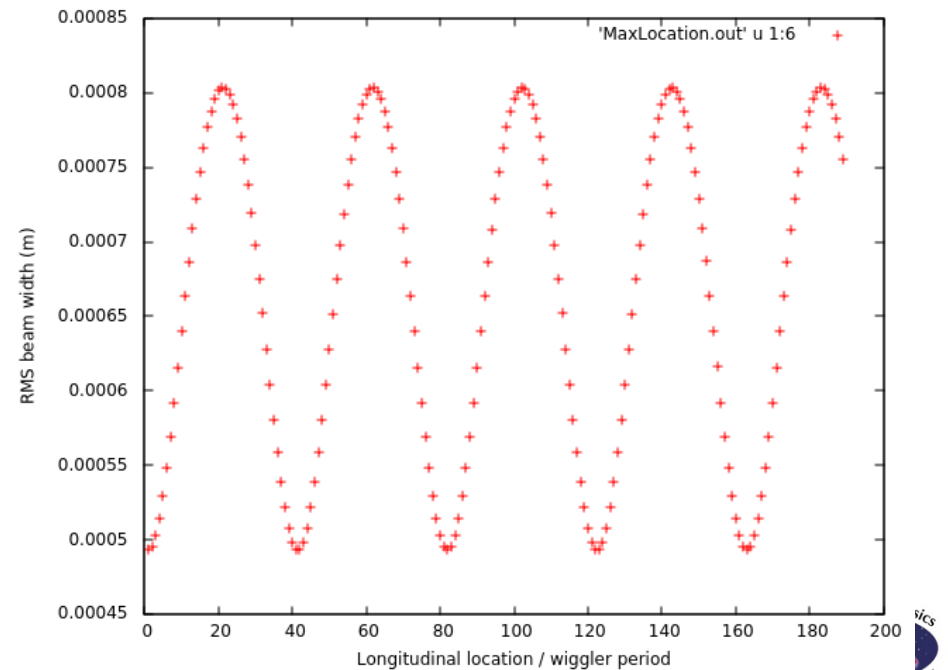- λw=10 cm
- a_w = 10
- λo=90.73 nm
- Mc = 70.6

(© Y. Jing, with code GENESIS)

3D Genesis simulation shows that the maximal gain in bunching factor is 18.7, which agrees with our estimation.

Avg bunching factor diff (Total - sase)

Longitudinal position [m]

Avg: Total - sase
Avg: Total

The FEL looks not saturated

Amplification of shot noise

Transverse beam size

Field Reduction due to Finite Transverse Modulation Size

$$\rho(\vec{r}) = \rho_o(r) \cdot \cos(kz);$$

$$\Delta\varphi = 4\pi\rho \Rightarrow \varphi(\vec{r}) = \varphi_o(r) \cdot \cos(kz);$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_o}{dr}\right) - k^2\varphi_o = 4\pi\rho_o(r)$$
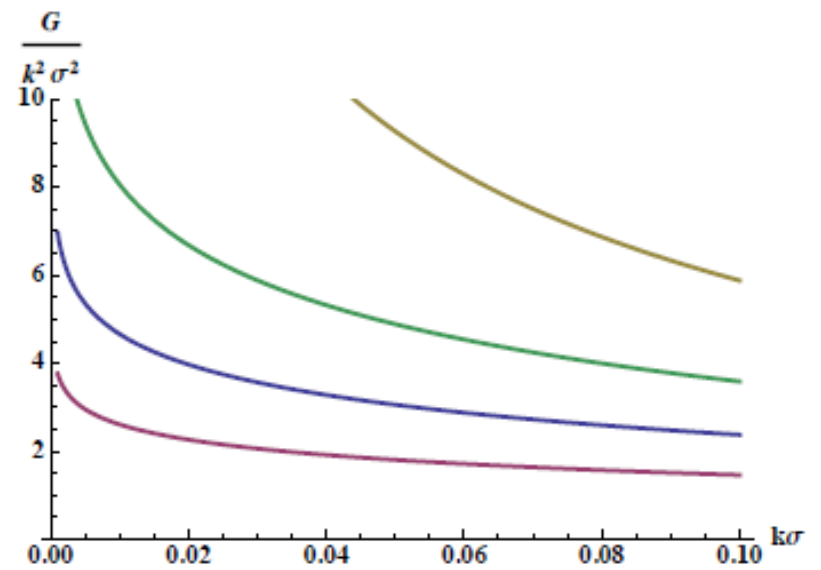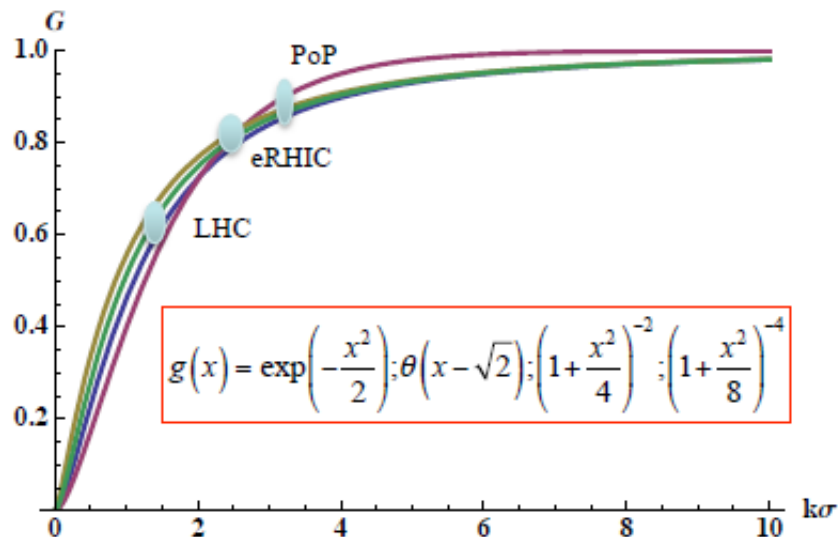
$$\rho(r) = \rho(0) \cdot g(r/\sigma)$$

$$E_{zo}(r=0) \propto -\frac{4\pi\tilde{q}}{\sigma^2}G(k_{cm}\sigma)$$

$$\varphi(\vec{r}) = -4\pi\cos(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$
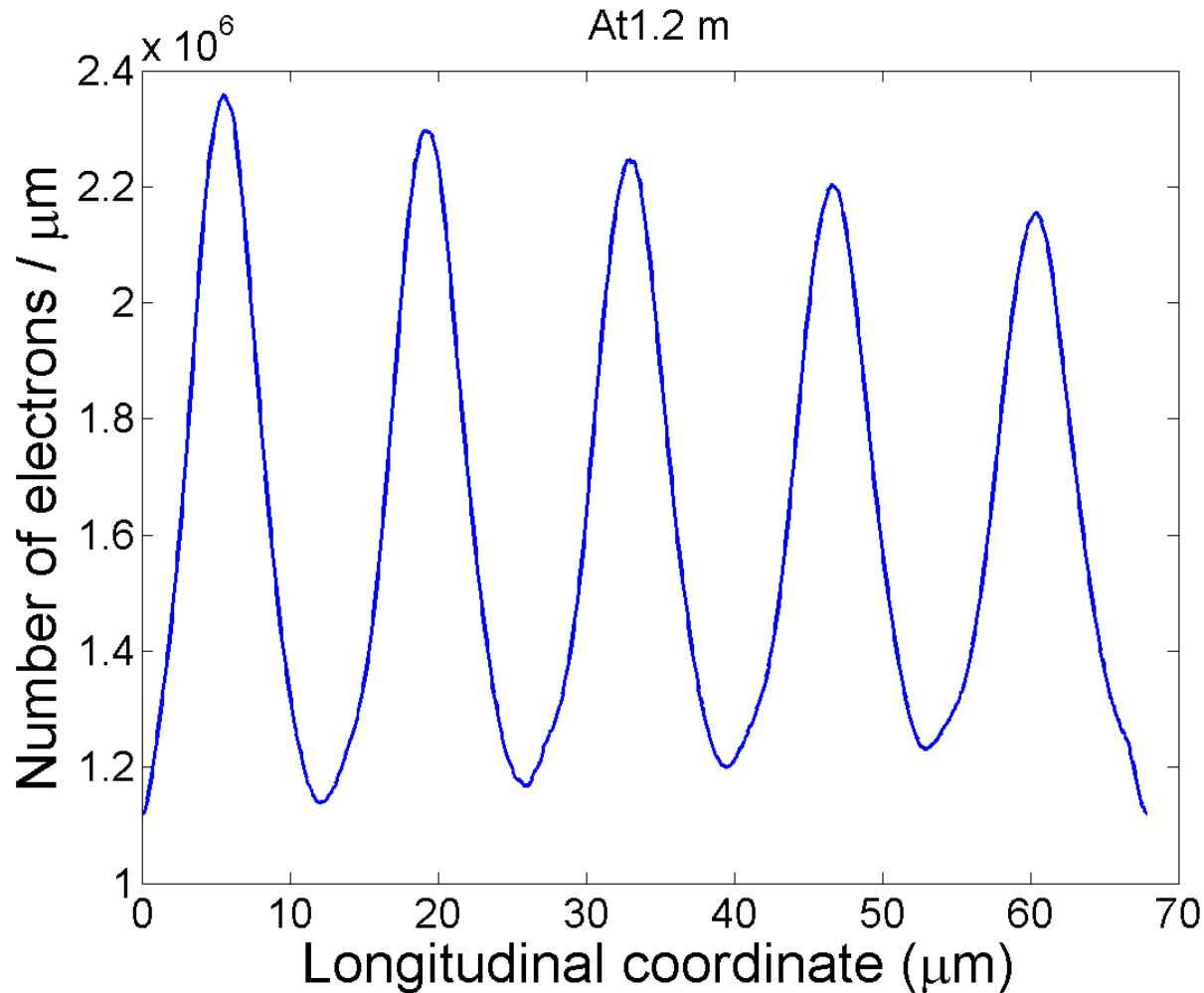
$$E_z = -\frac{\partial\varphi}{\partial z} = -4\pi k\sin(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$E_r = -\frac{\partial\varphi}{\partial r} = 4\pi k\cos(kz)\left\{I_1(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi - K_1(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$k_{cm}\sigma_\perp = \frac{k_o}{\gamma_o}\sqrt{\frac{\beta_\perp\varepsilon_{n\perp}}{\gamma_o}} = \sqrt{\gamma_o}\sqrt{\beta_\perp\varepsilon_{n\perp}}\frac{k_w}{2(1+a_w^2)}$$

$$g(x) = \exp\left(-\frac{x^2}{2}\right); \theta(x-\sqrt{2}); \left(1+\frac{x^2}{4}\right)^{-2}; \left(1+\frac{x^2}{8}\right)^{-4}$$

Background electron line density at the entrance of the kicker

Analytical Tools for Kicker I

- Dynamic equation in Kicker is very similar to that in the modulator except the initial modulation in 6D phase space dominates the process. For κ-2 velocity distribution, the electron density perturbation is determined by:

$$\frac{d^2}{dt^2}\widetilde{R}_1\!\left(\vec{k},t\right)+\omega_p^2\widetilde{R}_1\!\left(\vec{k},t\right)=Z_i\omega_p^2 e^{-\lambda\left(\vec{k},\vec{v}_0\right)t}-\omega_p^2\int_{-\infty}^{\infty}\widetilde{f}_1\!\left(\vec{k},\vec{v},0\right)e^{-\lambda\left(\vec{k},\vec{v}\right)t}d^3v$$

with $\quad \widetilde{R}_1\!\left(\vec{k},t\right)\equiv\widetilde{n}_1\!\left(\vec{k},t\right)e^{-\lambda\left(\vec{k}\right)t}-\int_{-\infty}^{\infty}\widetilde{f}_1\!\left(\vec{k},\vec{v},0\right)e^{-\lambda\left(\vec{k},\vec{v}\right)t}d^3v \quad$ and

$$\lambda\!\left(\vec{k},\vec{v}\right)\equiv i\vec{k}\cdot\vec{v}-\sqrt{\left(k_x\sigma_x\right)^2+\left(k_y\sigma_y\right)^2+\left(k_z\sigma_z\right)^2}$$

The solution of this inhomogeneous 2nd order differential equation reads

$$\widetilde{R}_1\!\left(\vec{k},t\right)=c_1\cos\!\left(\omega_p t\right)+c_2\sin\!\left(\omega_p t\right)$$

$$+\frac{1}{\omega_p}\int_{-\infty}^{\infty}\frac{\omega_p e^{-\lambda t}+\lambda\sin\!\left(\omega_p t\right)-\omega_p\cos\!\left(\omega_p t\right)}{\lambda^2+\omega_p^2}\widetilde{f}_1\!\left(\vec{k},\vec{v},0\right)d^3v$$
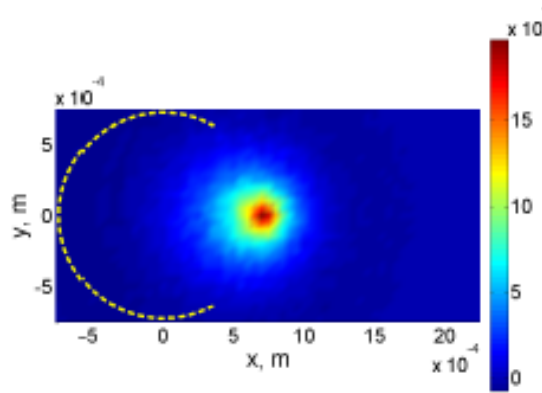
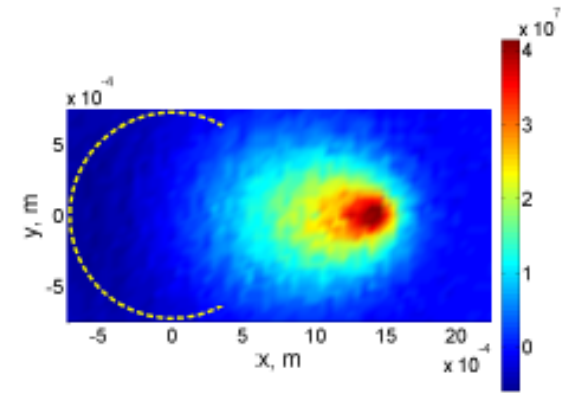# Simulation Tools for the Modulation Process II

(© J. Ma, with code SPACE)

– Simulation results for a continuous focusing channel (Beam is matched and transverse beam size does not vary.)

- Modulation is less effective for an off-centered ion. For an ion sitting at 1σ away from transverse electron beam center, the longitudinal density modulation reduces by ~40%.
- The transverse density modulation profile induced by an off-centered ion is significantly different from that induced by an ion at beam center
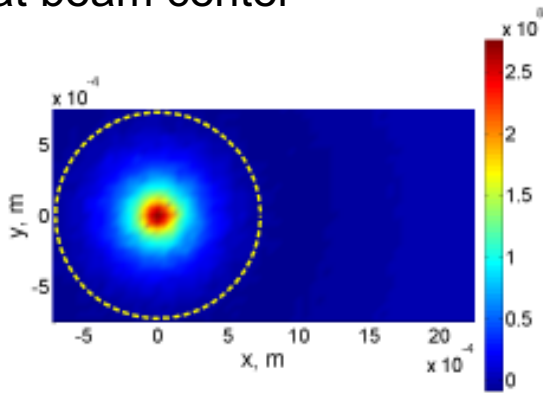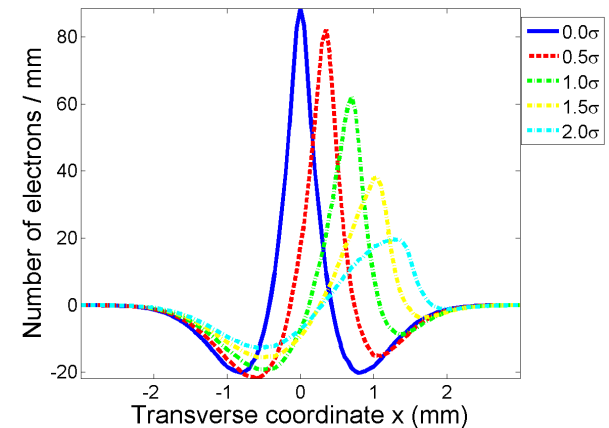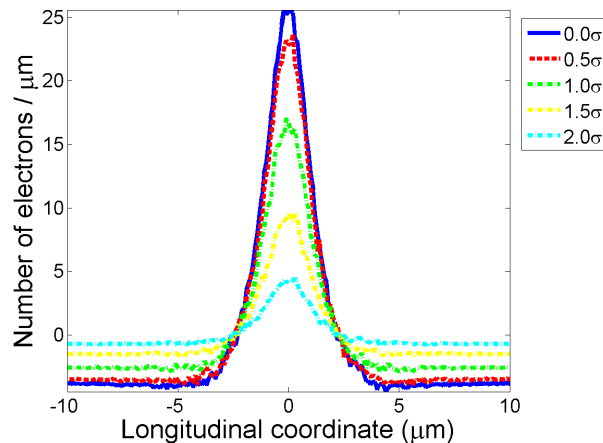

(a) Ion at center

(b) Ion at $x = 1\sigma$

(c) Ion at $x = 2\sigma$

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Simulation tools for FEL amplifier

- We use GENESIS 1.3 to simulate the amplification process in the FEL amplifier.

- Following the approach of perturbative trajectories, we run two sets of FEL simulation: one with shot noise plus modulation induced by the ion and the other one with shot noise only. The wave-packet due to the ion is extracted from the difference of the two sets of simulation. The plots are results for 20 MeV electrons.
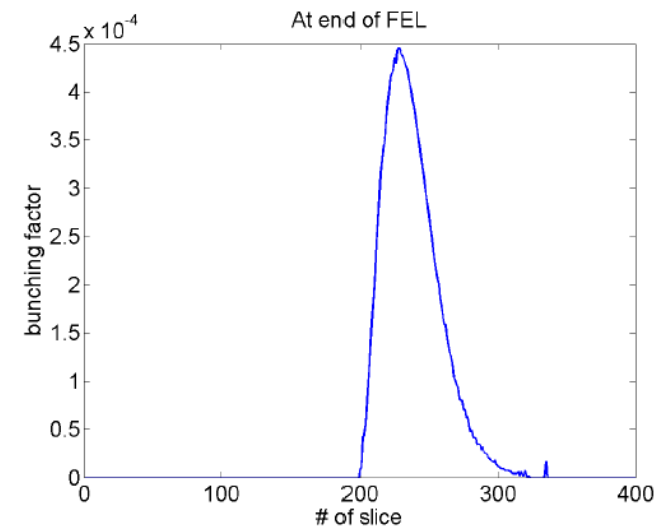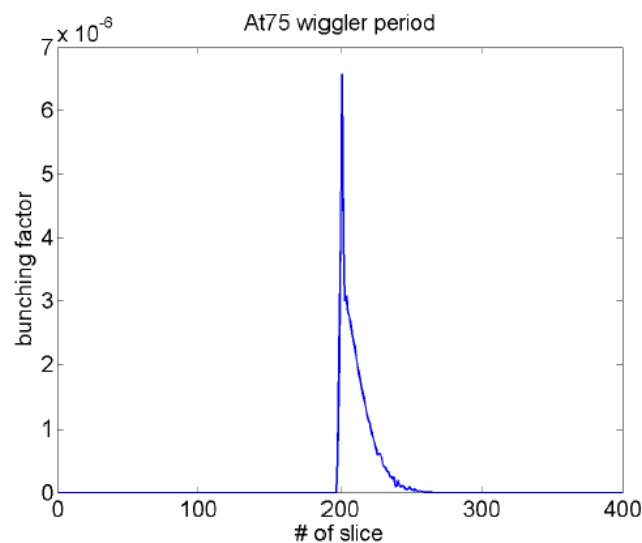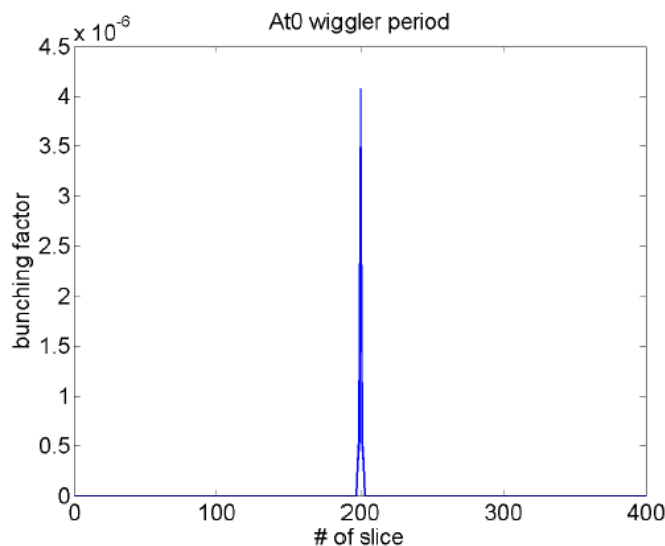
Simulation tools for kicker

(© J. Ma, with code SPACE)

- The macro-particles from GENESIS simulation are imported into SPACE for the kicker simulation. Background line density is 2e6/μm.
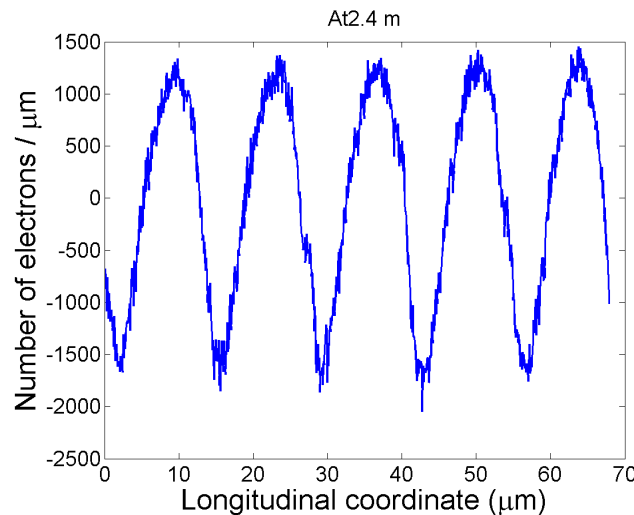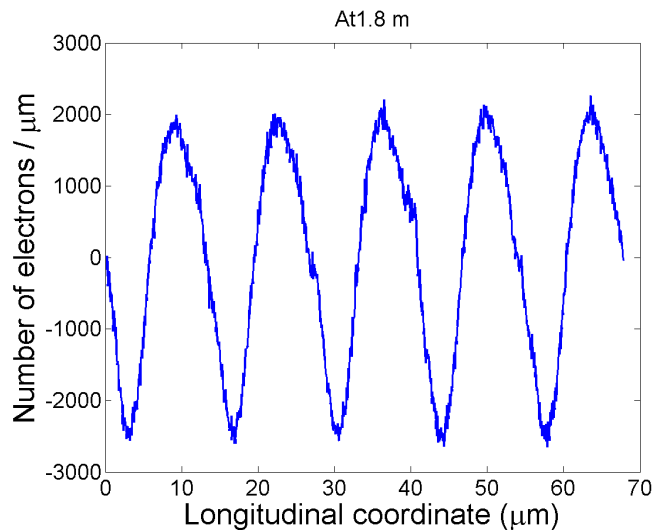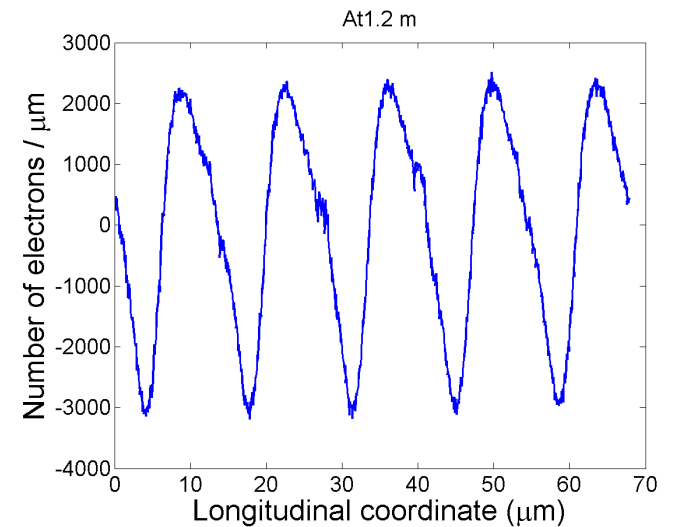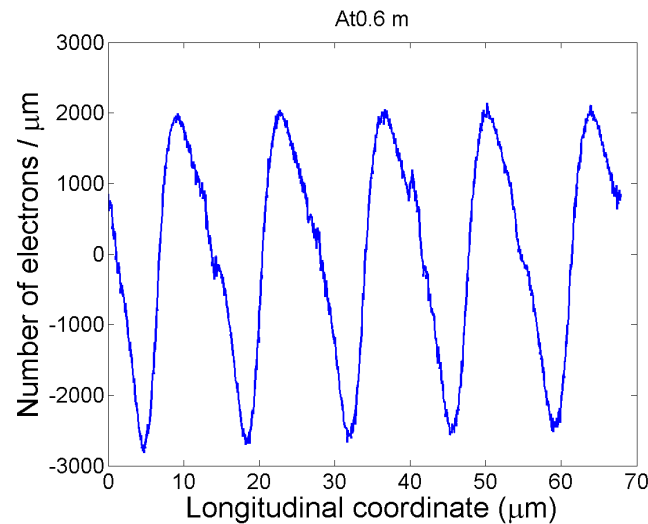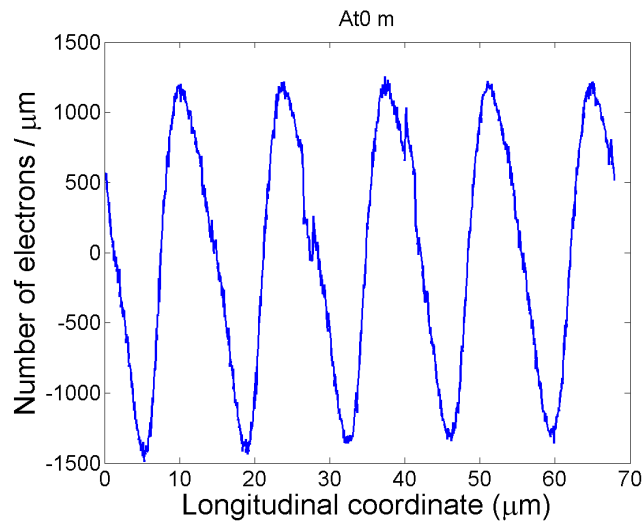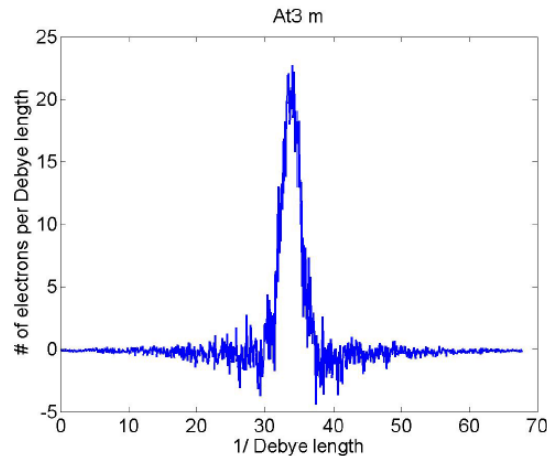

The evolution of the wave-packet shows similar behavior as that obtained from analytical model, i.e. the wave-packet amplitude increases initially and then starts to decrease.

Start-to-end simulation for the single pass
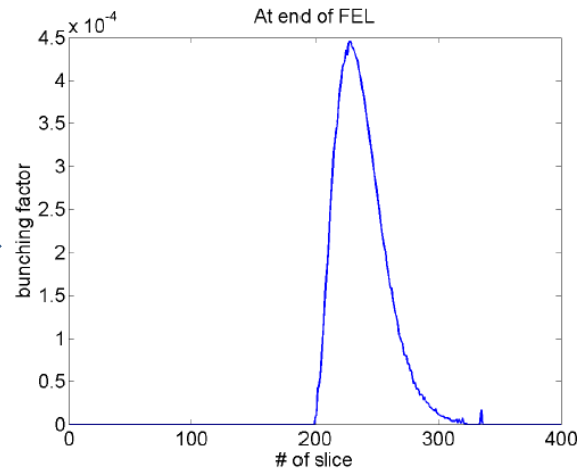## (© J. Ma, with code SPACE and GENESIS)

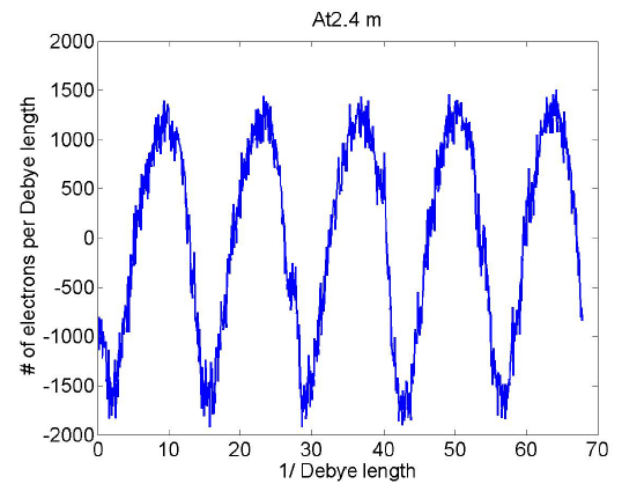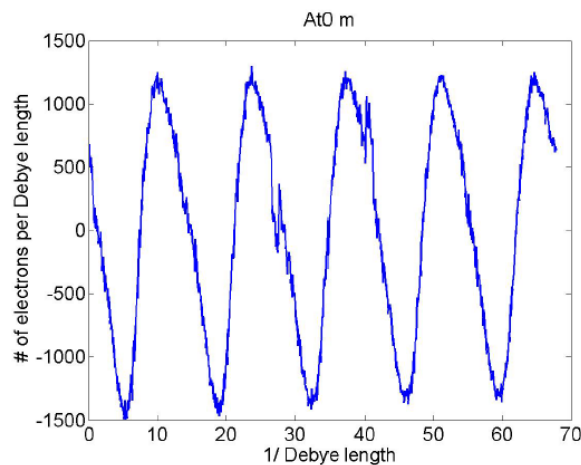- One example of start-to-end simulation (to cool 40 GeV Au)

Modulator   Amplifier

Kicker

How to evaluate CeC: the original recipe

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- Linear response of electron beam on perturbations – no saturation, superposition principle

$$\delta\vec{\mathbf{E}}_h = Ze \cdot \vec{\mathbf{G}}_{Eh}(\vec{r},\vec{r}_h,\gamma_h,t,t_h); \delta\vec{\mathbf{B}}_h = Ze \cdot \vec{\mathbf{G}}_{Bh}(\vec{r},\vec{r}_h,\gamma_h,t,t_h);$$

$$\delta\vec{\mathbf{E}}_e = -e \cdot \vec{\mathbf{G}}_{Ee}(\vec{r},\vec{r}_e,\gamma_e,t,t_e); \delta\vec{\mathbf{B}}_e = -e \cdot \vec{\mathbf{G}}_{Be}(\vec{r},\vec{r}_e,\gamma_e,t,t_e);$$

$$\vec{\mathbf{E}} = Ze \cdot \sum_h \vec{\mathbf{G}}_{Eh}(\vec{r},\vec{r}_h,\gamma_h,t,t_h) - e \cdot \sum_e \vec{\mathbf{G}}_{Ee}(\vec{r},\vec{r}_e,\gamma_e,t,t_e);$$

$$\vec{\mathbf{B}} = Ze \cdot \sum_h \vec{\mathbf{G}}_{Bh}(\vec{r},\vec{r}_h,\gamma_h,t,t_h) - e \cdot \sum_e \vec{\mathbf{G}}_{Be}(\vec{r},\vec{r}_e,\gamma_e,t,t_e)$$

$$\delta E_i = eZ\int \vec{\mathbf{E}} \cdot d\vec{r}_i; \quad \delta\vec{p}_i = eZ\int\left(\vec{\mathbf{E}} + \frac{[\vec{p}_i \times \vec{\mathbf{B}}]}{\gamma_i m}\right) \cdot dt;$$

- Evaluation of hadron distribution function using Fokker-Plank equation with both damping and diffusion terms
- Cooling transversely using coupling with longitudinal degrees of freedom

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How to evaluate CeC: the original recipe

Free Electron Lasers and High-energy Electron Cooling,
V. N. Litvinenko, Ya. S. Derbenev, 29th International Free Electron Laser Conference, Novosibirsk, Russia,
August 27-31, 2007

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- Linear response of electron beam on perturbations – no saturation, superposition principle

$$\delta E_i = eZ \int \vec{\mathbf{E}} \cdot d\vec{r}_i; \quad \delta \vec{p}_i = eZ \int \left(\vec{\mathbf{E}} + \frac{\left[\vec{p}_i \times \vec{\mathbf{B}} \right]}{\gamma_i m} \right) \cdot dt;$$

- Evaluation of hadron distribution function using Fokker-Plank equation with both damping and diffusion terms

$$\overline{f} = \langle \tilde{f} \rangle; \tilde{f} = \sum_h \delta\left(X - X_i(t) \right)$$

$$\frac{\partial \overline{f}(X,s)}{\partial t} + \frac{\partial}{\partial X_i} \left[\frac{dX_i(X,t)}{dt} \overline{f}(X,s) \right] - \frac{1}{2} \frac{\partial^2}{\partial X_i \partial X_k} \left[D_{ik}(X,t) \overline{f}(X,t) \right] = 0,$$

$$\left\langle \frac{dX_i(X,t)}{dt} \right\rangle = \frac{1}{\tau} \int (X_i - Z_i) \cdot W(Z,X|\tau,t) dZ = \frac{1}{T_o} \langle \delta X_i \rangle;$$

$$D_{ik}(X,t) = \frac{1}{2\tau} \int (X_i - Z_i)(X_k - Z_k) W(Z,X|\tau,t) dZ = \frac{1}{T_o} \langle \delta X_i \cdot \delta X_i \rangle.$$

- Cooling transversely using coupling with longitudinal degrees of freedom

$$\delta E_h(X_h) = (eZ)^2 \cdot g_{Eh}(X_h) - Ze^2 \cdot g_{Ee}(X_h).$$

$$X^T = \left\{ x, P_x, y, P_y, \tau = c(t_o - t), \delta = (E - E_o)/E_o \right\}$$

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