Hadronic vacuum polarisation: 3π contribution with analyticity constraints

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Hoferichter, Hoid, BK, JHEP 1908 (2019) 137

Motivation: $\pi^+\pi^-\pi^0$ with analyticity constraints

- second largest exclusive channel next to $\pi^+\pi^-$
- large discrepancy between different data integration analyses:

Channel	KNT18	DHMZ17	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^+\pi^-$	503.74 ± 1.96	506.70 ± 2.58	-2.96
$\pi^+\pi^-\pi^0$	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12
K^+K^-	23.00 ± 0.22	23.06 ± 0.41	-0.06
$K_S^0K_L^0$	13.04 ± 0.19	12.82 ± 0.24	0.22
Total	693.3 ± 2.5	693.1 ± 3.4	0.2

A. Keshavarzi, Mainz 2018

$\longrightarrow$ factor $\sim$10 smaller overall, absolute uncertainty comparable

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Channel	KNT18	FJ17	Difference
Data based channels ($0.318 \le \sqrt{s} \le 2$ GeV)			
$\pi^+\pi^-$	501.68 ± 1.71	502.16 ± 2.44	-0.48
$\pi^+\pi^-\pi^0$	47.83 ± 0.89	44.32 ± 1.48	3.51
$\pi^+\pi^-\pi^+\pi^-$	15.17 ± 0.21	14.80 ± 0.36	0.37
$\pi^+\pi^-\pi^0\pi^0$	19.80 ± 0.79	19.69 ± 2.32	0.11
K^+K^-	23.05 ± 0.22	21.99 ± 0.61	1.06
$K_S^0K_L^0$	13.05 ± 0.19	13.10 ± 0.41	-0.05
Total	693.27 ± 2.46	688.07 ± 4.14	5.20

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- independent cross-check with dispersion-theoretical amplitude:

  analyticity, unitarity, QCD constraints

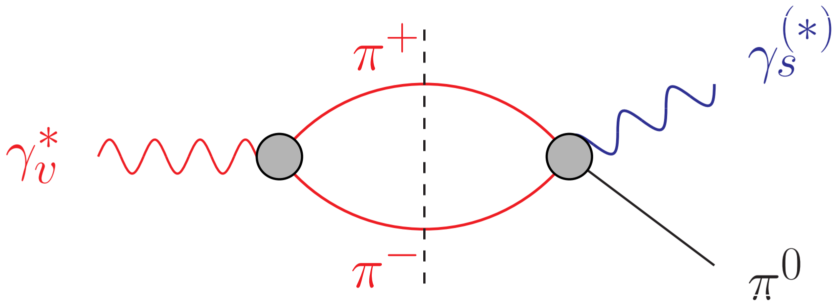
  analogous to $\pi^+\pi^-$ P. Stoffer's talk; Colangelo, Hoferichter, Stoffer 2018

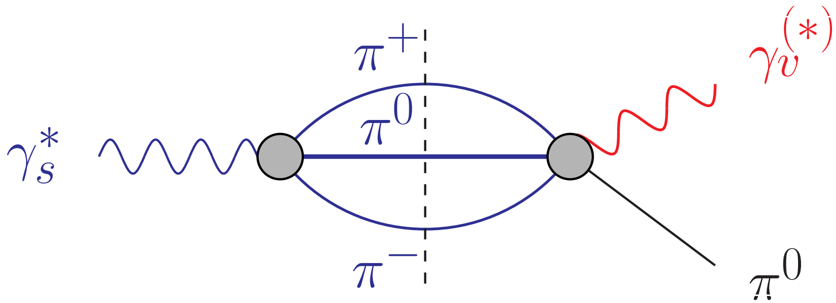
… based on dispersive analysis of $\pi^0 \to \gamma^*\gamma^*$

- isospin decomposition: B.-L. Hoid, Mainz 2018

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$

- leading hadronic intermediate states: Hoferichter et al. 2014





▷ isovector photon: 2 pions; isoscalar photon: 3 pions

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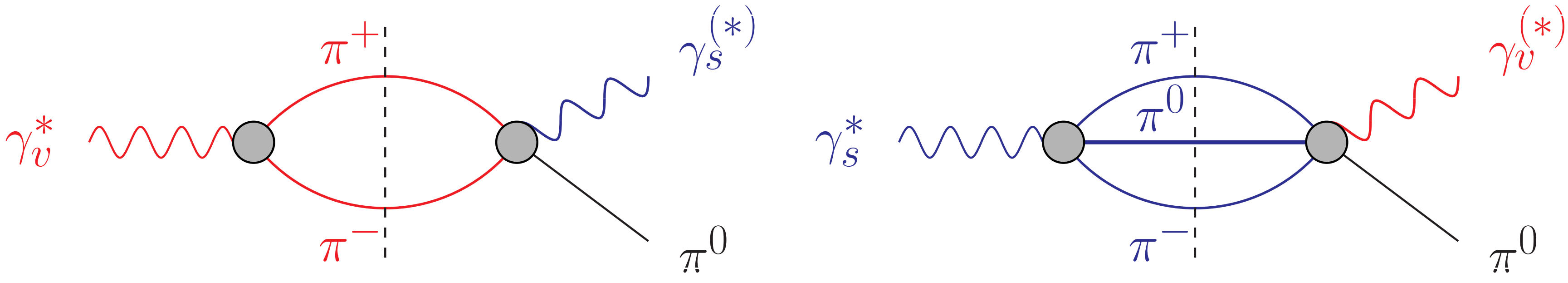

  - ▷ isovector photon: 2 pions; isoscalar photon: 3 pions
  - ▷ double-spectral-function representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$

$$\rho(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} \mathrm{Im}\left[F_\pi^{V*}(x) f_1(x,y)\right] + [x \leftrightarrow y]$$

$f_1(s, q^2)$: $\gamma^*(q^2)\pi \to \pi\pi$ P-wave Hoferichter et al. 2018

Dispersive representation $\gamma^* \to 3\pi$

- $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ amplitude:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

s,t,u: pion–pion invariant masses, $s+t+u = q^2 + 3M_\pi^2$

- "reconstruction theorem": neglect discontinuities in F-waves. . .
  $\longrightarrow$ decomposition into single-variable functions

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

- normalisation fixed from Wess–Zumino–Witten anomaly:

$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3}$$

- (s-channel) P-wave projection: $f_1(s,q^2) = \mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2)$
  $\hat{\mathcal{F}}(s,q^2)$: contribution from crossed channels $\mathcal{F}(t/u,q^2)$

Dispersive representation $\gamma^* \to 3\pi$

Unitarity relation for $\mathcal{F}(s, q^2)$:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i\{\underbrace{\mathcal{F}(s,q^2)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s,q^2)}_{\text{left-hand cut}}\} \times \theta(s - 4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

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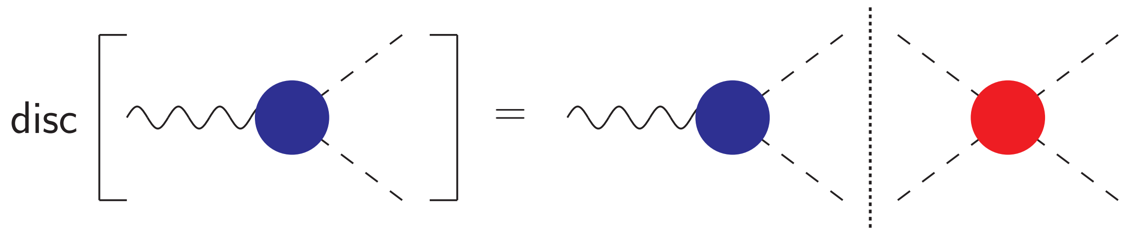
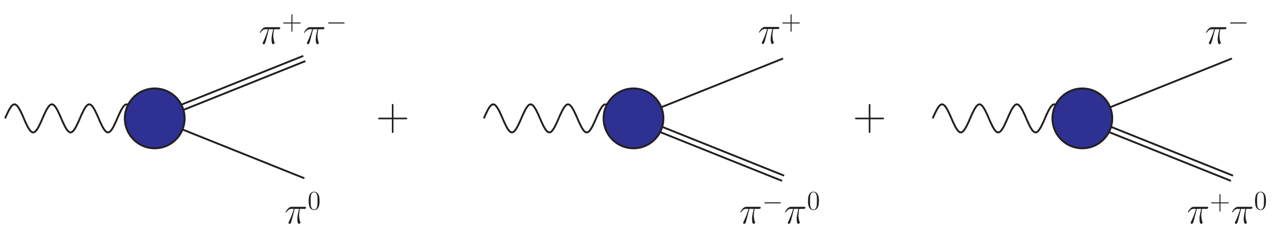


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\,, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$\mathcal{F}(s,t,u;q^2) =$ 

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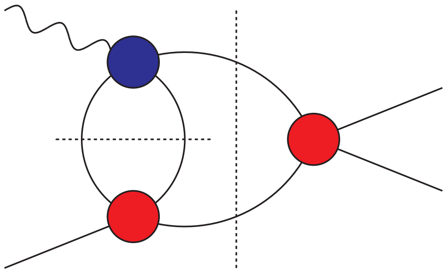
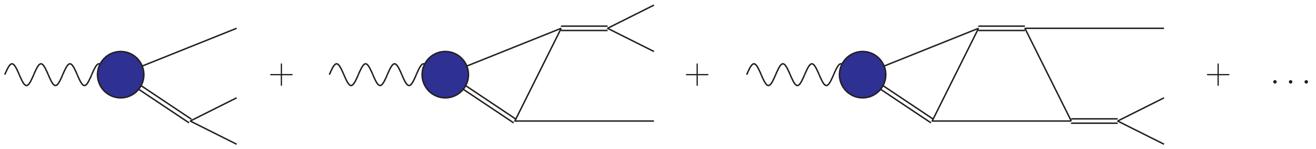


- inhomogeneities $\hat{\mathcal{F}}(s,q^2)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\left\{1 + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

$\mathcal{F}(s,q^2) =$ 

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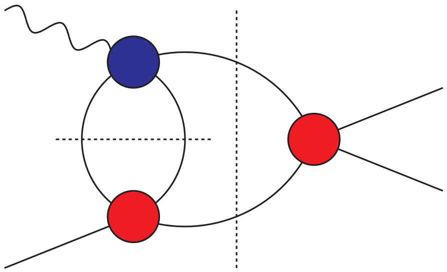

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- crossed-channel scatt. between s-, t-, u-channel (left-hand cuts)

Dispersive representation $\gamma^* \to 3\pi$

- parameterisation of subtraction function $a(q^2)$

  $\longrightarrow$ to be fitted to $e^+e^- \to 3\pi$ cross section data:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{\mathrm{thr}}^{\infty} ds' \frac{\mathrm{Im}\,\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

- $\mathcal{A}(q^2)$ includes resonance poles:

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\,\Gamma_V(q^2)} \qquad V = \omega, \phi, \omega', \omega''$$

  c_V real

- conformal polynomial (inelasticities); S-wave cusp eliminated:

$$C_n(q^2) = \sum_{i=1}^{n} c_i \Big(z(q^2)^i - z(0)^i \Big), \qquad z(q^2) = \frac{\sqrt{s_{\mathrm{inel}} - s_1} - \sqrt{s_{\mathrm{inel}} - q^2}}{\sqrt{s_{\mathrm{inel}} - s_1} + \sqrt{s_{\mathrm{inel}} - q^2}}$$

- exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\mathrm{Im}\,a(s')}{s'}$$

$e^+e^- \to 3\pi$ cross section data sets

experiment	region of $\sqrt{s}$ [GeV]	# data points	normalisation uncertainty
SND 2002	$[0.98, 1.38]$	67	5.0% or 5.4%
SND 2003	$[0.66, 0.97]$	49	3.4% or 4.5%
SND 2015	$[1.05, 1.80]$	31	3.7%
CMD-2 1995	$[0.99, 1.03]$	16	4.6%
CMD-2 1998	$[0.99, 1.03]$	13	2.3%
CMD-2 2004	$[0.76, 0.81]$	13	1.3%
CMD-2 2006	$[0.98, 1.06]$	54	2.5%
DM1 1980	$[0.75, 1.10]$	26	3.2%
ND 1991	$[0.81, 1.39]$	28	10% or 20%
DM2 1992	$[1.34, 1.80]$	10	8.7%
BaBar 2004	$[1.06, 1.80]$	30	all systematics

- normalisation-type systematics assumed 100% correlated
- avoid biased fit for empirical full covariance matrix
  $\longrightarrow$ iterative solution D'Agostini 1994, NNPDF 2010

Fit results

Parameters:

- resonance parameters M_ω, Γ_ω, M_ϕ, Γ_ϕ, c_ω, c_ϕ, $c_{\omega'}$, $c_{\omega''}$
- conformal parameters c_1, c_2, c_3
- energy rescaling $\sqrt{s} \to \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$
  $\longrightarrow$ far less an issue than for $\pi^+\pi^-$

Fit results

Parameters:

- resonance parameters $M_\omega, \Gamma_\omega, M_\phi, \Gamma_\phi, c_\omega, c_\phi, c_{\omega'}, c_{\omega''}$
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- energy rescaling $\sqrt{s} \to \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$

  $\longrightarrow$ far less an issue than for $\pi^+\pi^-$
- quality of the combined fit to all data:

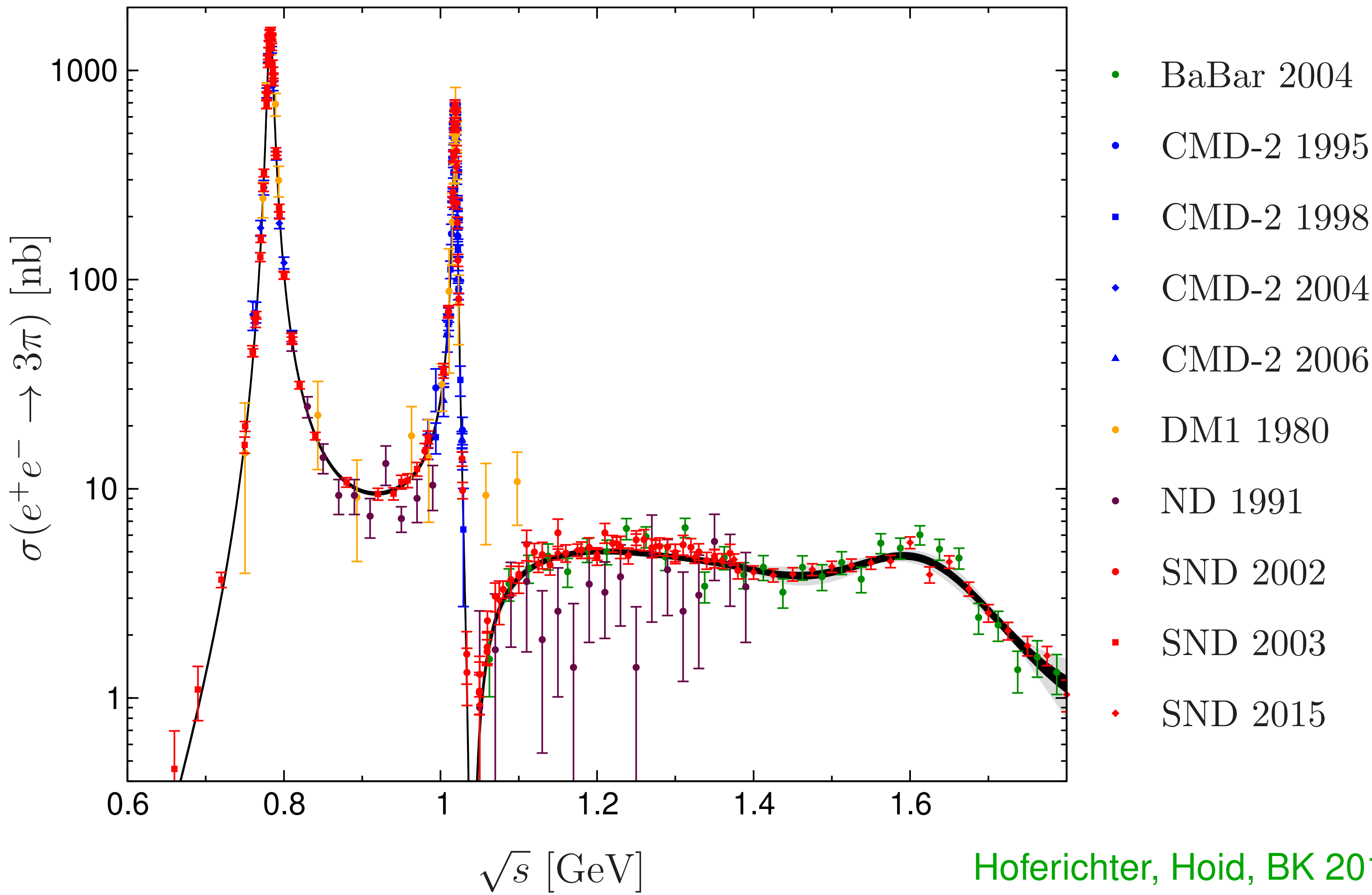
	this work		
χ^2/dof	430.8/305	=	1.41

▷ correlations increase χ^2/dof by $\sim$ 0.3

▷ significantly better fits to individual data sets

$\longrightarrow$ fit errors inflated by scale factor $S = \sqrt{\chi^2/\text{dof}}$

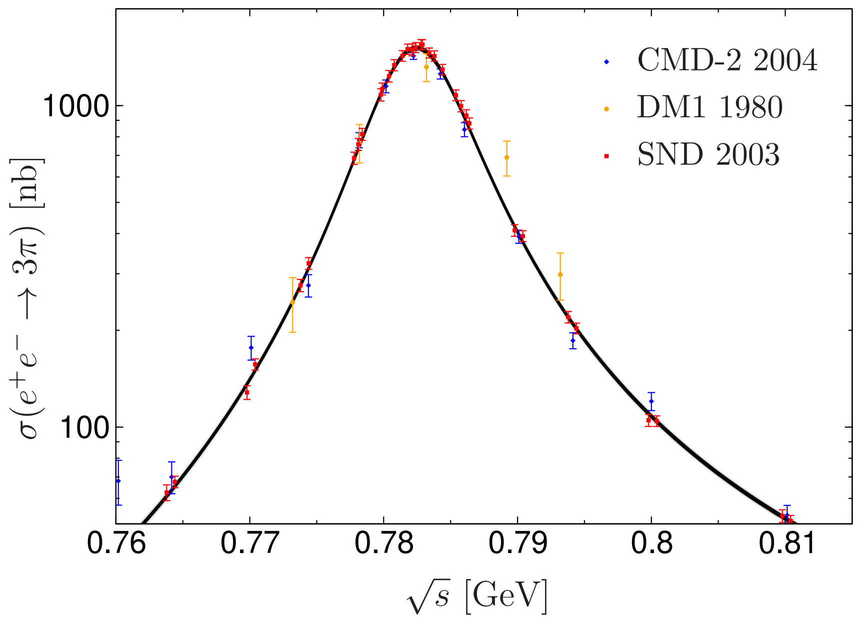
Fit results $e^+e^- \to 3\pi$ data up to 1.8 GeV

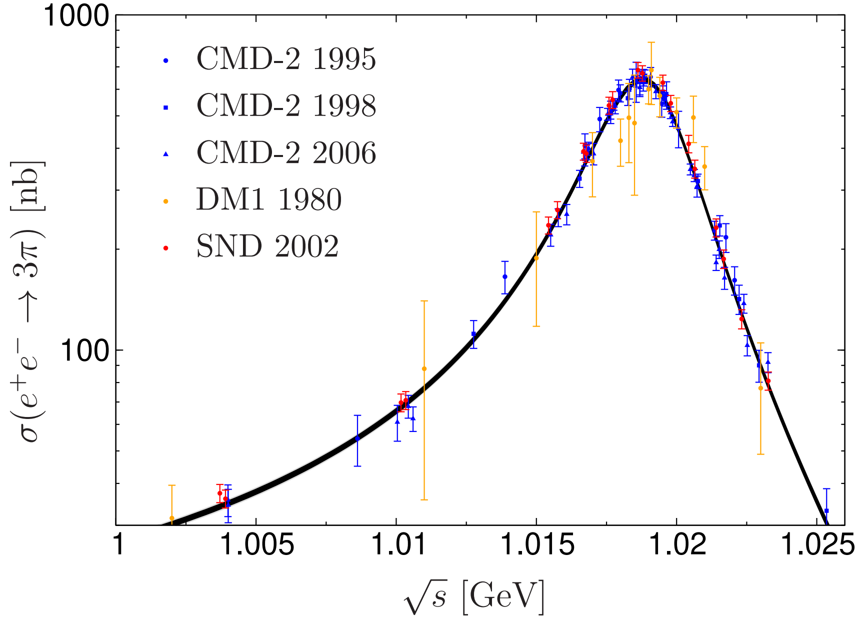


Hoferichter, Hoid, BK 2019

- black / gray bands represent fit and total uncertainties
- vacuum polarisation removed from the cross section

Fit results $e^+e^- \to 3\pi$: ω, ϕ peaks





- VP-subtraction: expect $\Delta M_\omega = -0.13$ MeV, $\Delta M_\phi = -0.26$ MeV

$M_\omega = 782.63(3)$ MeV $\quad\quad \Gamma_\omega = 8.71(6)$ MeV

$M_\phi = 1019.20(2)$ MeV $\quad\quad \Gamma_\phi = 4.23(4)$ MeV

. . . vs. PDG:

$M_\omega = 782.65(12)$ MeV $\quad\quad \Gamma_\omega = 8.49(8)$ MeV

$M_\phi = 1019.461(16)$ MeV $\quad\quad \Gamma_\phi = 4.249(13)$ MeV

$\longrightarrow$ M_ω compatible with PDG, tension with $\pi\pi$ channel persists

Fit results: 3π contribution to HVP

- central result for the 3π contribution to HVP:

$$a_\mu^{3\pi}|_{\leq 1.8\,\text{GeV}} = 46.2(6)(6) \times 10^{-10} = 46.2(8) \times 10^{-10}$$

Hoferichter, Hoid, BK 2019

- interpolation errors main discrepancy between different groups:

Davier et al. 2017, 2019	Keshavarzi et al. 2018
$46.20(1.45)$	$47.70(89)$

$\longrightarrow$ linear interpolation overestimates narrow resonances

- consistently above values as small as

$$a_\mu^{3\pi}|_{\leq 2.0\,\text{GeV}} = 44.3(1.5) \times 10^{-10}$$

Jegerlehner 2017

Results: HVP combination 2π and 3π

- combination with the 2π channel: Colangelo, Hoferichter, Stoffer 2018

$$a_\mu^{2\pi}|_{\leq 1.0\,\text{GeV}} + a_\mu^{3\pi}|_{\leq 1.8\,\text{GeV}} = 541.2(2.7) \times 10^{-10}$$

$\longrightarrow$ 80% of HVP cross-checked
imposing analyticity and unitarity constraints

- addition of the rest of HVP: Davier et al. 2017, Keshavarzi et al. 2018

$$a_\mu^{\text{HVP}} = 692.3(3.3) \times 10^{-10}$$

- combined with all other contributions (and old HLbL estimate): reaffirms $(g-2)_\mu$ anomaly at the level of 3.4σ

Summary / Outlook

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- dispersion-theoretical global fit function for 3π channel

  $\longrightarrow$ incorporates analyticity, unitarity, and low-energy constraints

- provides independent analysis of second largest HVP channel

- main tension in 3π HVP resolved

- tension to M_ω as extracted from 2π channel confirmed

Outlook

- final-state radiation in $e^+e^- \to 3\pi(\gamma)$ Prabhu et al.

- could do $\pi^0\gamma$ similarly — probably too small to matter