UNIVERSITÄT BERN

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AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Pseudoscalar-Meson Contribution from Schwinger's Sum Rule

Franziska Hagelstein (AEC Bern)

in collaboration with

V. Pascalutsa and V. Biloshytskyi (JGU Mainz)

FH and VP, PRL 120 (2018) 072002 and 1907.06927 (2019)

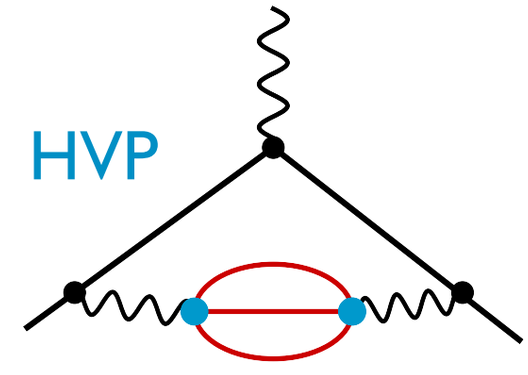
DATA-DRIVEN DISPERSIVE APPROACH TO HVP

- HVP is calculated with a systematic data-driven dispersive approach:

$$a^{\mathrm{HVP}} = \frac{\alpha^2}{3\pi^2} \int_{m_\pi^2}^{\infty} \mathrm{d}s \, \frac{R_\gamma^{\mathrm{had}}(s)\, K(s/m^2)}{s}$$

$$K(s/m^2) = \int_0^1 \mathrm{d}x \, \frac{x^2(1-x)}{x^2 + (1-x)\, s/m^2}$$

$$R_\gamma^{\mathrm{had}}(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \mathrm{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$



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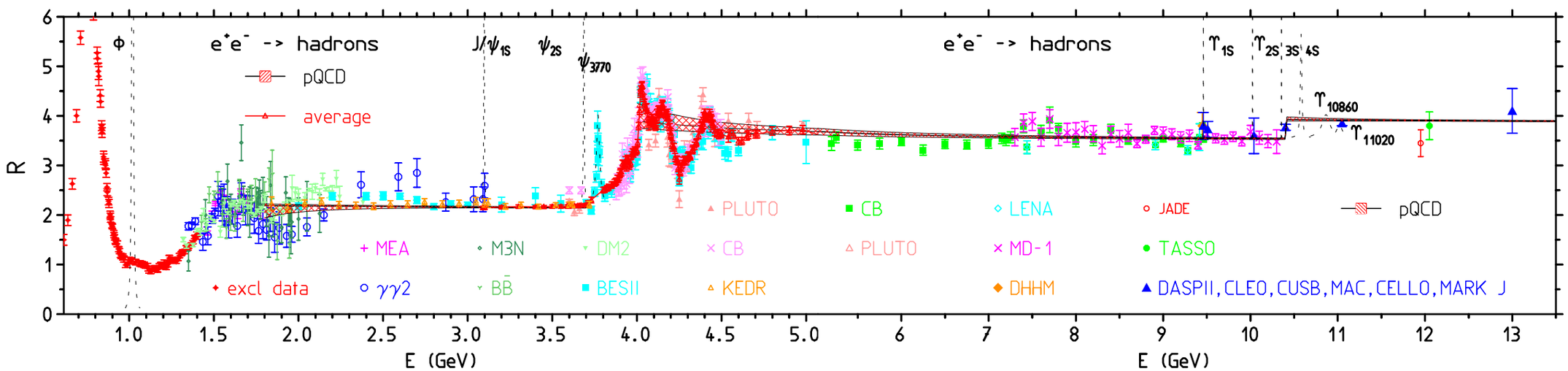

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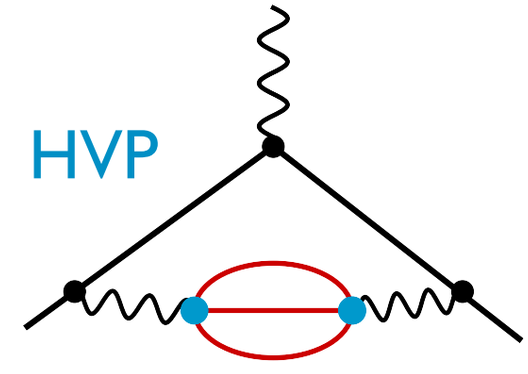
HVP



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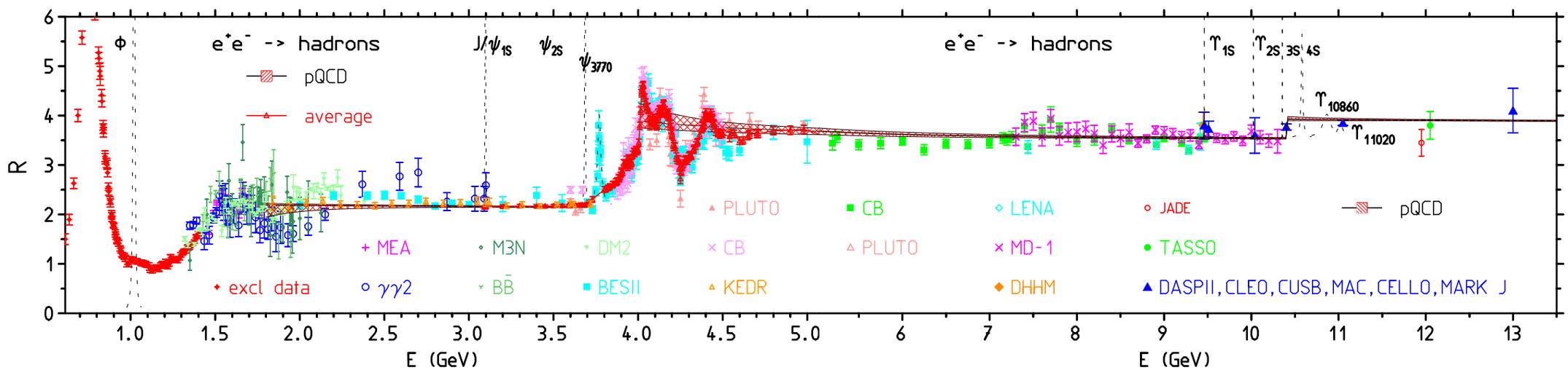
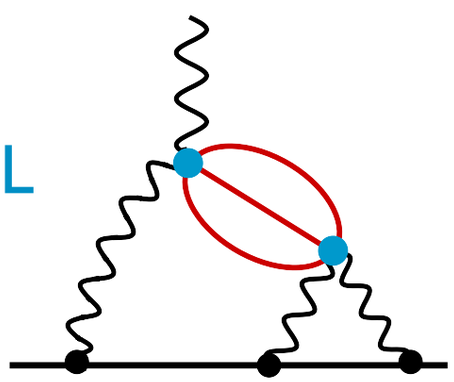


- Is there an exact dispersive formula which treats HVP and HLbL (and everything else) in the same way?

HLbL



GERASIMOV DRELL HEARN SUM RULE

Gerasimov—Drell—Hearn sum rule

$$I_{\mathrm{GDH}} = \frac{2\pi^2\alpha}{m^2}a^2 = \int_{\nu_0}^{\infty} \mathrm{d}\nu \frac{\Delta\sigma_{\mathrm{abs}}(\nu)}{\nu}$$

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$a_\mu \approx 0.0011659209(6)$ [BNL]

- GDH sum rule for the muon:
  - Huge cancelation requires measurements with incredible accuracy
    - l.h.s.: HVP starts at $\mathcal{O}(\alpha^2)$, I_{GDH} starts at $\mathcal{O}(\alpha^5)$
    - r.h.s.: hadronic photo-production cross section starts at $\mathcal{O}(\alpha^3)$

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$a_p \approx 1.7929$ and
$I_{\mathrm{GDH}} = 204.784481\ \mu b$ [CODATA]

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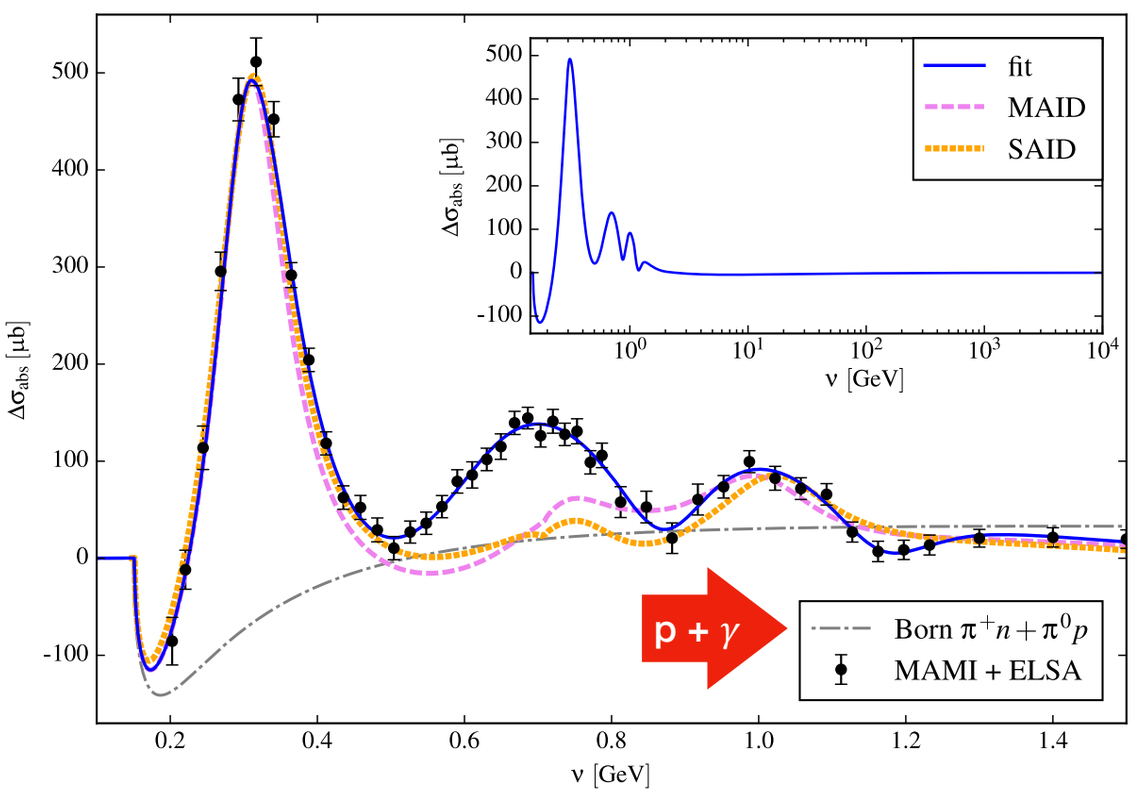

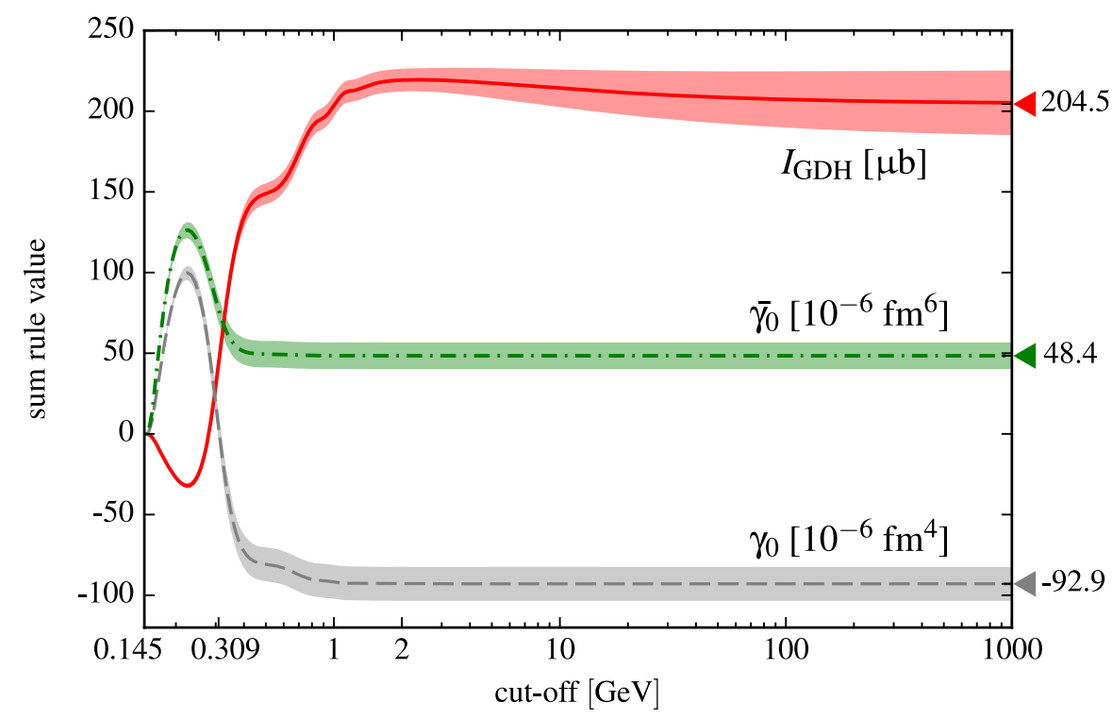

THE SCHWINGER SUM RULE (1975)

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anomalous magnetic moment (a.m.m.)
$a = \frac{1}{2}(g-2)_\mu$

muon mass m

photon lab-frame energy ν and virtuality $Q^2 = -q^2$

$$a = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$$

fine-structure constant $\alpha \approx 1/137$

photo-absorption threshold ν_0

longitudinal-transverse photo-absorption cross section σ_{LT}

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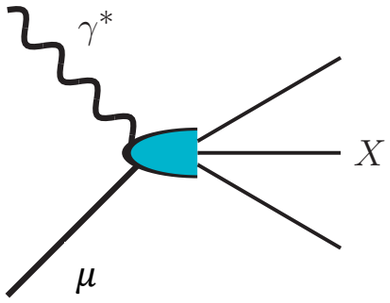
- Linear relation between g-2 and a single experimental observable — the photo-absorption cross section



inelastic cross section

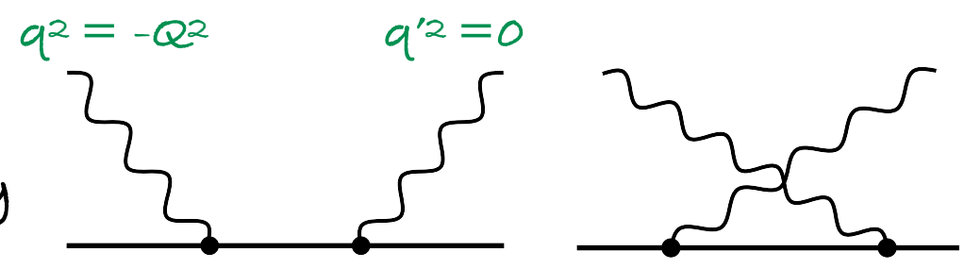
$X = \mu\gamma, \mu\gamma\gamma, \mu\pi^0, \mu\gamma\pi^0, \ldots$

THE SCHWINGER TERM

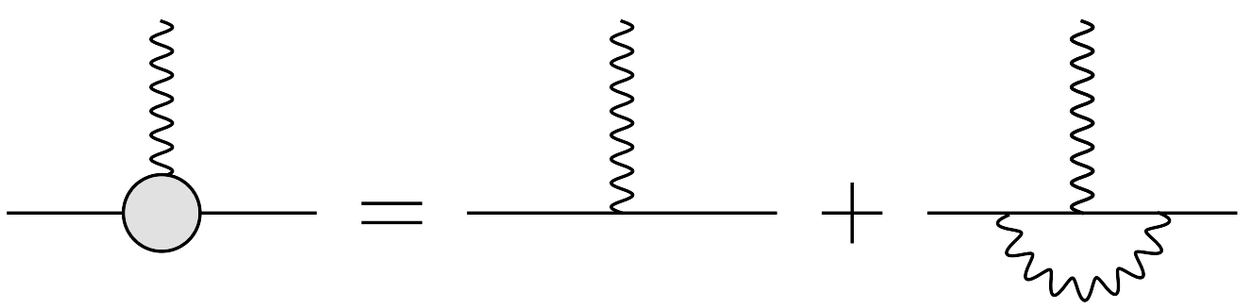
- Schwinger sum rule: $a = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$

- Input: longitudinal-transverse photo-absorption cross section

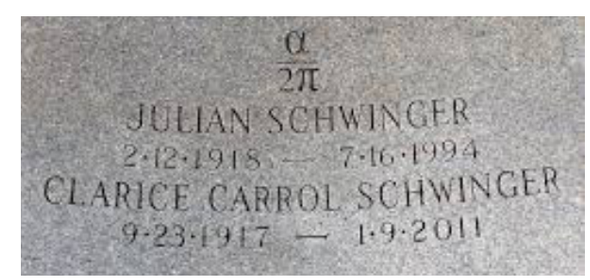
tree-level QED
Compton scattering



$$\sigma_{LT}^{\gamma^* \mu \to \gamma \mu}(\nu, Q^2) = \frac{\pi \alpha^2 Q \, (s - m^2)^2}{4 m^3 \nu^2 \left(\nu^2 + Q^2\right)} \left(-2 - \frac{m(m+\nu)}{s} + \frac{3m + 2\nu}{\sqrt{\nu^2 + Q^2}} \operatorname{arccoth} \frac{m+\nu}{\sqrt{\nu^2 + Q^2}} \right)$$



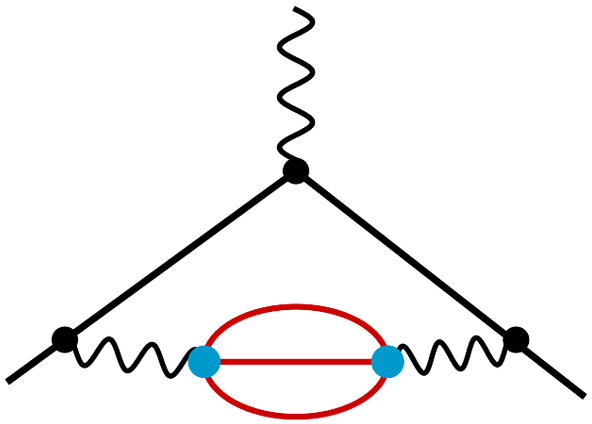
$F_2(0) = a$ $\quad$ $a^{(0)} = 0$ $\quad$ $a^{(1)} = \alpha/2\pi$



Schwinger term —
the leading QED result

HVP — STANDARD FORMULA

- Hadronic vacuum polarization: 2 Data-driven approaches based on dispersion theory
  - A) Standard Formula
  - B) Schwinger Sum Rule



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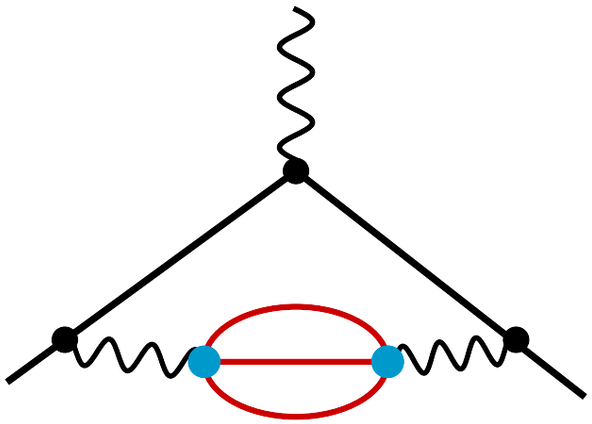
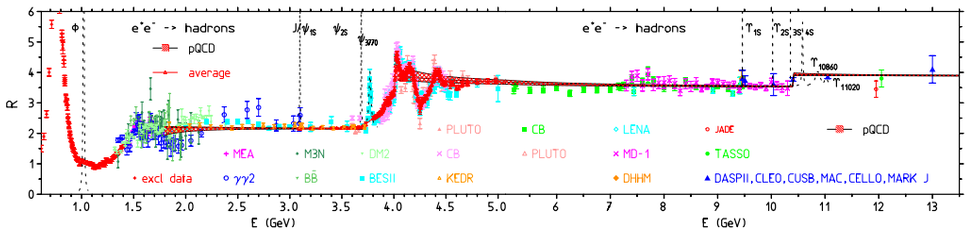


$$a^{\mathrm{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \operatorname{Im} \Pi^{\mathrm{had}}(s) \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + (1-x)(s/m^2)}$$

$$\operatorname{Im} \Pi^{\mathrm{had}}(s) = \frac{s}{4\pi\alpha} \sigma(\gamma^* \to \text{anything}) = \frac{\alpha}{3} R_\gamma^{\mathrm{had}}(s)$$

photon selfenergy

decay rate of a virtual timelike photon into hadrons



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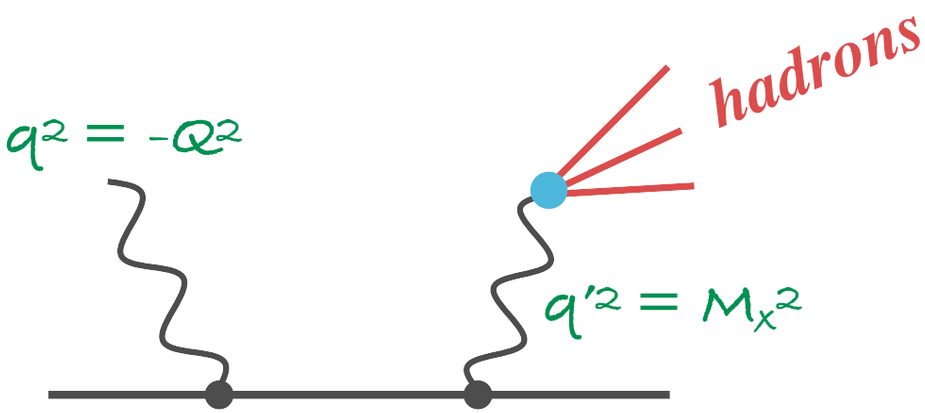

+ crossed diagram

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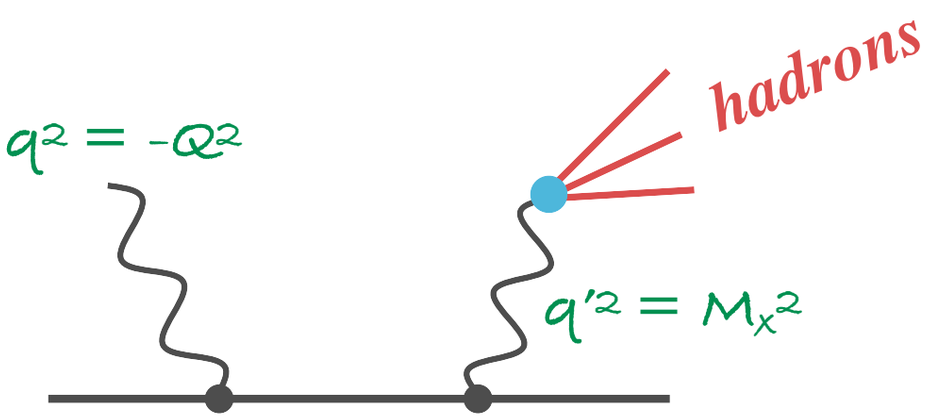

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- Cross section of hadron production through timelike Compton scattering:

factories into: $$\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}M_X^2}{M_X^2}\,\sigma(\gamma\mu \to \gamma^*\mu)\,\mathrm{Im}\,\Pi_X(M_X^2)$$

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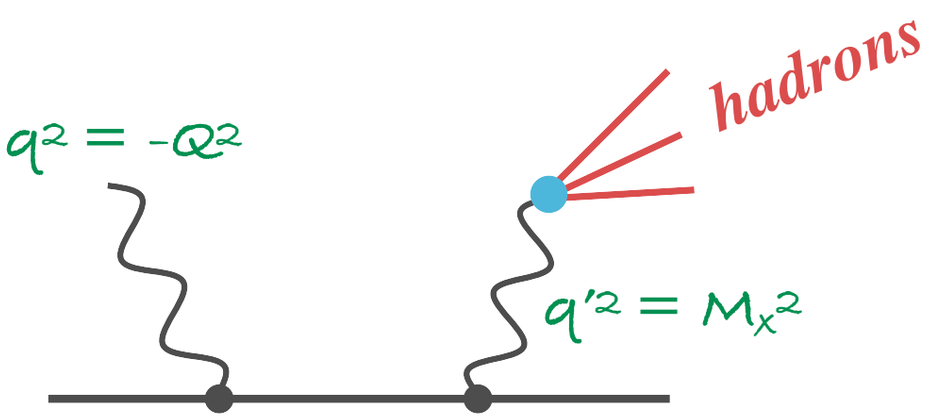

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- Timelike Compton scattering cross section:

$$\boxed{\left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0} = \frac{\pi\alpha^2}{2m^2\nu^3}\left[-(5s+m^2+M_X^2)\lambda + (s+2m^2-2M_X^2)\log\frac{\beta+\lambda}{\beta-\lambda}\right]}$$

$$\beta = (s+m^2-M_X^2)/2s, \quad s = m^2+2m\nu$$

$$\lambda = (1/2s)\sqrt{[s-(m+M_X)^2]\,[s-(m-M_X)^2]}$$

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- HVP from the Schwinger sum rule with the cross section of hadron production through timelike Compton scattering:

$$a = \frac{m^2}{\pi^2 \alpha} \int_{4m_\pi^2}^{\infty} \mathrm{d}M_X^2 \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{1}{Q} \frac{\mathrm{d}\sigma_{LT}^{\gamma\mu \to \mu X}(\nu, Q^2)}{\mathrm{d}M_X^2} \right]_{Q^2=0}$$

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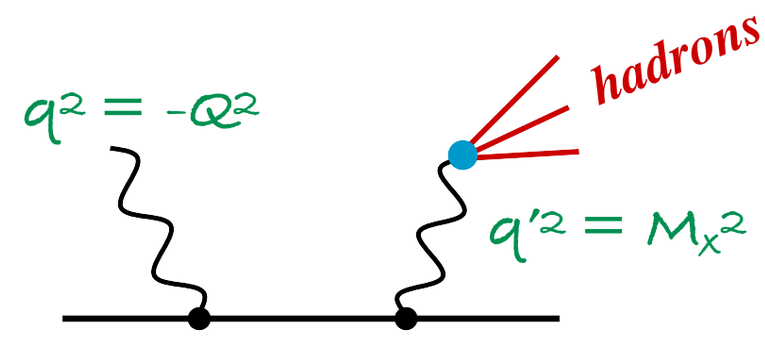

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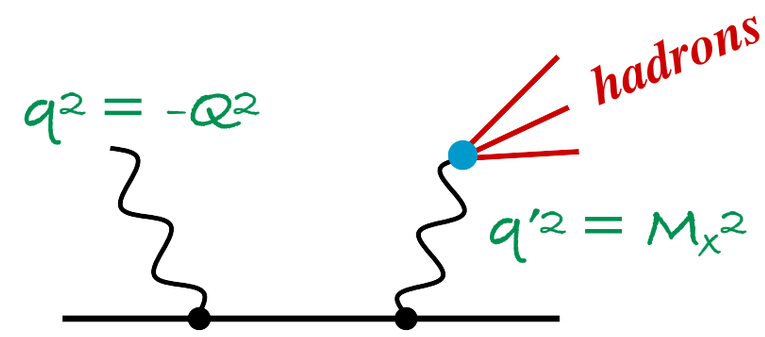

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kernel function: $\frac{\alpha}{\pi} K(M_X^2/m^2) \equiv \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + (1-x)(M_X^2/m^2)}$

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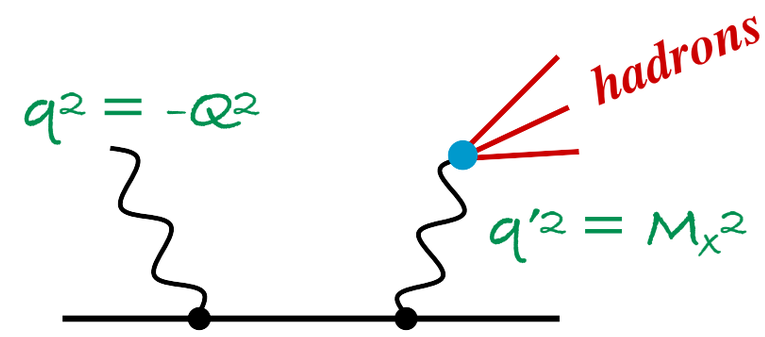

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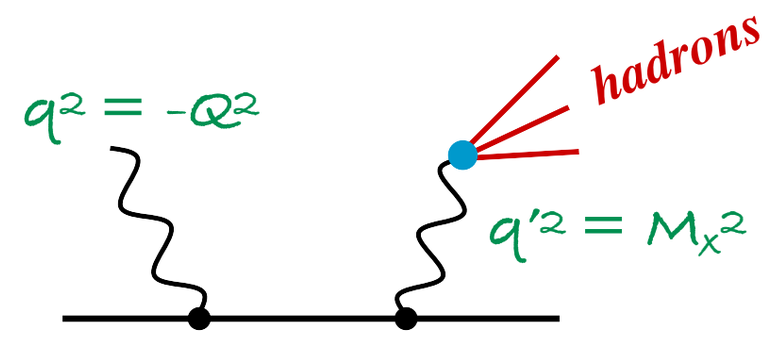

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for $M_X = 0$, we find $K(0) = 1/2$, and therefore the Schwinger term: $a^{(1)} = \alpha/2\pi$

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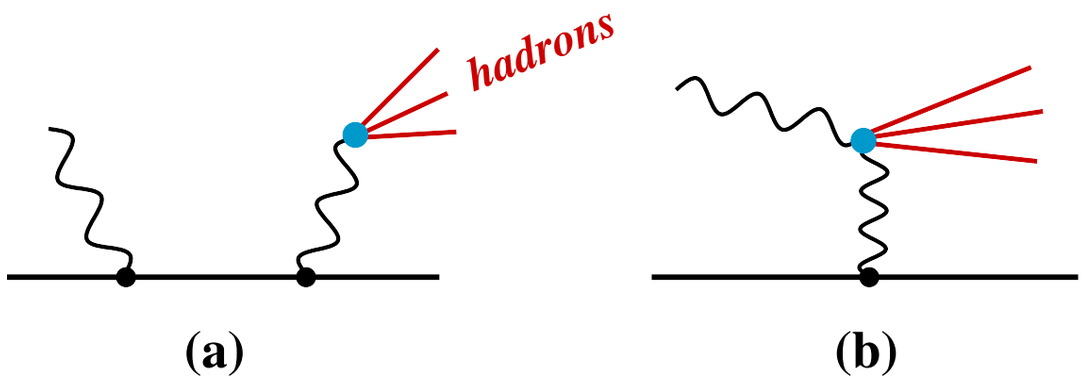
PHOTOABSORPTION CROSS SECTIONS

- Hadron photo-production off the muon



(a) Timelike Compton scattering

(b) Primakoff effect

$\mu\gamma \to \mu + \text{hadrons}$

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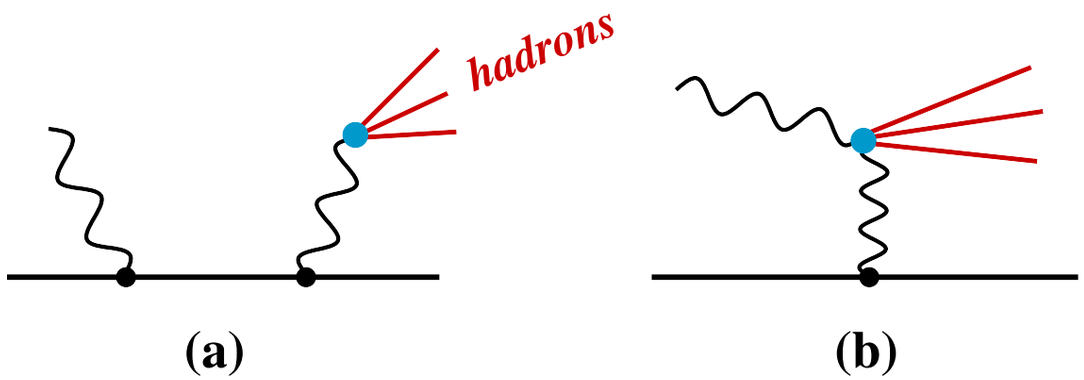

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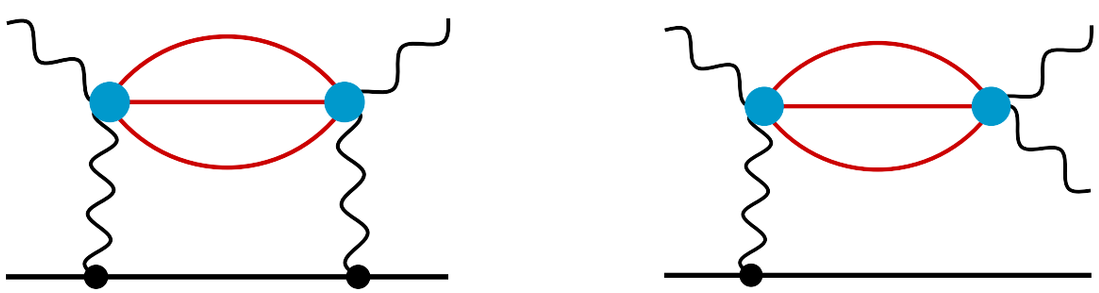
- Electromagnetic channels — HLbL contribution to Compton scattering



$\mu\gamma \to \mu\gamma$

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PSEUDOSCALAR-MESON CONTRIBUTION

Consider 4 channels at $\mathcal{O}(a^3)$

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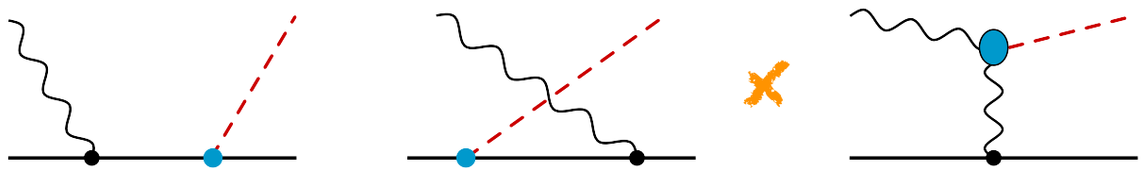

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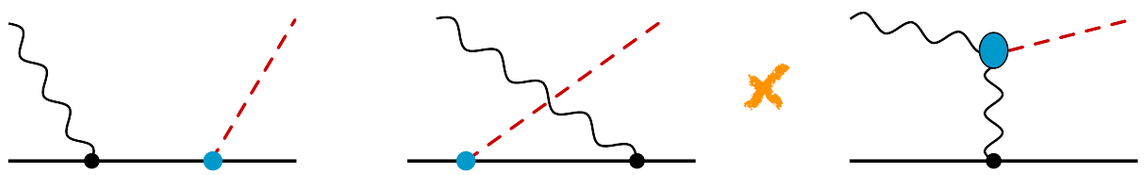

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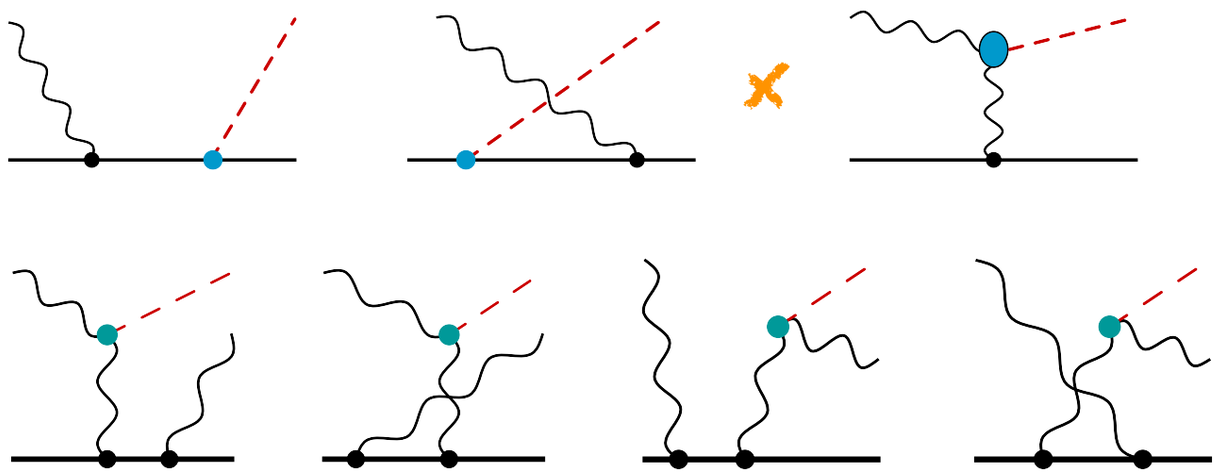

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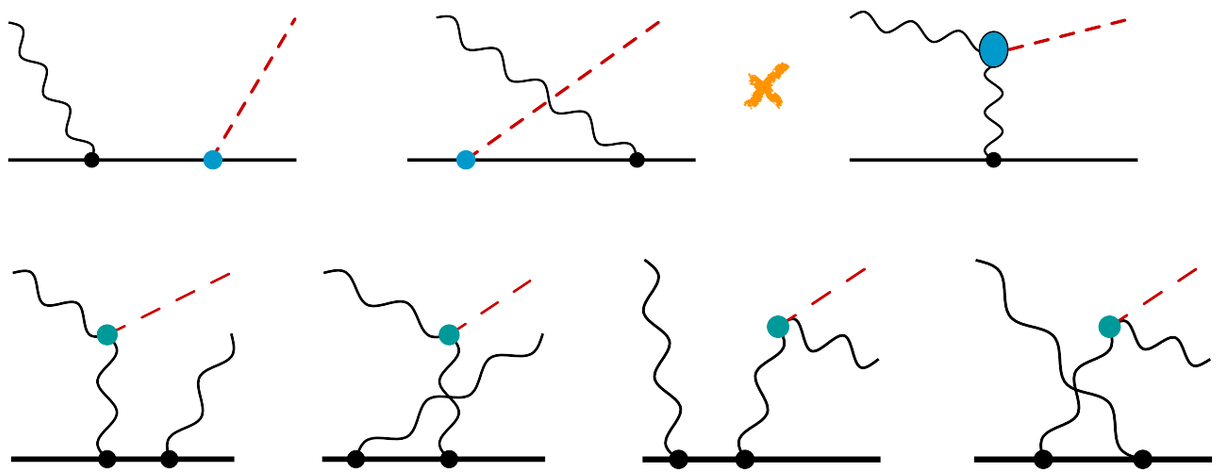

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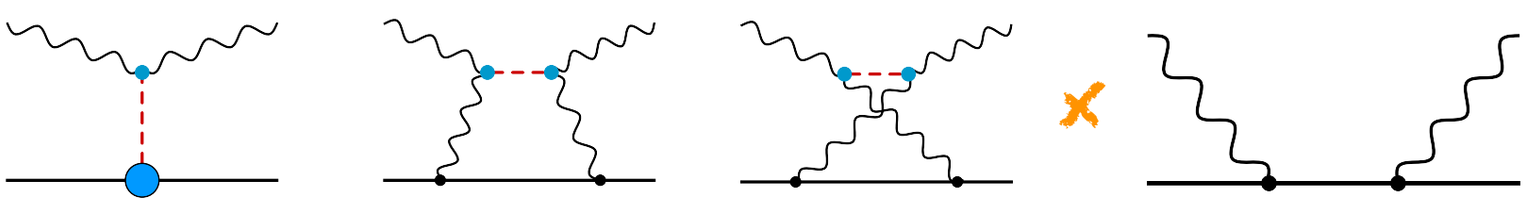

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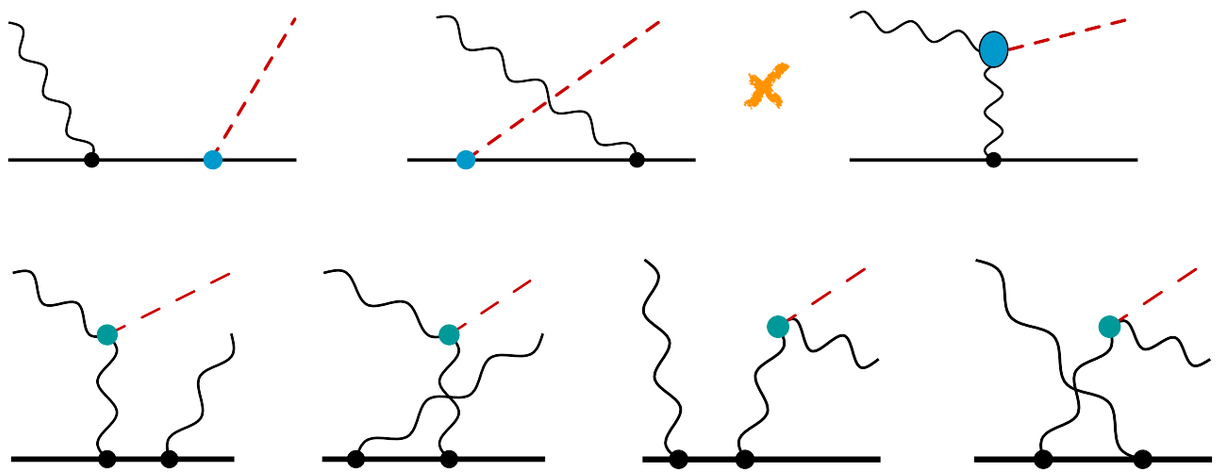

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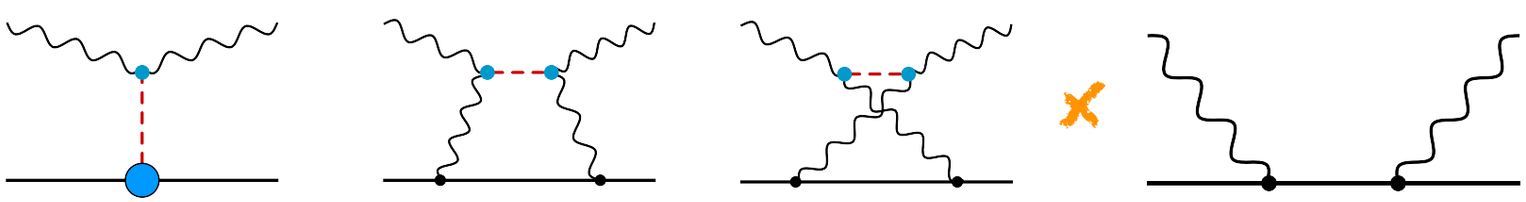

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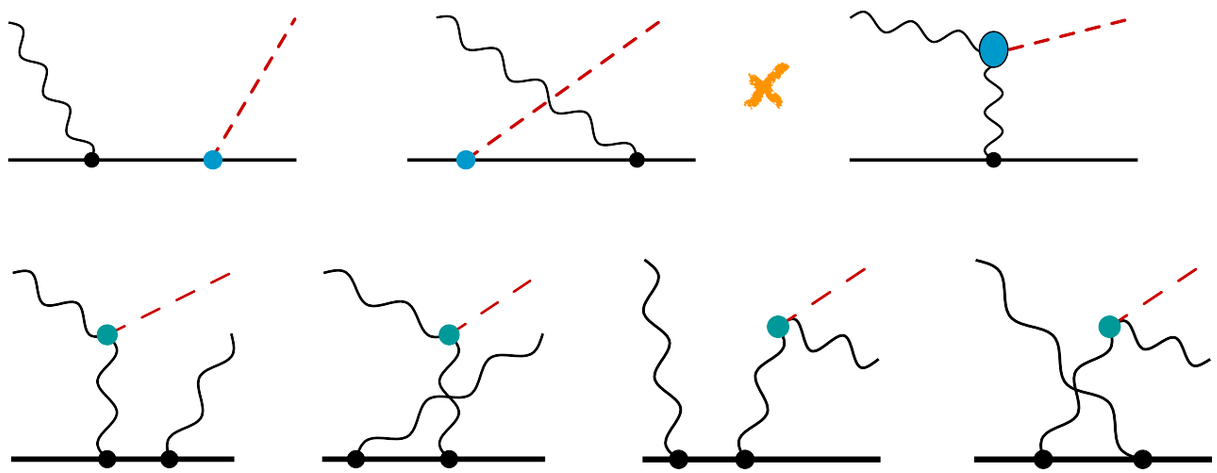

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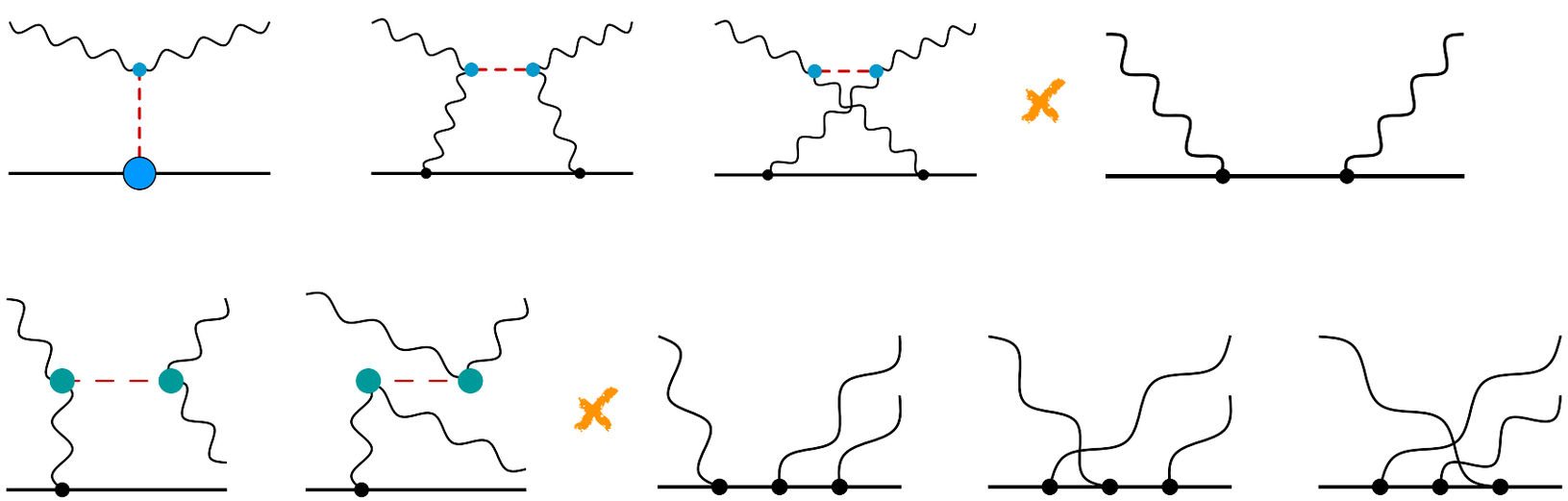

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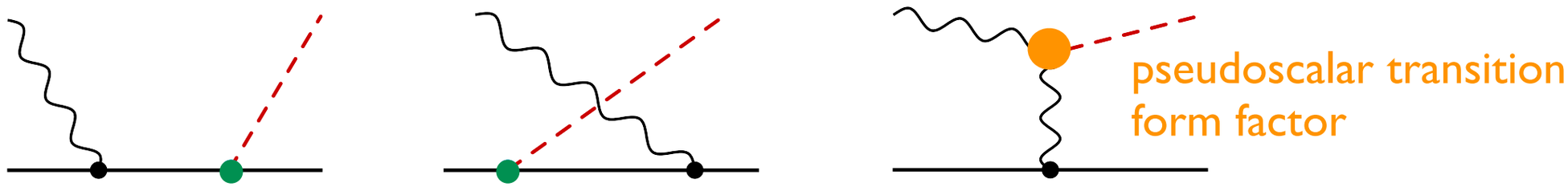

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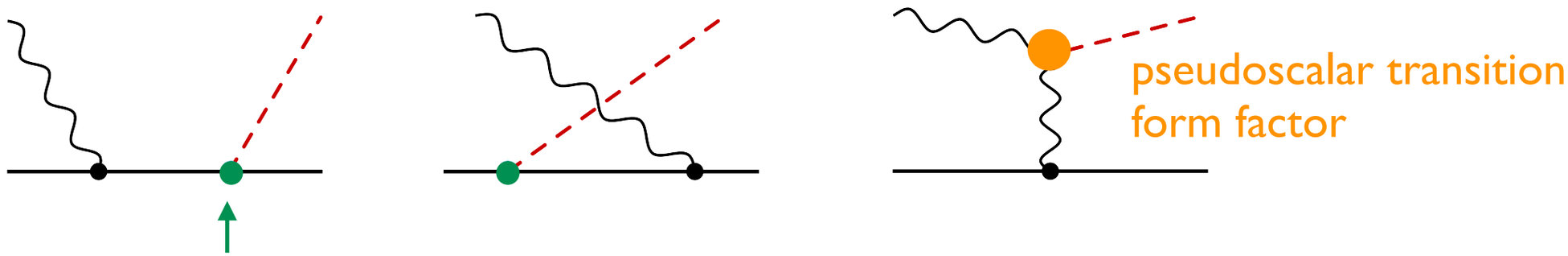

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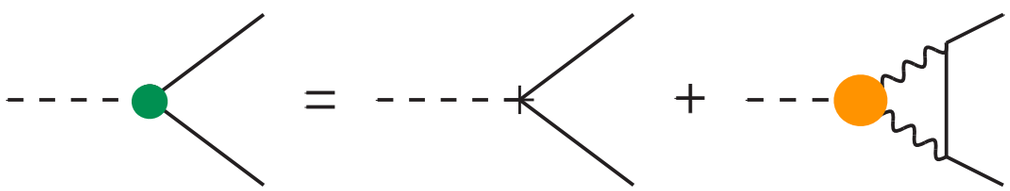

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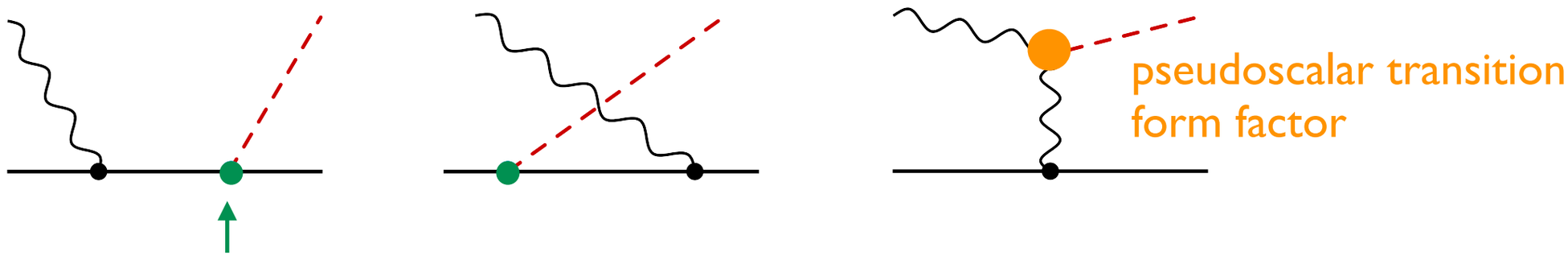

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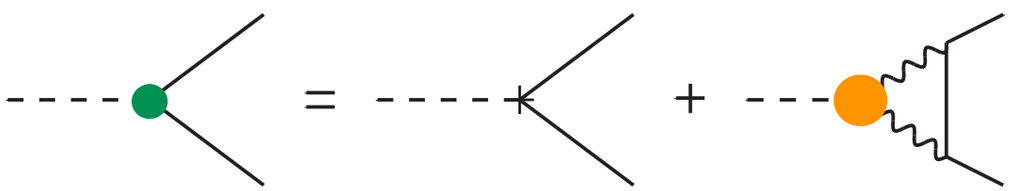

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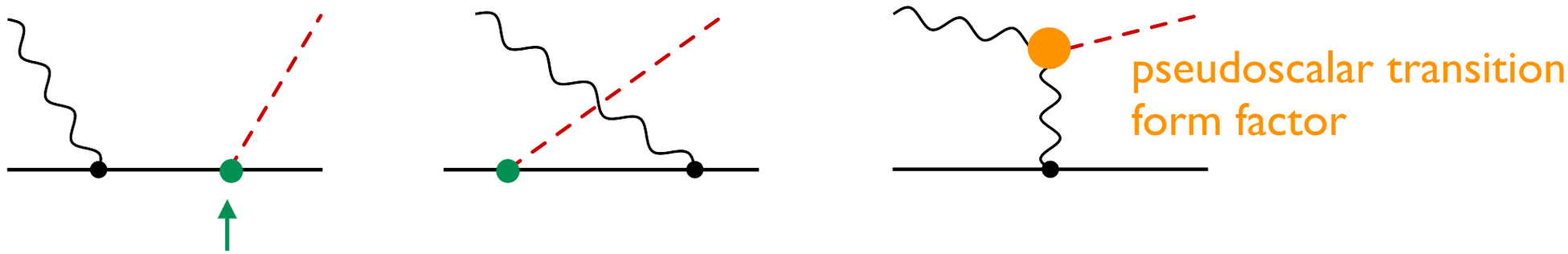

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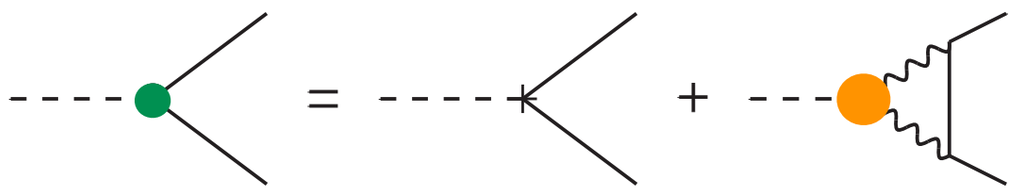

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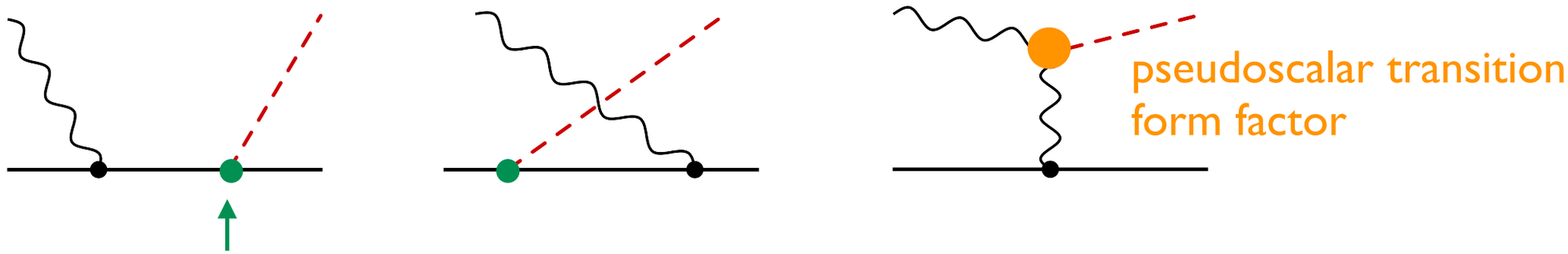

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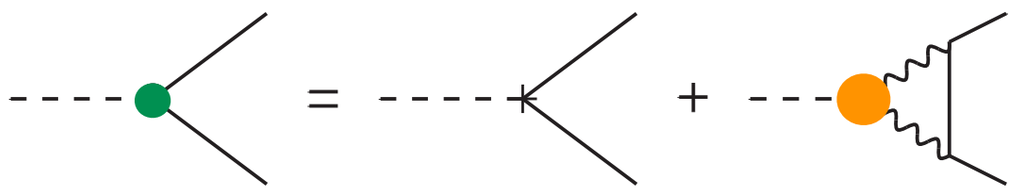

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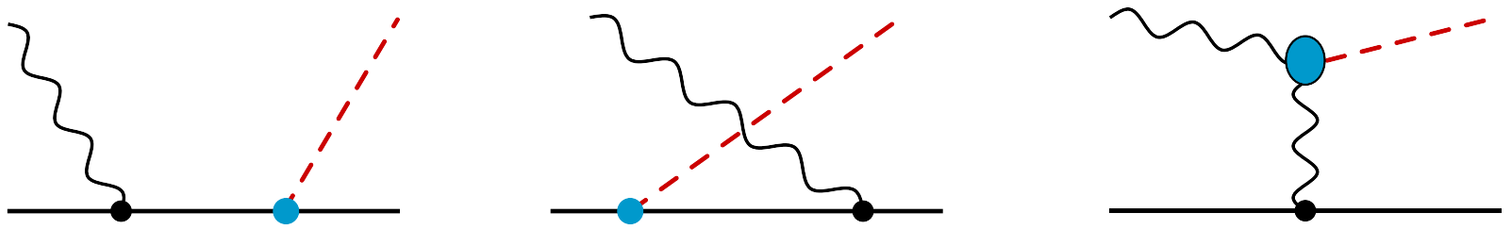
PSEUDOSCALAR PRODUCTION CHANNEL



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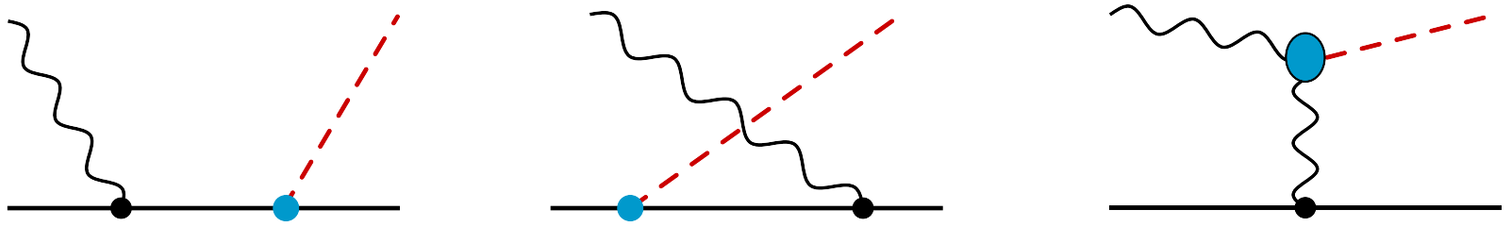

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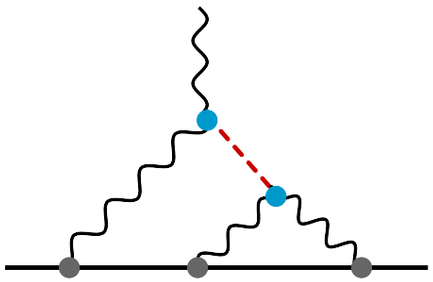

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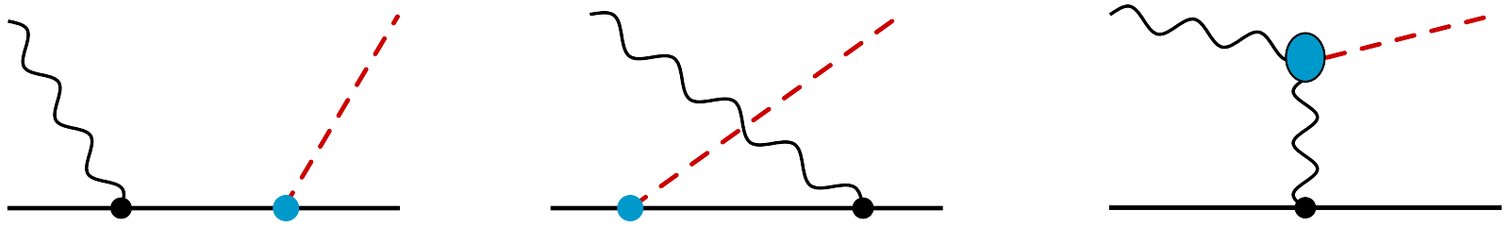

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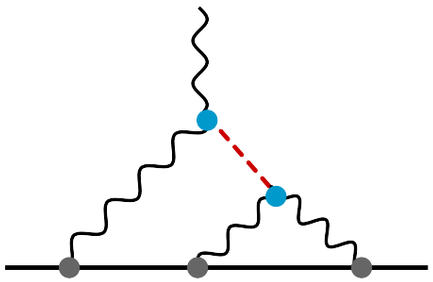

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- Contribution of the low-energy region (until ν = 2 GeV): $a_\mu^{(\gamma\mu\to\mu\pi)} = 8.4^{+2.1}_{-1.9}\times 10^{-10}$

DISPERSIVE APPROACHES TO HLBL

- Other dispersive approaches study different structures and apply different cuts

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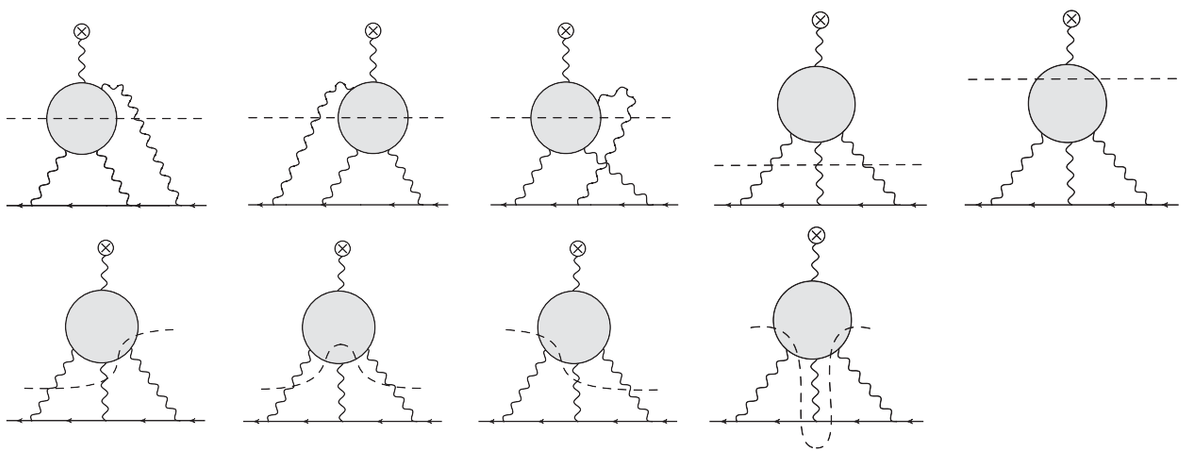

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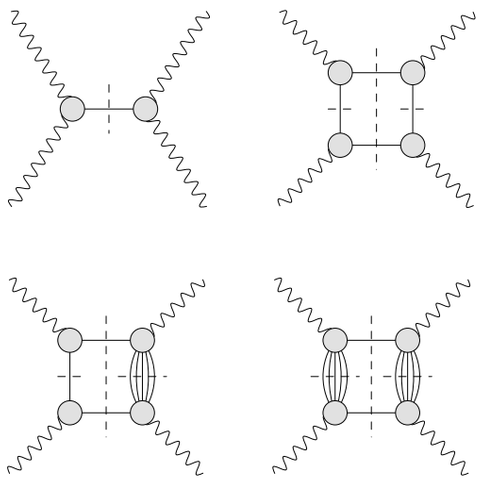

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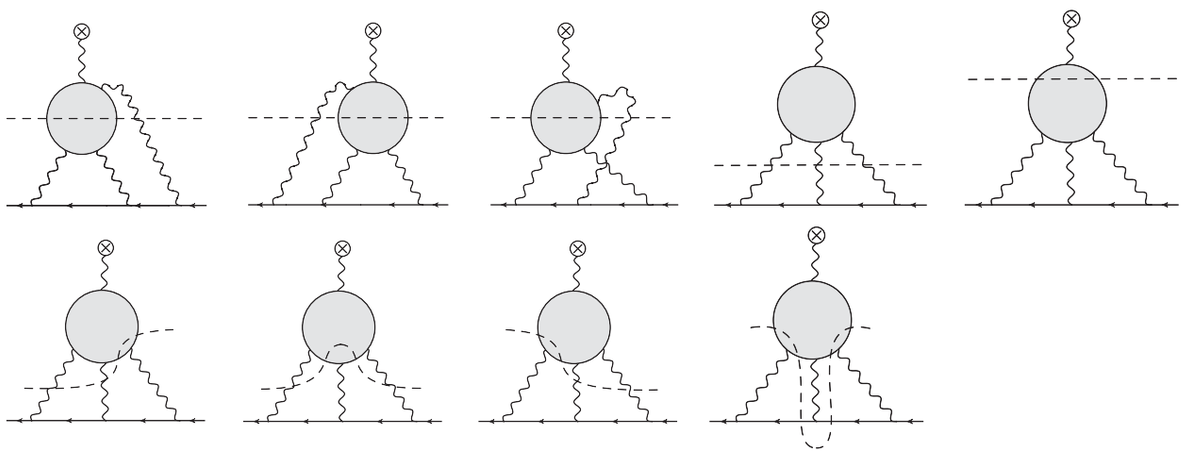

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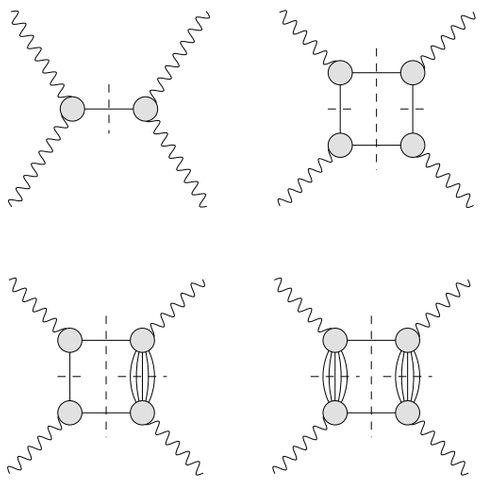

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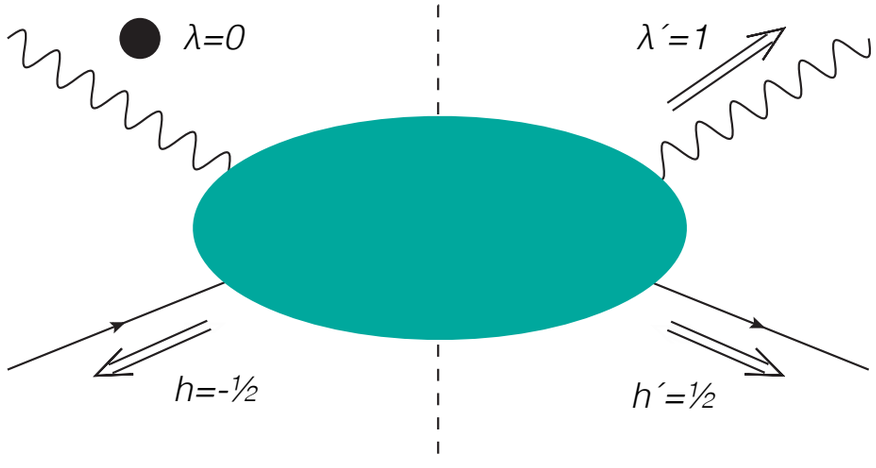
$$\frac{d\sigma_{LT}}{dt} = \frac{1}{16\pi(s-m_\mu^2)^2}\sum_{\lambda' h'}\frac{1}{\sqrt{2}}\mathrm{Re}\left(\mathscr{M}^*_{\lambda' h' 1\,1/2}\mathscr{M}_{\lambda' h' 0\,-1/2}\right)$$



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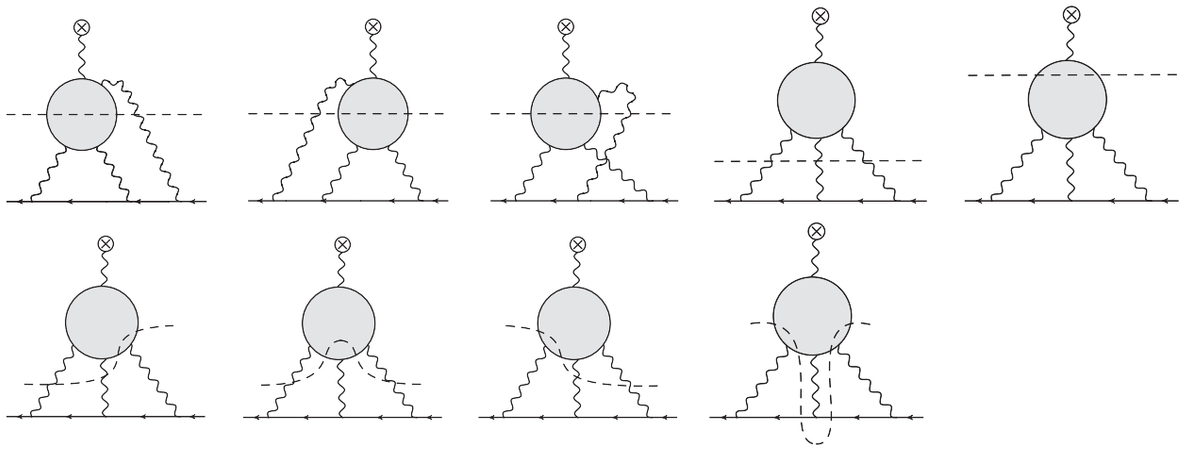

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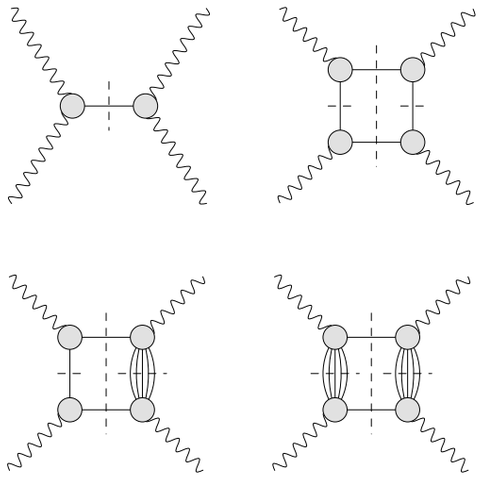
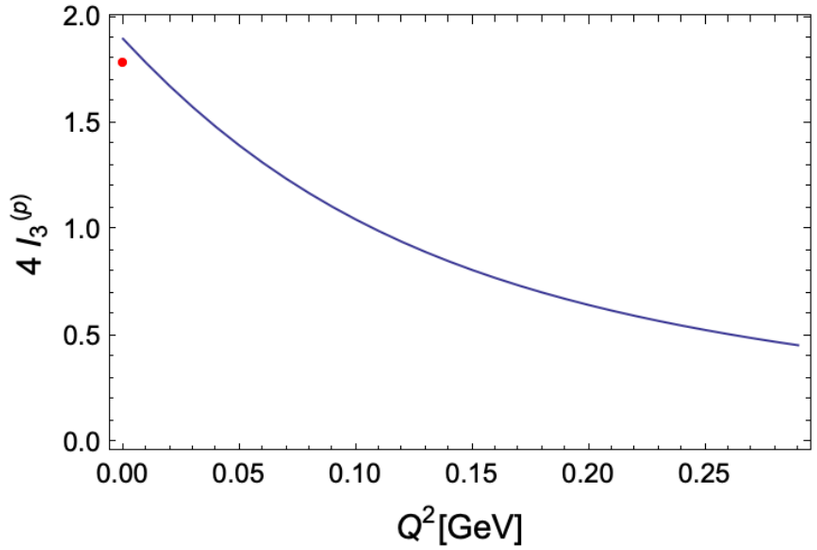
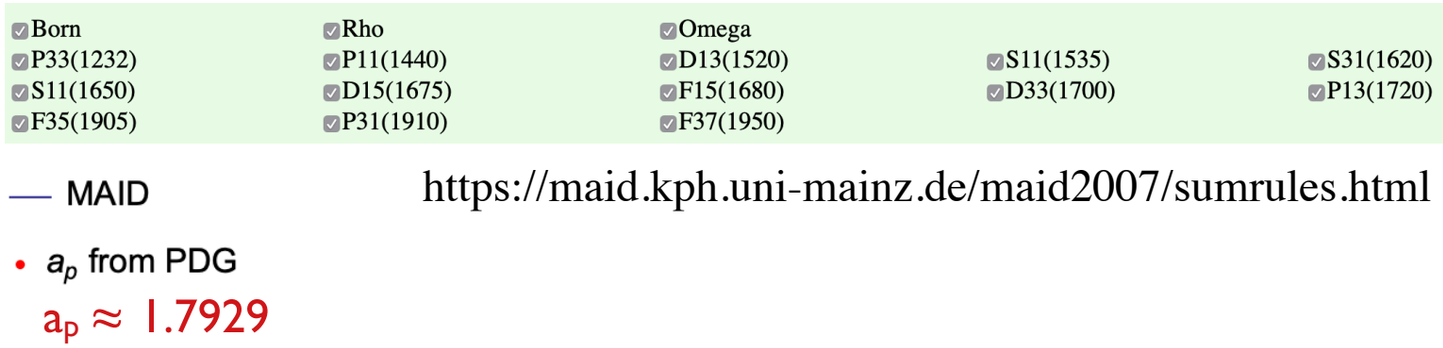


G. Colangelo, et al., JHEP 1509 (2015) 074.

- Schwinger sum rule: σ_{LT} is a response function involving a spin flip and therefore not positive-definite ⟶ individual channels can contribute with different signs





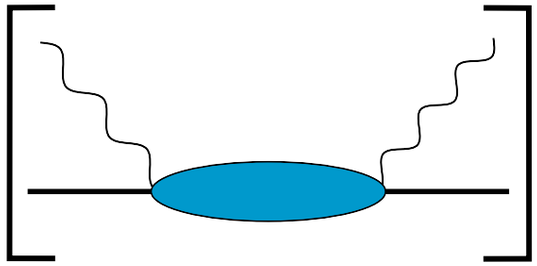
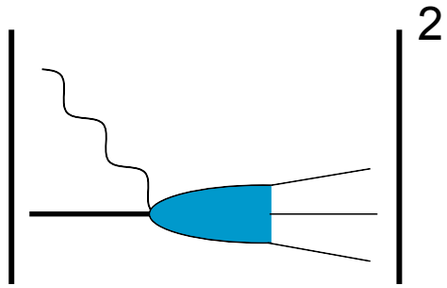
https://maid.kph.uni-mainz.de/maid2007/sumrules.html

SUMMARY AND CONCLUSION

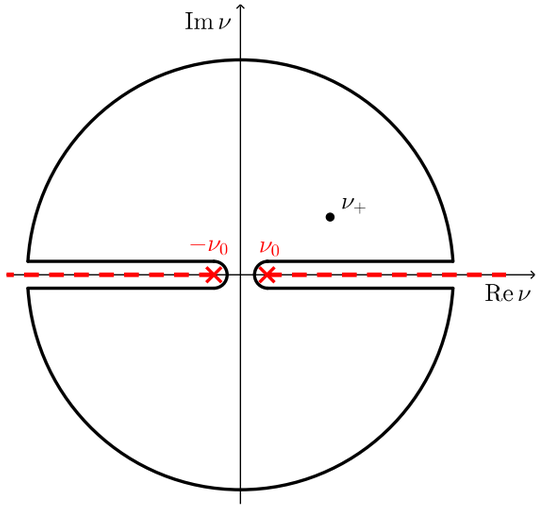
- The Schwinger sum rule is a new promising tool for data-driven evaluations of hadronic corrections (both HVP and HLbL) to the muon g-2
- Reproduces $\alpha/2\pi$ and the standard HVP formula
- Hadron photo-production contribution can be measured directly
- Pseudoscalar production gives a contribution to a_μ which is a factor of 2 to 3 larger than the conventional pseudoscalar-pole calculations
- Compton scattering channel gives a contribution to a_μ which is more than one order of magnitude smaller than the pseudoscalar production contribution

Back-up Slides

MODEL-INDEPENDENT SUM RULES

- Optical theorem (unitarity): Im  $\propto$  ^{2}

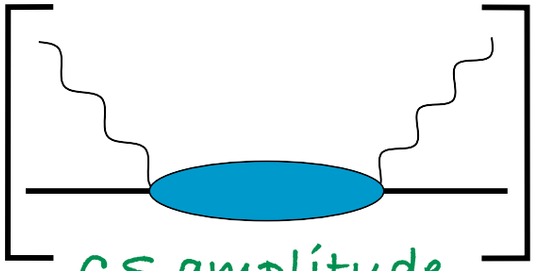
- Dispersion relation (analyticity/causality, crossing symmetry):



Cauchy formula: $f(\nu_+) = \frac{1}{2\pi i} \oint_C \mathrm{d}\xi \, \frac{f(\xi)}{\xi - \nu_+}$

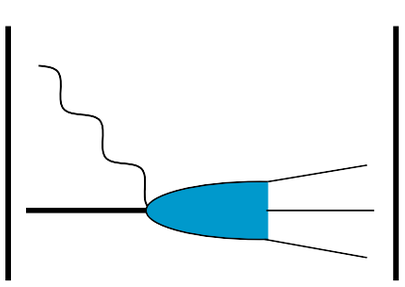
- Forward Compton scattering sum rules:

Low-energy expansion:
charge, g-2, polarizabilities



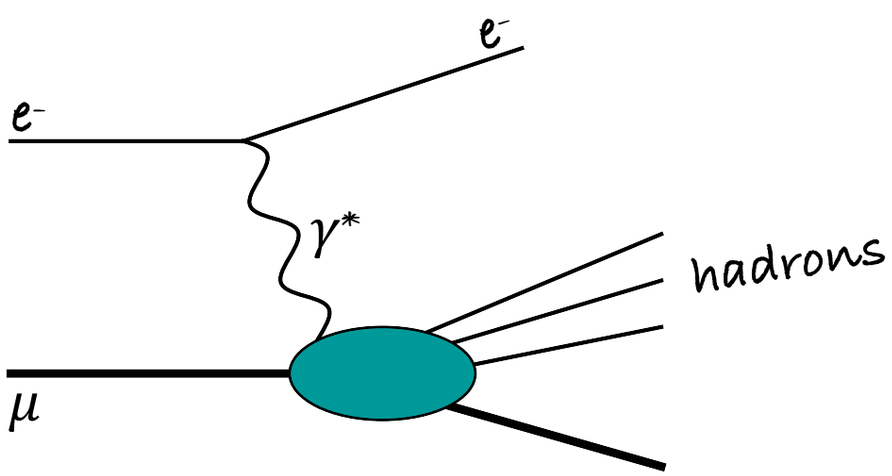
CS amplitude

$$(\nu, Q^2) = \frac{2}{\pi} \int_{\nu_0}^{\infty} \mathrm{d}\nu' \frac{\nu'}{\nu'^2 - \nu^2}$$

 ^{2} (ν', Q^2)

Cross sections, structure functions

SPIN STRUCTURE FUNCTIONS



$$a = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$$

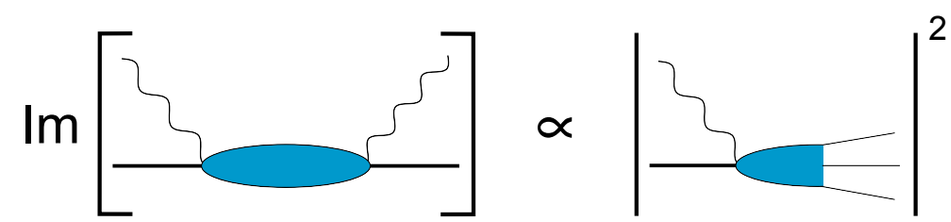
$$= \lim_{Q^2 \to 0} \frac{8m^2}{Q^2} \int_0^{x_0} \mathrm{d}x \, [\bar{g}_1 + \bar{g}_2](x, Q^2)$$

muon spin structure functions g_1 and g_2

Bjorken variable: $x = \frac{Q^2}{2m\nu}$

- Spin-dependent forward doubly-virtual Compton scattering:

$$T_A^{\mu\nu}(q,p) = -\frac{1}{M} \gamma^{\mu\nu\alpha} q_\alpha \, S_1(\nu, Q^2) + \frac{Q^2}{M^2} \gamma^{\mu\nu} S_2(\nu, Q^2)$$



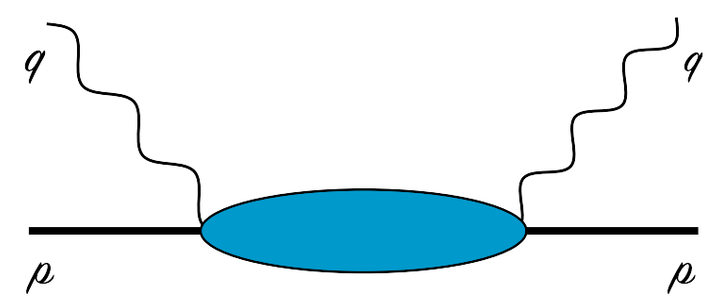
- Optical theorem:

$$\mathrm{Im}\, S_1(\nu, Q^2) = \frac{4\pi^2 \alpha}{\nu} g_1(x, Q^2) = \frac{M\nu^2}{\nu^2 + Q^2} \left[\frac{Q}{\nu} \sigma_{LT} + \sigma_{TT} \right] (\nu, Q^2)$$

$$\mathrm{Im}\, S_2(\nu, Q^2) = \frac{4\pi^2 \alpha M}{\nu^2} g_2(x, Q^2) = \frac{M^2 \nu}{\nu^2 + Q^2} \left[\frac{\nu}{Q} \sigma_{LT} - \sigma_{TT} \right] (\nu, Q^2)$$

SCHWINGER, GDH AND BC SUM RULES

- Sum rules are model-independent relations based on very general principles:
  - Analyticity/causality (dispersion relations)
  - Unitarity (optical theorem)
  - Crossing symmetry



- Some sum rules of Compton scattering off a spin-1/2 particle:

$$(1+a)a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}}{Q} - \frac{\sigma_{TT}}{\nu}\right]_{Q^2=0}$$

Burkhardt—Cottingham sum rule (1970) $\int_0^1 \mathrm{d}x\, g_2(x,Q^2) = 0$

$$\ominus \qquad a^2 = -\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{TT}(\nu)}{\nu}$$

Gerasimov—Drell—Hearn sum rule (1966)

$$a = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

Schwinger sum rule (1975)

linear dependence