

Lattice HVP: Status/Plan RBC/UKQCD

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Aaron & Mattia joined since 2018 paper

Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

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We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_{\mu}^{\text{HVP LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R -ratio data, we significantly improve the precision to $a_{\mu}^{\text{HVP LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_{\mu}^{\text{HVP LO}}$.

Pure lattice result and dispersive result with reduced $\pi\pi$ dependence (window method)

Lattice QCD – Time-Moment Representation

Starting from the vector current $J_\mu(x) = i \sum_f Q_f \bar{\Psi}_f(x) \gamma_\mu \Psi_f(x)$ we may write

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

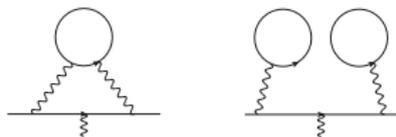
$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle$$

and w_t capturing the photon and muon part of the HVP diagrams (Bernecker-Meyer 2011).

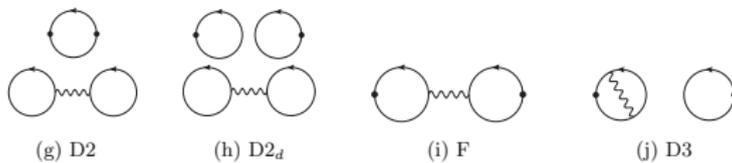
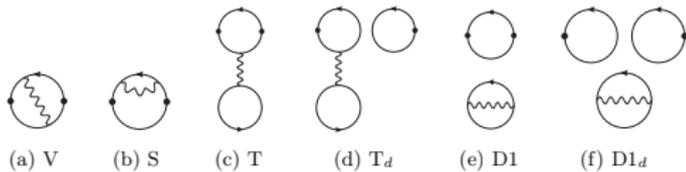
The correlator $C(t)$ is computed in lattice QCD+QED at physical pion mass with non-degenerate up and down quark masses including up, down, strange, and charm quark contributions. The missing bottom quark contributions are computed in pQCD.

Diagrams

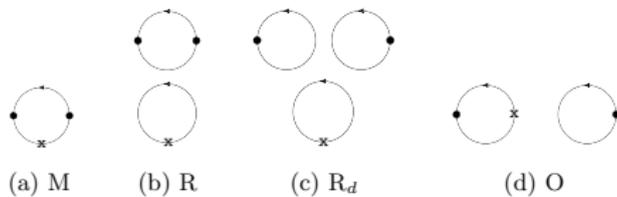
Isospin
limit



QED
corrections



Strong
isospin
breaking



Window method (implemented in RBC/UKQCD 2018)

We therefore also consider a window method. Following Meyer-Bernecker 2011 and smearing over t to define the continuum limit we write

$$a_\mu = a_\mu^{\text{SD}} + a_\mu^{\text{W}} + a_\mu^{\text{LD}}$$

with

$$a_\mu^{\text{SD}} = \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)],$$

$$a_\mu^{\text{W}} = \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)],$$

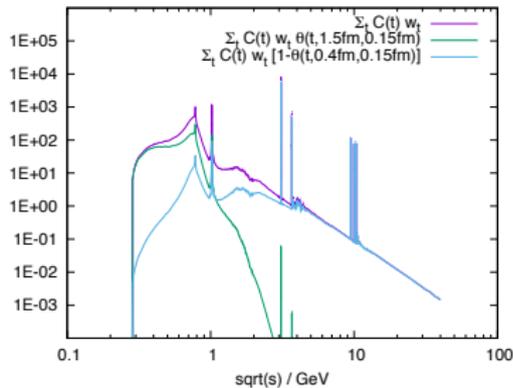
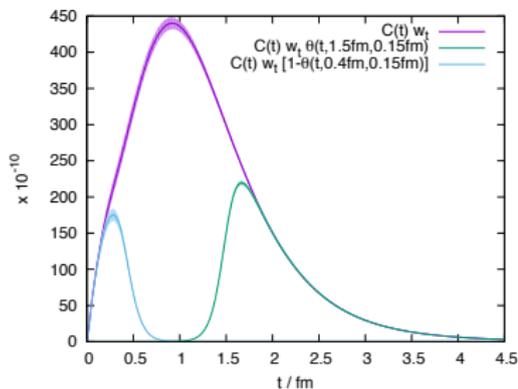
$$a_\mu^{\text{LD}} = \sum_t C(t) w_t \Theta(t, t_1, \Delta),$$

$$\Theta(t, t', \Delta) = [1 + \tanh [(t - t')/\Delta]] / 2.$$

In this version of the calculation, we use

$C(t) = \frac{1}{12\pi^2} \int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t}$ with $R(s) = \frac{3s}{4\pi\alpha^2} \sigma(s, e^+e^- \rightarrow \text{had})$
to compute a_μ^{SD} and a_μ^{LD} .

How does this translate to the time-like region?



Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.

The pure lattice calculation of RBC/UKQCD 2018:

$$\begin{aligned} 10^{10} \times a_{\mu}^{\text{HVP LO}} &= 715.4(18.7) \\ &= 715.4(16.3)_S(7.8)_V(3.0)_C(1.9)_A(3.2)_{\text{other}} \end{aligned}$$

(S) statistics, (V) finite-volume errors, (C) the continuum limit extrapolation, (A) scale setting uncertainty;
other \supset neglected diagrams for QED and SIB, estimate of bottom quark contribution

Statistical noise mostly from isospin symmetric light quark connected (14.2) and disconnected (3.3), QED (5.7), SIB (4.3)

RBC/UKQCD 2019 update (in preparation):

- ▶ Improved methodology
- ▶ A lot of new data

Improved methodology

Improved statistics and systematics – Bounding Method

BMW/RBC/UKQCD 2016

The correlator in finite volume

$$C(t) = \sum_n |\langle 0|V|n\rangle|^2 e^{-E_n t}.$$

We can bound this correlator at each t from above and below by the correlators

$$\tilde{C}(t; T, \tilde{E}) = \begin{cases} C(t) & t < T, \\ C(T)e^{-(t-T)\tilde{E}} & t \geq T \end{cases}$$

for proper choice of \tilde{E} . We can chose $\tilde{E} = E_0$ (assuming $E_0 < E_1 < \dots$) to create a strict upper bound and any \tilde{E} larger than the local effective mass to define a strict lower bound.

Therefore if we had precise knowledge of the lowest $n = 0, \dots, N$ values of $|\langle 0|V|n\rangle|$ and E_n , we could define a new correlator

$$C^N(t) = C(t) - \sum_{n=0}^N |\langle 0|V|n\rangle|^2 e^{-E_n t}$$

which we could bound much more strongly through the larger lowest energy $E_{N+1} \gg E_0$. New method: do a GEVP study of FV spectrum to perform this subtraction.

GEVP operators ($l = 1, l_3 = 0, \mathbf{p}_{\text{tot}} = \vec{0}, T_1^-$): 2pi (1,2,3,4 units of momentum), 4pi (two different), local and smeared vector currents

- ▶ Two data sets: n-pi operators made out of improved ($\gamma_5, \gamma_5\gamma_t, \gamma_5\gamma_i e^{i\vec{p}\vec{x}}$) or unimproved (γ_5) pion operators

- ▶ Operators automatically generated for given representation: Aaron Meyer will make his code public soon



- ▶ Automatically contract operators to diagrams:
<https://github.com/lehner/Wick>
- ▶ Automatically evaluate diagrams using A2A/Distillation data:
<https://github.com/lehner/Contractor>

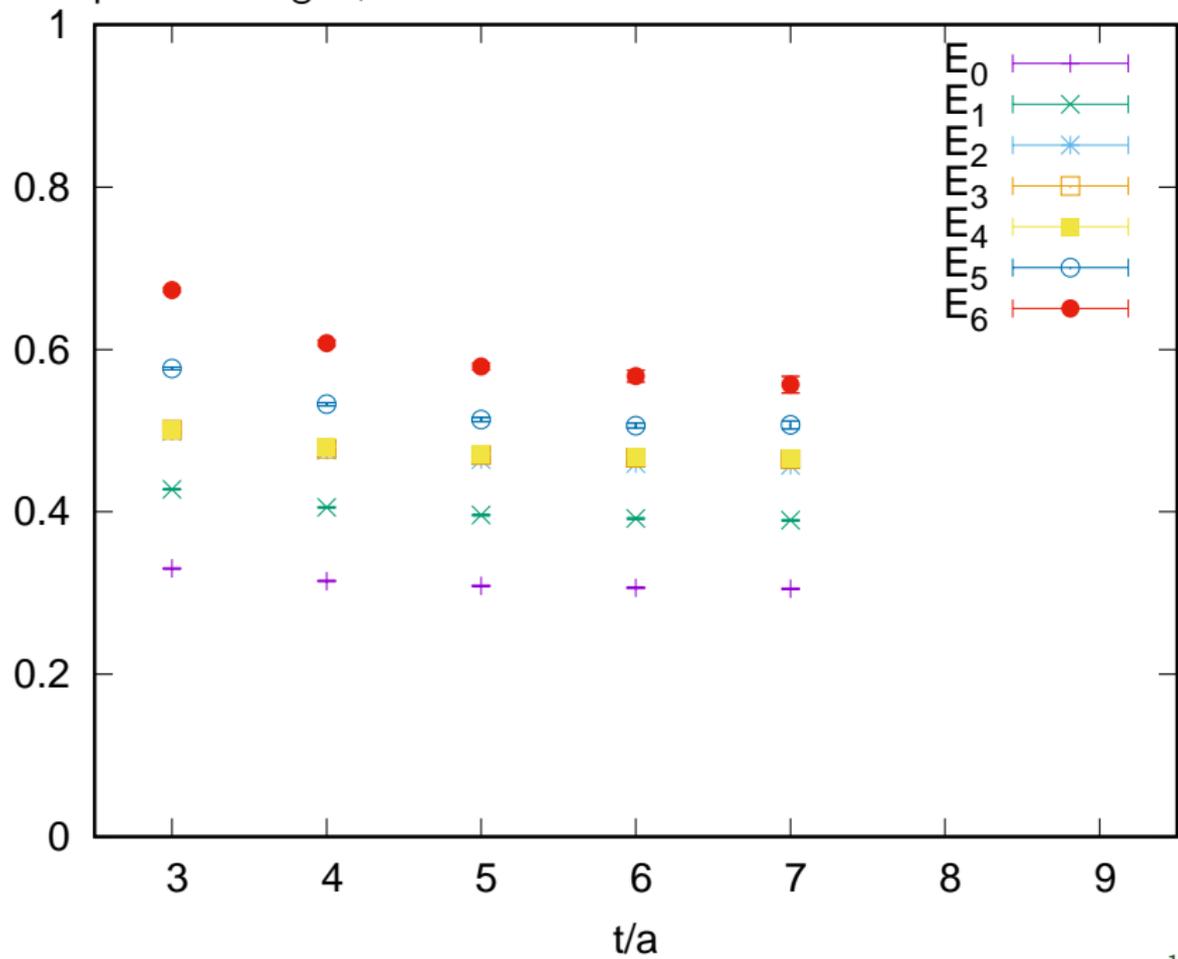
974 contractions for 4pi-4pi:

2/3:

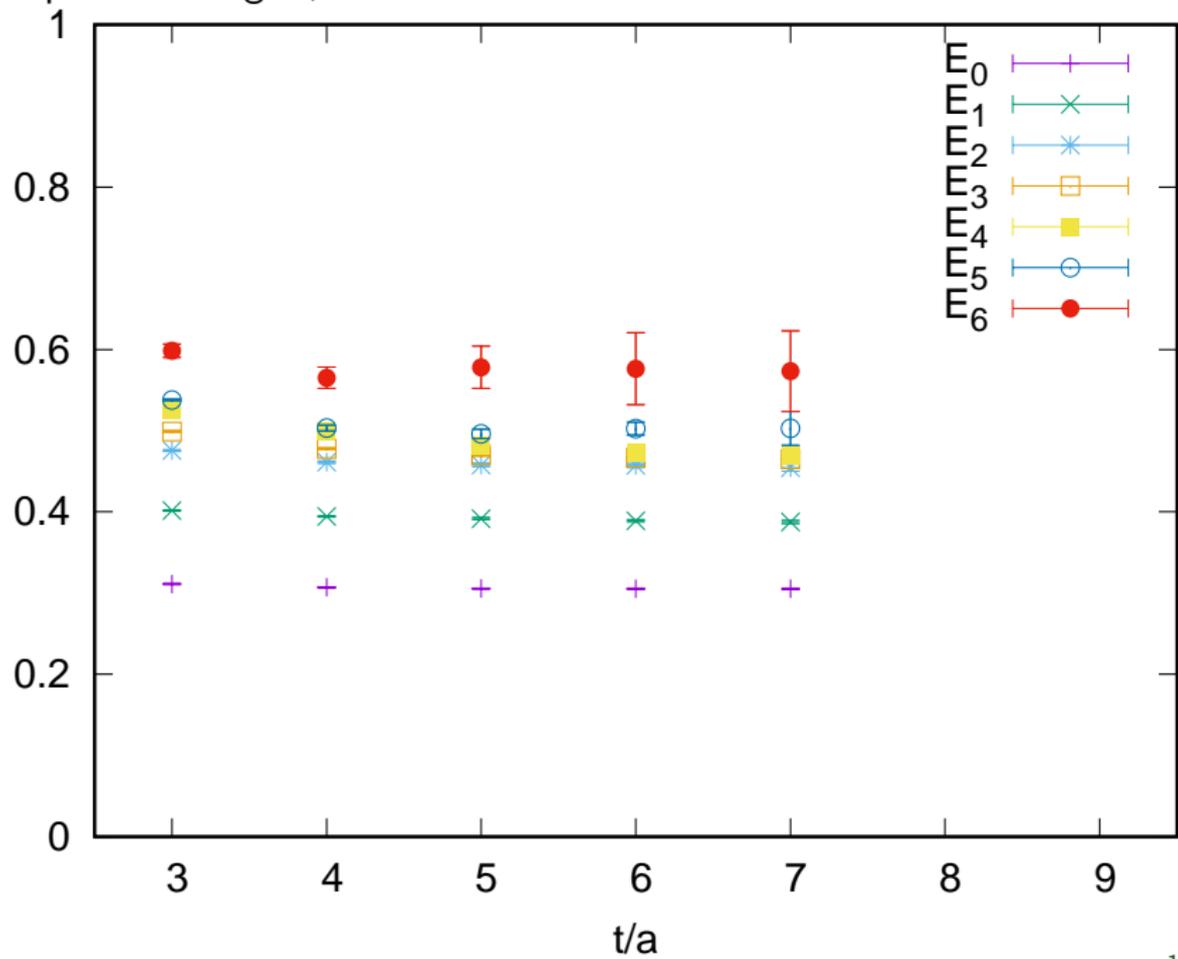


974 contractions for 4pi-4pi:

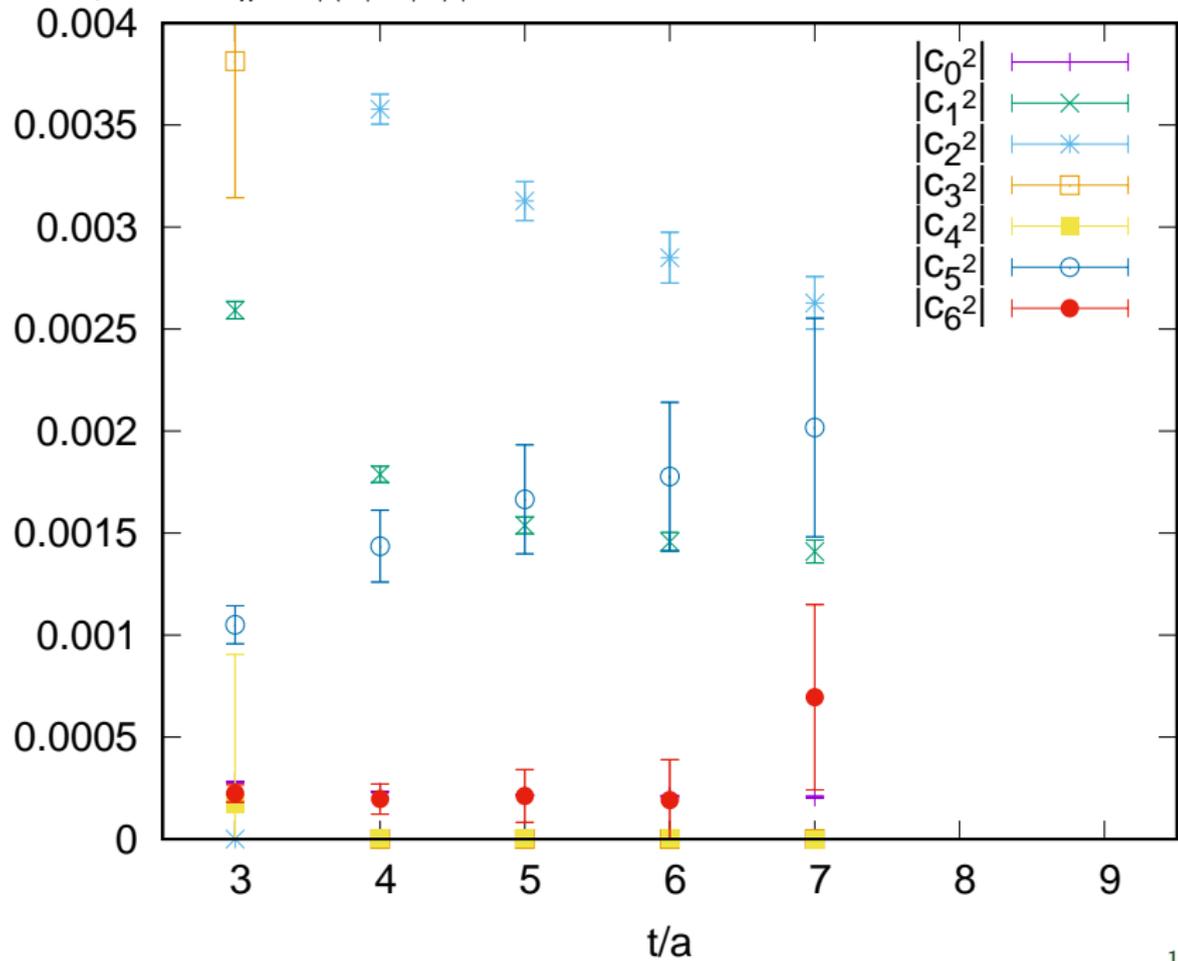
Unimproved energies, $a^{-1} = 1.73 \text{ GeV}$



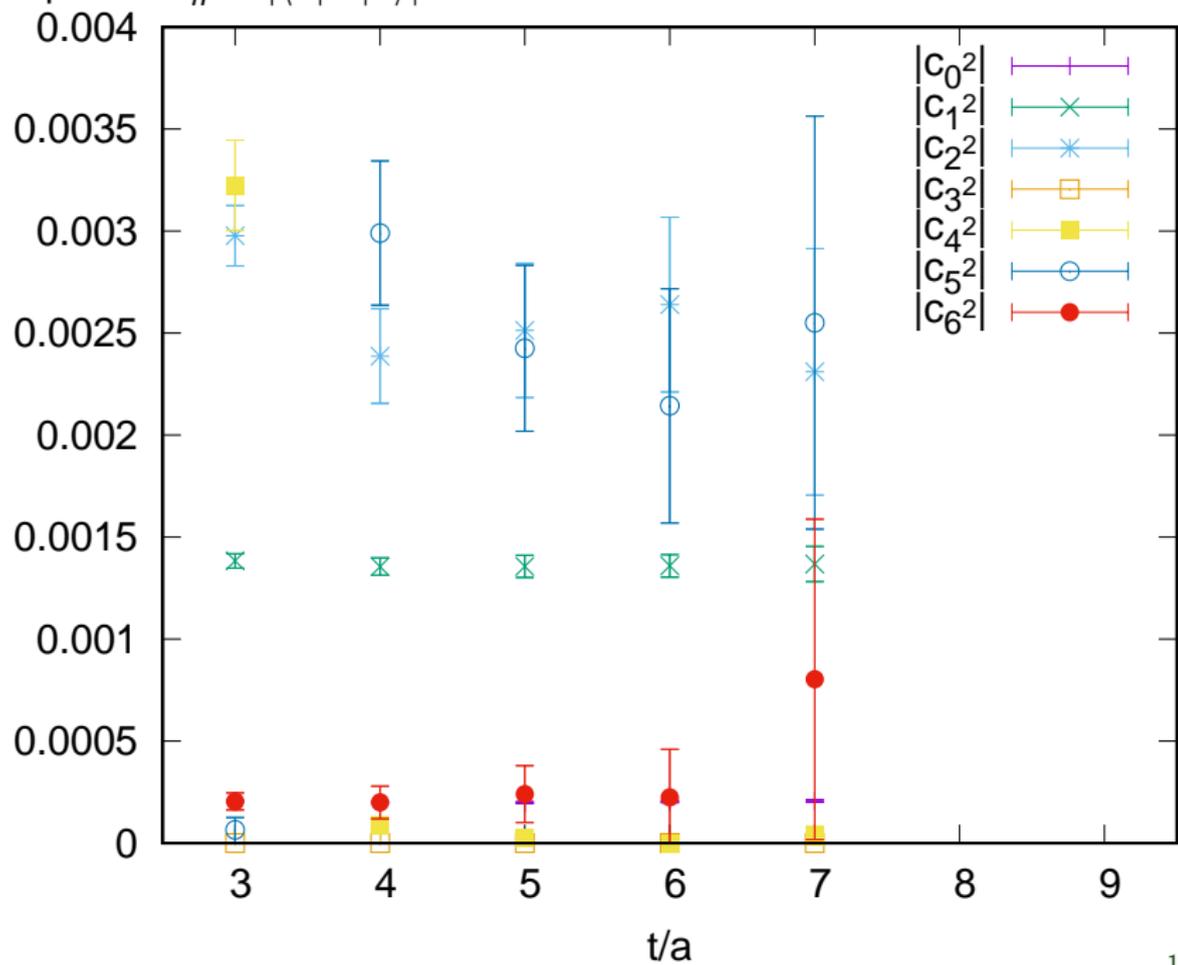
Improved energies, $a^{-1} = 1.73$ GeV

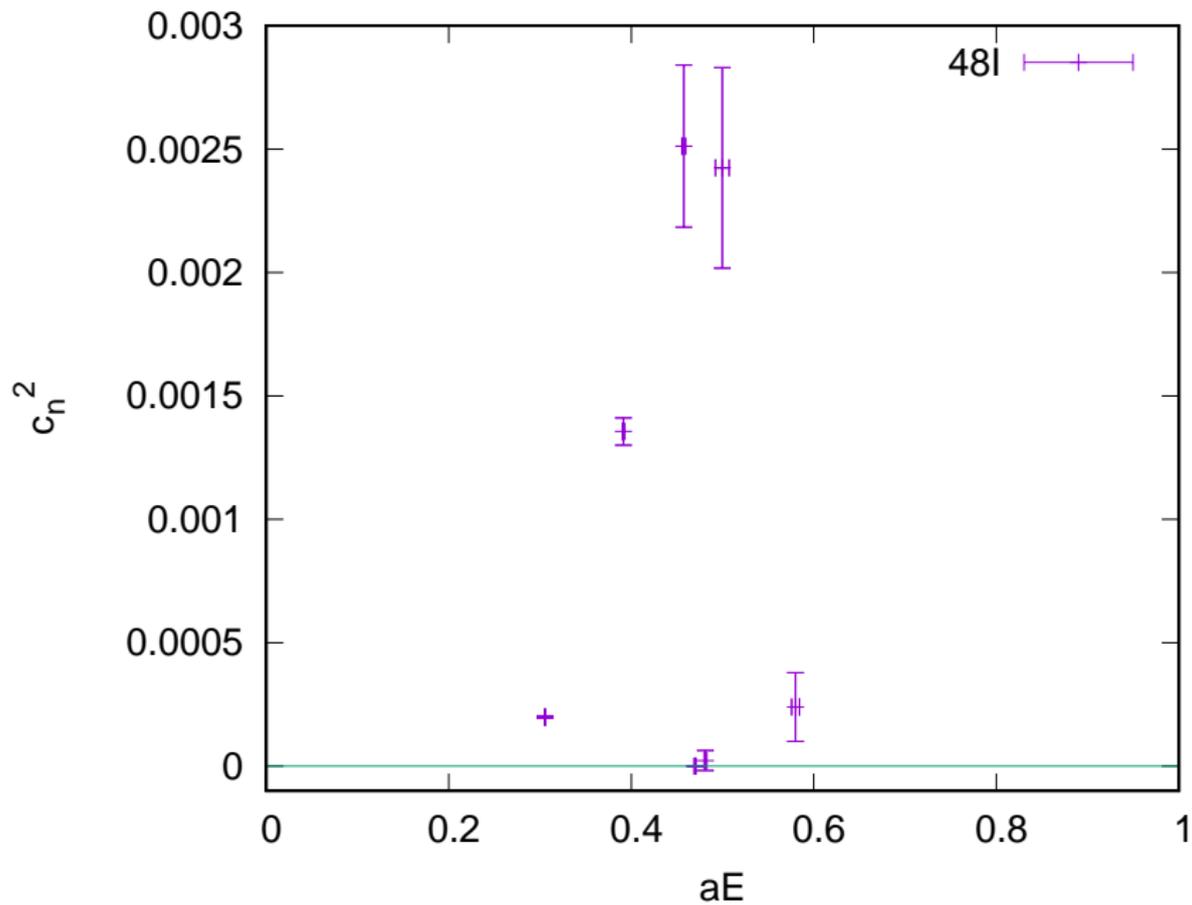


Unimproved $c_n^2 = |\langle 0|V|n\rangle|^2$

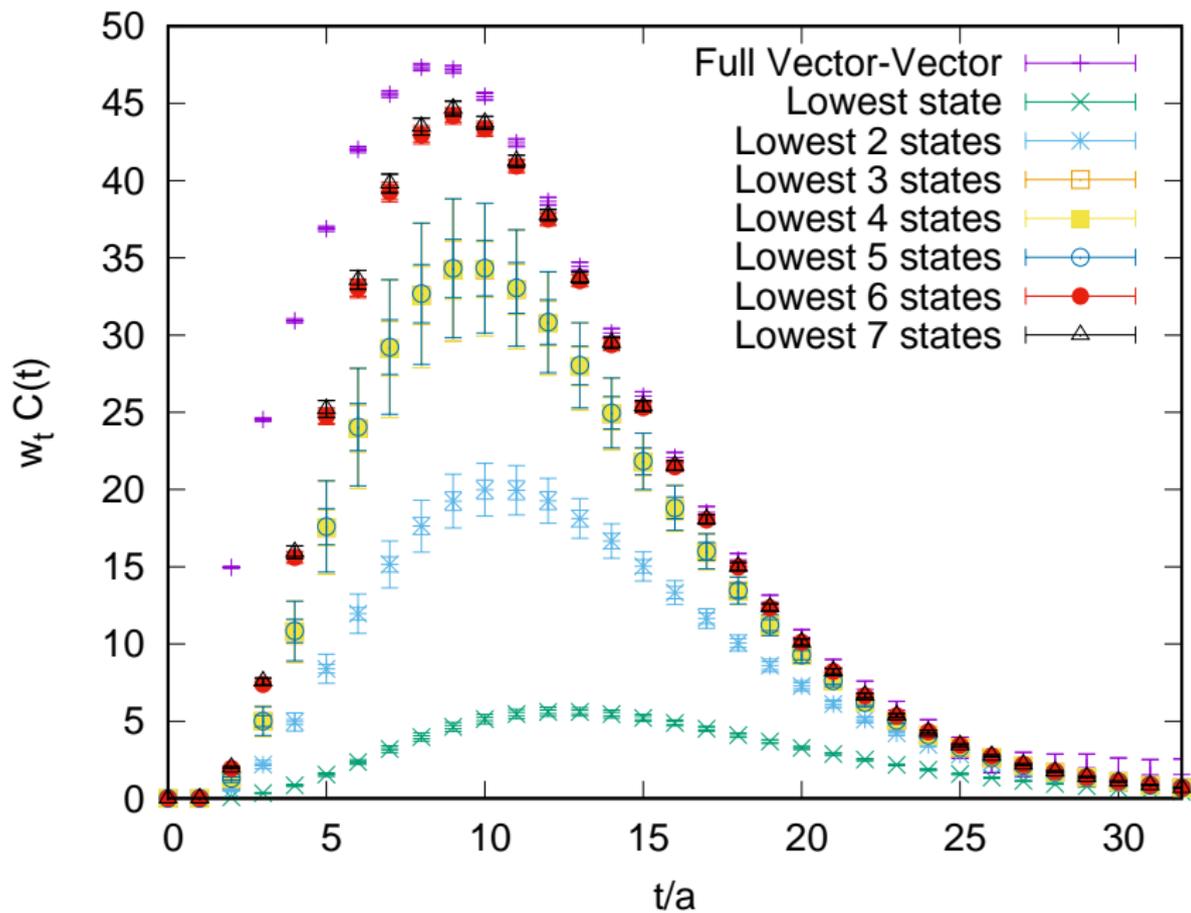


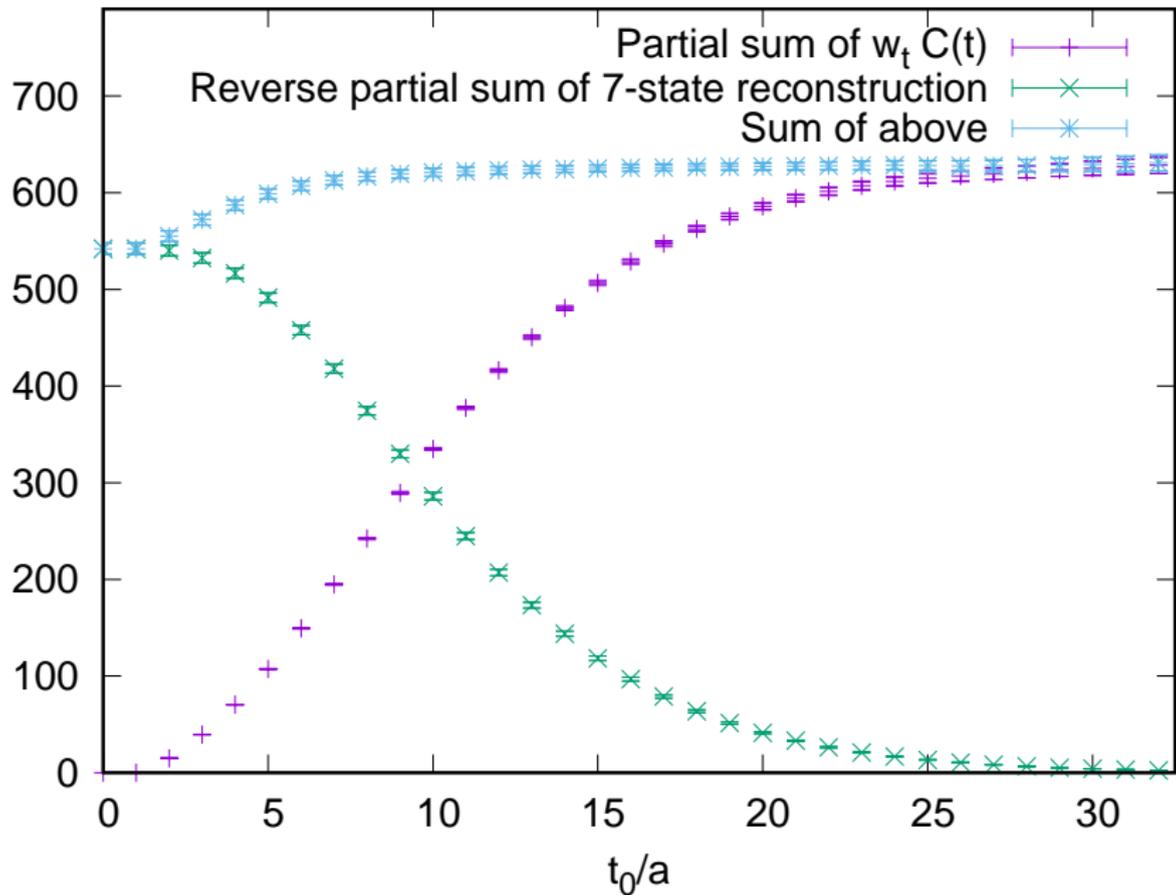
Improved $c_n^2 = |\langle 0|V|n\rangle|^2$

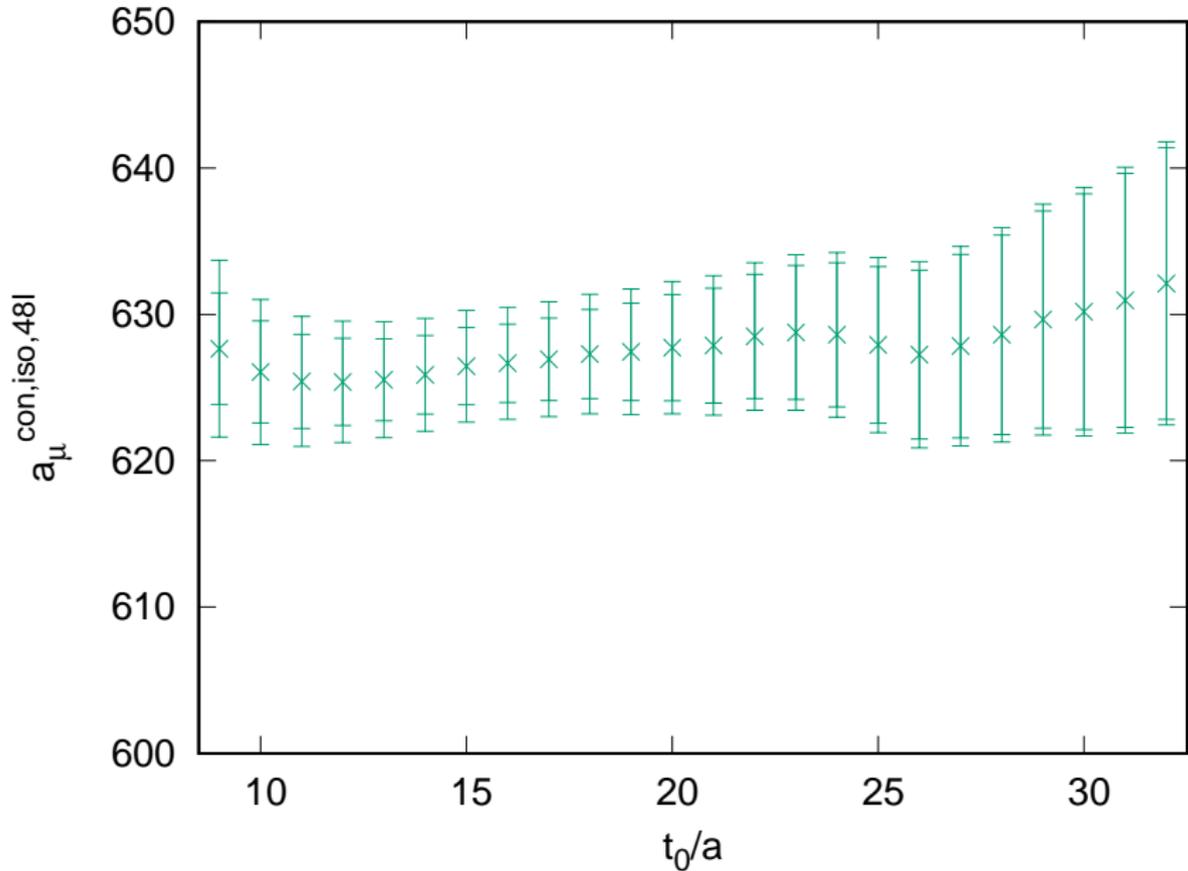




$$a^{-1} = 1.73 \text{ GeV}$$







$$a_\mu^{ud, conn, iso, 48l} 10^{10} = 626.5(2.6)_S(0.4)_{Z_V}(2.6)_{a^{-1}}(0.5)_{\text{Bnd}}(0.5)_{\text{Exc.st.}}$$

Compare to previous statistical error for light-connected-iso: 14.2

Improved systematics – compute finite-volume effects from first-principles

RBC/UKQCD study of QCD at **physical pion mass** at three different volumes:

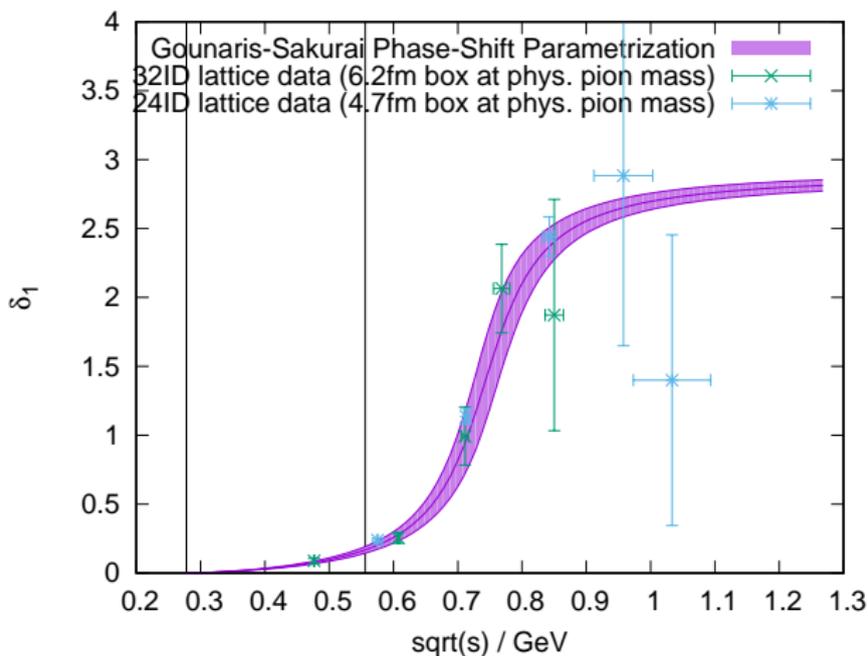
$$L = 4.66 \text{ fm}, L = 5.47 \text{ fm}, L = 6.22 \text{ fm}$$

Results for light-quark isospin-symmetric connected contribution:

▶ $a_\mu(L = 6.22 \text{ fm}) - a_\mu(L = 4.66 \text{ fm}) = 12.2 \times 10^{-10}$ (sQED),
 $21.6(6.3) \times 10^{-10}$ (lattice QCD)

- ▶ Need to do better than sQED in finite-volume

First constrain the p -wave phase shift from our $L = 6.22$ fm physical pion mass lattice:



$$E_\rho = 0.766(21) \text{ GeV (PDG } 0.77549(34) \text{ GeV)}$$

$$\Gamma_\rho = 0.139(18) \text{ GeV (PDG } 0.1462(7) \text{ GeV)}$$

GSL² finite-volume results compared to sQED and lattice

GSL² method of Meyer 2012

Results for light-quark isospin-symmetric connected contribution:

- ▶ FV difference between $a_\mu(L = 6.22 \text{ fm}) - a_\mu(L = 4.66 \text{ fm}) = 12.2 \times 10^{-10}$ (sQED), $21.6(6.3) \times 10^{-10}$ (lattice QCD), $20(3) \times 10^{-10}$ (GSL²)
- ▶ GSL² prediction agrees with actual FV effect measured on the lattice, sQED is in slight tension, two-loop FV ChPT to be compared next [Bijnens and Relefors 2017](#)
- ▶ Use GSL² to update FV correction of [Phys. Rev. Lett. 121, 022003 \(2018\)](#): $a_\mu(L \rightarrow \infty) - a_\mu(L = 5.47 \text{ fm}) = 16(4) \times 10^{-10}$ (sQED), $22(1) \times 10^{-10}$ (GSL²); sQED error estimate based on [Bijnens and Relefors 2017, table 1](#).
- ▶ Compare also to Hansen-Patella 2019 [1904.10010](#): $a_\mu(L \rightarrow \infty) - a_\mu(L = 5.47 \text{ fm}) \approx 14 \times 10^{-10}$, effect of neglected $e^{-\sqrt{2}m_\pi L}$ significant; currently computed

Other improvements (I) - SIB:

- ▶ Calculate both mass derivative with insertion of scalar operators as well as fit to valence pion mass dependence

Other improvements (II) - Disconnected diagrams:

- ▶ Study volume-dependence and continuum limit, much more statistics; build “tadpole” fields following our [PRL116\(2016\)232002](#)

Other improvements (III) - QED continuum limit and scale:

- ▶ QED analysis of 2018 paper now with continuum limit, new Z_V determination for consistency check (vector charge of pion), improved Ω mass analysis

Other improvements (IV) - QED from HLbL data:

- ▶ HLbL point-source data: see talk by Luchang on Wednesday
- ▶ HVP QED from re-analysis of HLbL point-source data (see also τ project and Mattia's talk, 1811.00508) reduces statistical noise by $\approx 10\times$ for V and S
- ▶ Infinte-volume and continuum limit also for diagram V, S, and F
- ▶ First results for T, D1, and R; other sub-leading in preparation



New data set

Ensembles at physical pion mass:

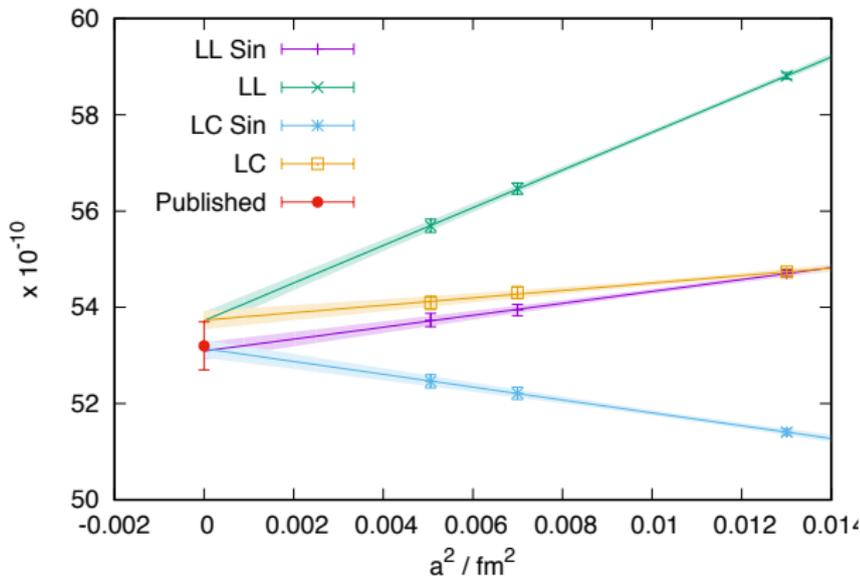
48l (1.73 GeV, 5.5fm), 64l (2.359 GeV, 5.4fm), 24ID (1 GeV, 4.7fm), 32ID (1 GeV, 6.2fm), 48ID (1 GeV, 9.3fm), 32IDf (1.37 GeV, 4.6fm)

RBC/UKQCD 2019 (data for light quarks, changes from 2018):

- ▶ A2A data for connected isospin symmetric: 48l (127 conf \rightarrow 400 conf), 64l (160 conf \rightarrow 250 conf), 24ID (new 130 conf, multi mass), 32ID (new 88 conf, multi mass)
- ▶ A2A data (tadpole fields) for disconnected: 48l (33 conf), 24ID (new 260 conf, multi mass), 32IDf (new 103 conf)
- ▶ QED and SIB corrections to meson and Ω masses, Z_V : 48l (30 conf) and 64l (new 30 conf)
- ▶ QED and SIB from HLbL point sources on 48l, 24ID, 32ID, 32IDf (on order of 20 conf each, 2000 points per config)
- ▶ Distillation data on 48l (33 conf), 64l (in progr.), 24ID (33 conf), 32ID (11 conf, multi-mass)
- ▶ New Ω mass operators (excited states control): 48l (130 conf)

Add $a^{-1} = 2.77$ GeV lattice spacing

- ▶ Third lattice spacing for strange data ($a^{-1} = 2.77$ GeV with $m_\pi = 234$ MeV with sea light-quark mass corrected from global fit):



- ▶ For light quark need new ensemble at physical pion mass. Started run on Summit Machine at Oak Ridge this year ($a^{-1} = 2.77$ GeV with $m_\pi = 139$ MeV).

Data is generated, now analysis/cross-checks (will have at least two independent analyses for each component)

Components:

- ▶ Light-quark conn. isospin symmetric
- ▶ QED update of diagrams V,S,F
- ▶ QED update of Z_V , $m_{\pi}^{+,0}$, $m_K^{+,0}$, m_{Ω} correction
- ▶ FV corrections (GSL²)
- ▶ Third lattice spacing for strange (and light)
- ▶ Disconnected
- ▶ SIB connected+disconnected
- ▶ Additional systematic error estimates

Conclusions and Outlook

- ▶ We have all ingredients to make a first-principles lattice computation of HVP with error $O(5 \times 10^{-10})$
- ▶ With lattice precision improvements, window method will be able to weigh in on BaBar/KLOE
- ▶ Data for next paper is ready and analysis is progressing fast
- ▶ Timeline: complete analysis/cross-checks of individual contributions in the next month. Then combine to $a_{\mu}^{HVP LO}$.
- ▶ arXiv before October 15, 2019 not realistic, not due to a lack of effort; still a good chance before end of year

Backup

Dispersive method - e^+e^- status

Recent results by [Keshavarzi et al. 2018](#), [Davier et al. 2017](#):

Channel	This work (KNT18)	DHMZ17 [78]	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	-3.40
$\pi^+\pi^-\pi^0$ (data + ChPT)	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
...			
Total	693.3 ± 2.5	693.1 ± 3.4	0.2

Good agreement for total, individual channels disagree to some degree.

Muon g-2 Theory Initiative workshops recently held at Fermilab, [KEK](#), UConn, and [Mainz](#), intend to facilitate discussions and further understanding of these tensions.

One difference: treatment of correlations, impactful in particular in case when not all experimental data agrees

Gounaris-Sakurai-Lüscher method [H. Meyer 2012, Mainz 2017]

- ▶ Produce FV spectrum and matrix elements from phase-shift study (Lüscher method for spectrum and amplitudes, GS for phase-shift parametrization)
- ▶ This allows for a prediction of FV effects beyond chiral perturbation theory given that the phase-shift parametrization captures all relevant effects (can be checked against lattice data)
- ▶ This method is now being employed by ETMC, Mainz, and RBC/UKQCD.

Dispersive method - τ status

Experiment	$a_\mu^{\text{had,LO}}[\pi\pi, \tau] (10^{-10})$	
	$2m_{\pi^\pm} - 0.36 \text{ GeV}$	$0.36 - 1.8 \text{ GeV}$
ALEPH	$9.80 \pm 0.40 \pm 0.05 \pm 0.07$	$501.2 \pm 4.5 \pm 2.7 \pm 1.9$
CLEO	$9.65 \pm 0.42 \pm 0.17 \pm 0.07$	$504.5 \pm 5.4 \pm 8.8 \pm 1.9$
OPAL	$11.31 \pm 0.76 \pm 0.15 \pm 0.07$	$515.6 \pm 9.9 \pm 6.9 \pm 1.9$
Belle	$9.74 \pm 0.28 \pm 0.15 \pm 0.07$	$503.9 \pm 1.9 \pm 7.8 \pm 1.9$
Combined	$9.82 \pm 0.13 \pm 0.04 \pm 0.07$	$506.4 \pm 1.9 \pm 2.2 \pm 1.9$

Davier et al. 2013: $a_\mu^{\text{had,LO}}[\pi\pi, \tau] = 516.2(3.5) \times 10^{-10} (2m_{\pi^\pm} - 1.8 \text{ GeV})$

Compare to e^+e^- :

- ▶ $a_\mu^{\text{had,LO}}[\pi\pi, e^+e^-] = 507.1(2.6) \times 10^{-10}$ (DHMZ17, $2m_{\pi^\pm} - 1.8 \text{ GeV}$)
- ▶ $a_\mu^{\text{had,LO}}[\pi\pi, e^+e^-] = 503.7(2.0) \times 10^{-10}$ (KNT18, $2m_{\pi^\pm} - 1.937 \text{ GeV}$)

Here treatment of isospin-breaking to relate matrix elements of $V_\mu^{I=1, I_3=1}$ to $V_\mu^{I=1, I_3=0}$ crucial. Progress towards a first-principles calculation from LQCD+QED, see 1811.00508.

Regions of precision (R-ratio data here is from **Fred Jegerlehner** 2017)

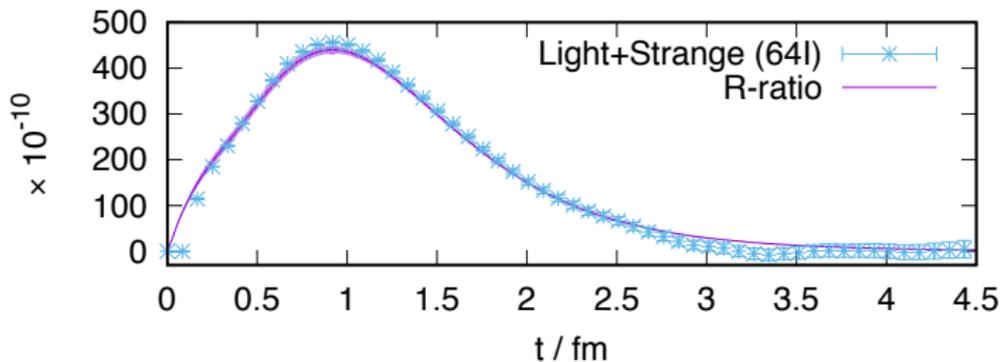


FIG. 4. Comparison of $w_t C(t)$ obtained using R-ratio data [1] and lattice data on our 64I ensemble.

The precision of lattice data deteriorates exponentially as we go to large t , however, is precise at intermediate distances. The R-ratio is very precise at long distances.

Note: in this plot a direct comparison of R-ratio and lattice data is not appropriate. Continuum limit, infinite-volume corrections, charm contributions, and IB corrections are missing from lattice data shown here.

We perform the calculation as a perturbation around an isospin-symmetric lattice QCD computation with two degenerate light quarks with mass m_{light} and a heavy quark with mass m_{heavy} tuned to produce a pion mass of 135.0 MeV and a kaon mass of 495.7 MeV.

The correlator is expanded in the fine-structure constant α as well as $\Delta m_{\text{up, down}} = m_{\text{up, down}} - m_{\text{light}}$, and $\Delta m_{\text{strange}} = m_{\text{strange}} - m_{\text{heavy}}$. We write

$$C(t) = C^{(0)}(t) + \alpha C_{\text{QED}}^{(1)}(t) + \sum_f \Delta m_f C_{\Delta m_f}^{(1)}(t) + \mathcal{O}(\alpha^2, \alpha \Delta m, \Delta m^2).$$

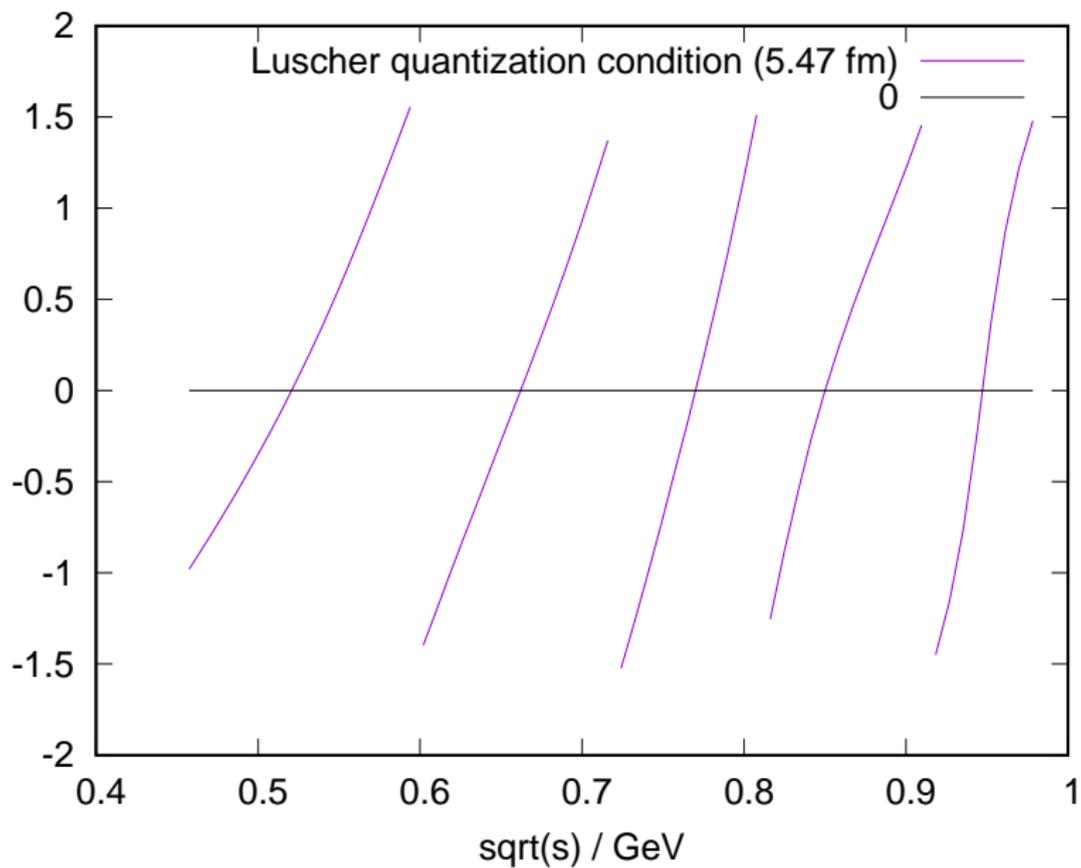
The correlators of this expansion are computed in lattice QCD with dynamical up, down, and strange quarks. We compute the missing contributions to a_μ from charm sea quarks in perturbative QCD (RHAD) by integrating the time-like region above 2 GeV and find them to be smaller than 0.3×10^{-10} .

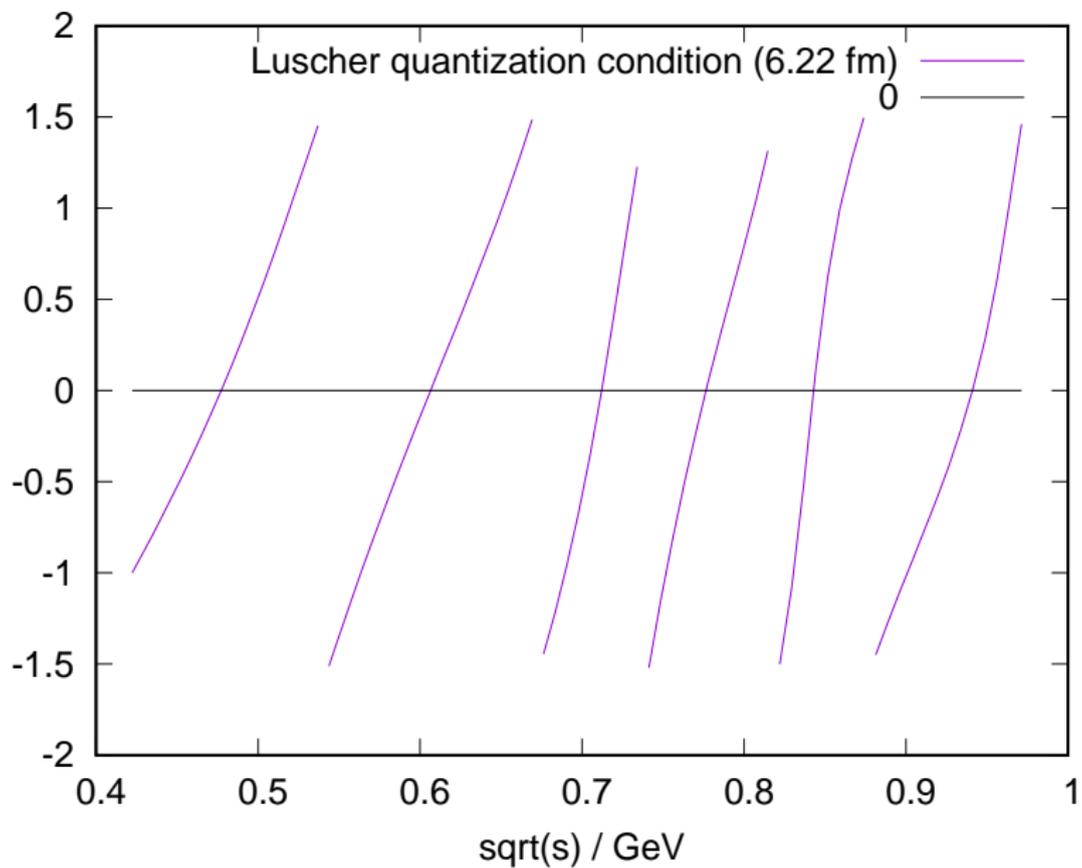
We tune the bare up, down, and strange quark masses m_{up} , m_{down} , and m_{strange} such that the π^0 , π^+ , K^0 , and K^+ meson masses computed in our calculation agree with the respective experimental measurements. The lattice spacing is determined by setting the Ω^- mass to its experimental value.

We perform the lattice calculations for the light quark contributions using RBC/UKQCD's 48l and 64l lattice configurations with lattice cutoffs $a^{-1} = 1.730(4)$ GeV and $a^{-1} = 2.359(7)$ GeV and a larger set of ensembles with up to $a^{-1} = 2.774(10)$ GeV for the charm contribution.

From the parameter tuning procedure on the 48l we find $\Delta m_{\text{up}} = -0.00050(1)$, $\Delta m_{\text{down}} = 0.00050(1)$, and $\Delta m_{\text{strange}} = -0.0002(2)$.

The shift of the Ω^- mass due to the QED correction is significantly smaller than the lattice spacing uncertainty and its effect on $C(t)$ is therefore not included separately.

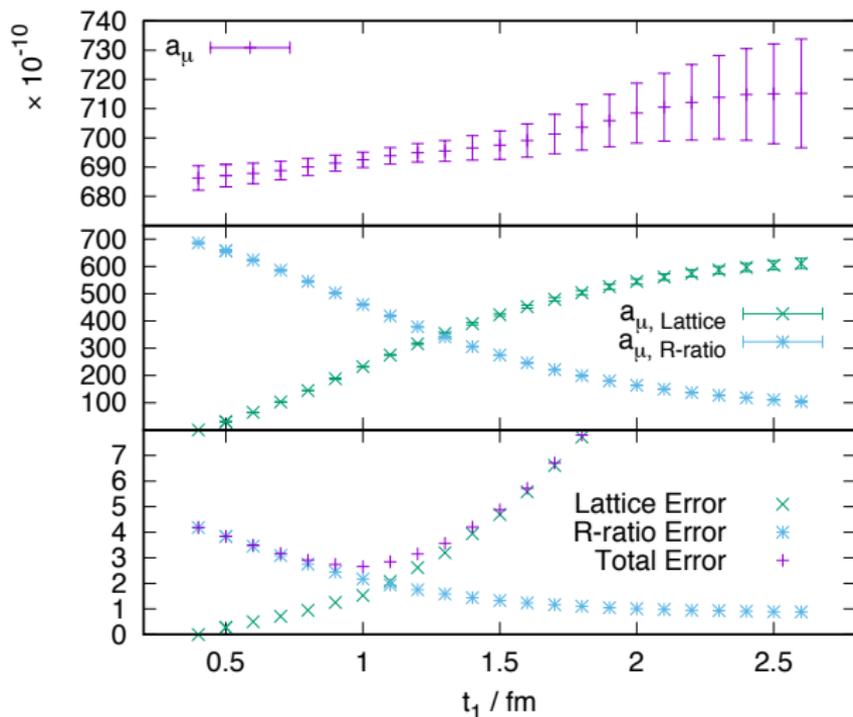




Consolidate continuum limit

Adding a finer lattice

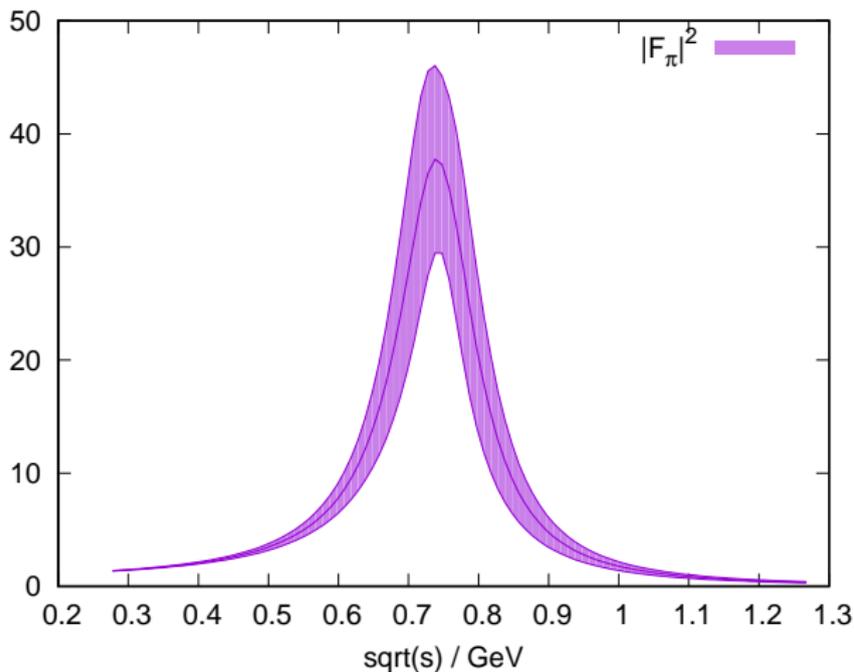
Window method with fixed $t_0 = 0.4$ fm



For $t = 1$ fm approximately 50% of uncertainty comes from lattice and 50% of uncertainty comes from the R-ratio. Is there a small slope? More in a few slides!

Can use this to check experimental data sets; see my KEK talk for more details

Predicts $|F_\pi(s)|^2$:



We can then also predict matrix elements and energies for our other lattices; successfully checked!