Electromagnetic finite-size effects to the hadronic vacuum polarisation

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September 13, 2019

HVP

- Need electromagnetic (EM) effects to reach (sub-)percent level accuracy
- We want EM finite-size effects on hadronic vacuum polarisation (HVP) [Bijnens, NHT, Portelli et al., PRD, 014508 (2019)]
- Vector 2-point function

$$\Pi_{\mu\nu}(q) = \int d^4x \, e^{iq\cdot x} \langle 0 | \, T\left[j_\mu(x) j_\nu^\dagger(0) \right] | 0 \rangle$$

$$\Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - q^2 \delta_{\mu\nu} \right) \Pi(q^2)$$

$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = \frac{1}{3q_0^2} \sum_{j=1}^{3} \left(\Pi_{jj}(q^2) - \Pi_{jj}(0) \right)$$

Finite volume corrections

- Finite volume effects depend on lattice size L

$$\text{Massive particles:} \quad e^{-mL}$$

$$\text{Massless particles:} \quad \frac{1}{L^a}$$

- Electromagnetic corrections to the HVP potentially dangerous
- Euclidean $q^2 \geq 0 \implies a \geq 2$, but what is it really?
- Can get at the FV effects with effective field theory

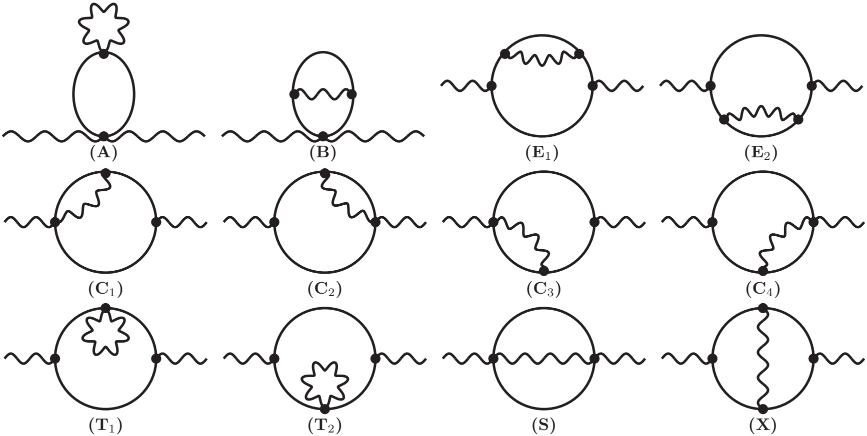
QED in finite volume

- Difficult to define charged states in finite volume with periodic boundary conditions (Gauss' law)
- Solution: QED_L [Hayakawa, Uno 2008]
- The global zero mode subtracted on every time slice, i.e.

$$\sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}$$

- We use scalar QED BUT include form factors
- Also, $q = (q_0, \mathbf{0})$

The HVP at NLO in the electromagnetic coupling



- Scalar QED: Pion loops dressed with photons
- Compare with lattice simulations and `VEGAS` MC evaluation of loop integrals in lattice perturbation theory

Actual calculation

- 2-loop integrals for each diagram U

$$\hat{\Pi} = 2\,\hat{\Pi}_E + 2\,\hat{\Pi}_T + \hat{\Pi}_S + \hat{\Pi}_X + 4\,\hat{\Pi}_C$$

$$\hat{\Pi}_U = \int \frac{d^4k}{(2\pi)^4}\frac{d^4\ell}{(2\pi)^4}\,\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Do energy integrals analytically

$$\hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0) = \int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- On the lattice momenta are discretised: $\mathbf{k} = (2\pi/L)\mathbf{n}$
but let pions be in IV

$$\text{Poisson:} \quad \sum_{\boldsymbol{\ell}} = \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} + \mathcal{O}\left(e^{-m_\pi L}\right)$$

Actual calculation

- Want the finite volume corrections, so for each diagram U we need to calculate

$$\Delta\hat{\Pi}_U(q^2) = \frac{1}{L^3}\Delta_n \int \frac{d^3\ell}{(2\pi)^3}\,\hat{\rho}_U\left(q_0^2, \frac{2\pi\,\mathbf{n}}{L}, \boldsymbol{\ell}\right) + \mathcal{O}\left(e^{-m_\pi L}\right)$$

$$\Delta_{\mathbf{n}} = \left(\sum_{\mathbf{n}}{}' - \int d^3\mathbf{n}\right)$$

- Soft photons probe FV effects $\implies$ Taylor expand in $1/L$
- Result will be in terms of $c_j = \Delta_{\mathbf{n}}\,|\mathbf{n}|^{-j}$ and $\Omega_{i,j}(z = q_0^2/m^2)$

$$\Omega_{i,j}(z) = \int dx\, x^2 \frac{1}{(1+x^2)^{i/2}(z+4(x^2+1))^j}$$

Finite volume corrections

- Diagram by diagram:

$$\begin{aligned}
\Delta\hat{\Pi}_E(z) =& \frac{c_1}{\pi m^2 L^2}\left(-\frac{4}{3}\,\Omega_{-1,3} + \frac{1}{2}\,\Omega_{1,2} + \frac{4}{3}\,\Omega_{1,3} - \frac{1}{4}\,\Omega_{3,1}\right) \\
& - \frac{c_0}{m^3 L^3}\left(-\frac{8}{3}\,\Omega_{0,3} + \frac{32}{3}\,\Omega_{0,4} + \frac{1}{16}\,\Omega_{2,2} + \frac{10}{3}\,\Omega_{2,3}\right. \\
& \left. - \frac{32}{3}\,\Omega_{2,4} - \frac{23}{128}\,\Omega_{4,1} + \frac{5}{16}\,\Omega_{4,2} - \frac{2}{3}\,\Omega_{4,3}\right) \\
\Delta\hat{\Pi}_C(z) =& \frac{c_1}{\pi m^2 L^2}\frac{1}{8}\,\Omega_{3,1} \\
& - \frac{c_0}{m^3 L^3}\left(\frac{8}{3}\,\Omega_{0,3} + \frac{1}{6}\,\Omega_{2,2} - \frac{8}{3}\,\Omega_{2,3} + \frac{1}{8}\,\Omega_{4,1} - \frac{1}{6}\,\Omega_{4,2}\right) \\
\Delta\hat{\Pi}_T(z) =& \frac{c_1}{\pi m^2 L^2}\frac{1}{4}\,\Omega_{3,1}
\end{aligned}$$

Finite volume corrections

$$\Delta\hat{\Pi}_S(z) = -\frac{c_1}{\pi m^2 L^2}\frac{1}{4}\,\Omega_{3,1} + \frac{c_0}{m^3 L^3}\left(2\,\Omega_{2,2} + \frac{1}{4}\,\Omega_{4,1}\right)$$

$$\begin{aligned}\Delta\hat{\Pi}_X(z) = &\frac{c_1}{\pi m^2 L^2}\left(\frac{8}{3}\,\Omega_{-1,3} - \Omega_{1,2} - \frac{8}{3}\,\Omega_{1,3} - \frac{1}{4}\,\Omega_{3,1}\right)\\ &- \frac{c_0}{m^3 L^3}\left(-\frac{128}{3}\,\Omega_{-2,4} - \frac{16}{3}\,\Omega_{0,3} + 64\,\Omega_{0,4} - \frac{11}{24}\,\Omega_{2,2} + \frac{20}{3}\,\Omega_{2,3}\right.\\ &\left.\qquad - \frac{64}{3}\,\Omega_{2,4} - \frac{17}{64}\,\Omega_{4,1} + \frac{29}{24}\,\Omega_{4,2} - \frac{4}{3}\,\Omega_{4,3}\right)\end{aligned}$$

Finite volume corrections

- In total:

$$\Delta\hat{\Pi}(q^2) = \frac{c_0}{m_\pi^3 L^3}\left(-\frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3}\right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-m_\pi L}\right)$$

- Suppression: Leading order is $1/L^3$
- Physics: Neutral current and photon far away sees no charge
- Universality: adding form factors yields same cancellation
- For $m_\pi L$ one has $1/(m_\pi L)^3 < 1.5\%$, so on $\mathcal{O}(\alpha) \sim 1\%$ corrections to the HVP it gives 0.02% (cf. $e^{-m_\pi L} \lesssim 1.8\%$ in QCD)

Charged currents

- Include π^0 in charged current

$$\Delta\hat{\Pi}_{\text{charged}}\left(q^2\right) = -\frac{c_0}{m^3L^3}\left(-\frac{13}{24}\Omega_{2,2}+\frac{20}{9}\Omega_{2,3}-\frac{15}{64}\Omega_{4,1}+\frac{7}{24}\Omega_{4,2}+\frac{4}{9}\Omega_{4,3}\right)$$
$$+\frac{c_1}{m^2L^2\pi}\left(-\frac{8}{3}\Omega_{-1,3}+\Omega_{1,2}+\frac{8}{3}\Omega_{1,3}+\frac{1}{8}\Omega_{3,1}\right)$$
$$+\mathcal{O}\left(\frac{1}{L^4},e^{-m_\pi L}\right)$$

- No cancellation of $1/L^2$ here

Numerical validation

- Want to validate the results with numerical methods
- Action $S[\phi, A] = S_\phi[\phi, A] + S_A[A]$

$$S_\phi\left[\phi, A\right] = \frac{a^4}{2}\sum_x \left[\sum_\mu \left|D_\mu \phi(x)\right|^2 + m_0^2 \left|\phi(x)\right|^2\right] = \frac{a^4}{2}\sum_x \phi^*(x)\Delta\phi(x),$$

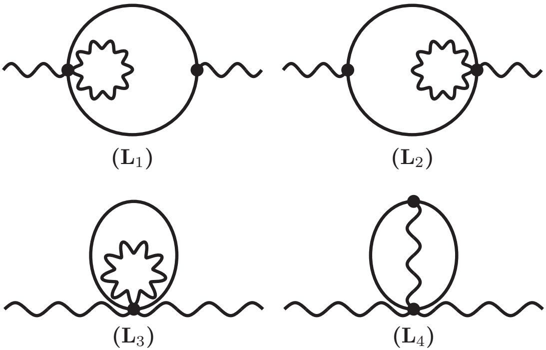
$$S_A[A] = \frac{a^4}{2}\sum_{x,\mu}\left[\sum_\nu \frac{1}{2}F_{\mu\nu}(x)^2 + [\delta_\mu A_\mu(x)]^2\right] = -\frac{a^4}{2}\sum_{x,\mu} A_\mu(x)\delta^2 A_\mu(x)\,,$$

$$\Delta = m^2 - \sum_\mu D_\mu^* D_\mu$$

- $\Pi_{\mu\nu}$ is discrete Fourier transform of $C_{\mu\nu}\left(x\right) \equiv \left\langle V_\mu\left(x\right) V_\nu\left(0\right)\right\rangle$

Numerical validation

- Additional lattice diagrams



Numerical validation

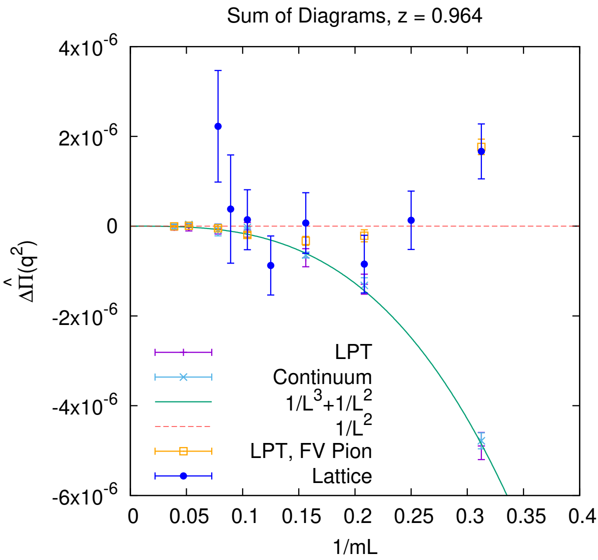
Simulation details:

- Eight different volumes with L between 16 and 64;
  Time extent $T = 128$
- $z = q^2/m_0^2 = 0.964$

Lattice perturbation theory (LPT):

- `Cuba VEGAS` Monte Carlo integration for both FV and IV pions
- We do this for several lattice spacings and then make a continuum extrapolation

Numerical validation

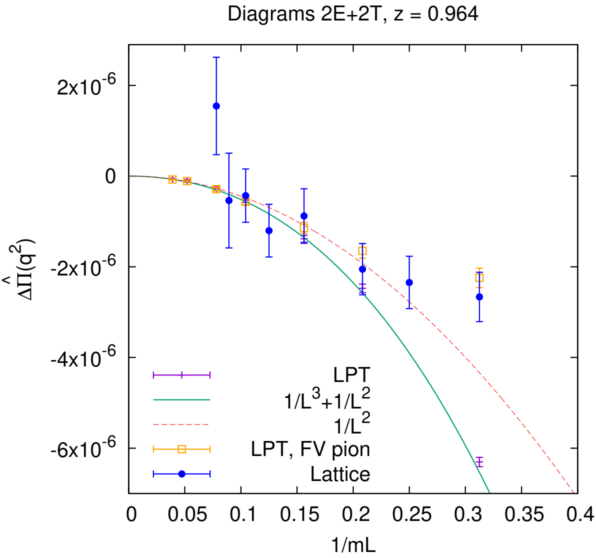


Conclusions

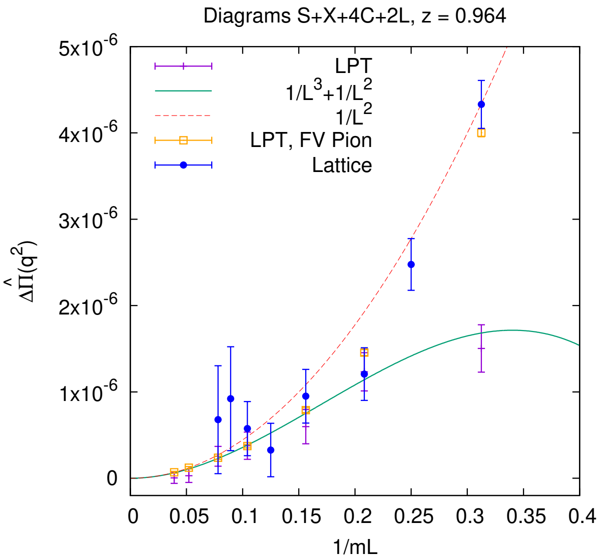
- Finite volume corrections smaller than anticipated: $c_0/(m_\pi L)^3$
- For $mL \gtrsim 4$ the effects are very small 10^{-6} (cf. LO HVP $10^{-3} \sim 10^{-5}/\alpha$)
- FV effects in principle negligible in foreseeable future
- Whole analytic calculation automatised: Mathematica notebook in supp. material, can be used also for other FV effect calculations

Back-up slides

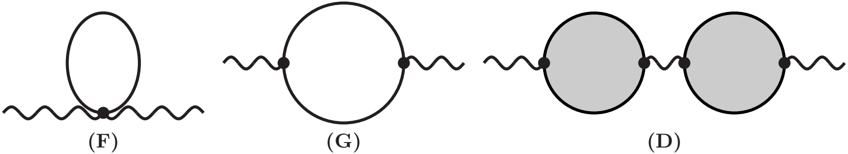
Numerical validation



Numerical validation



LO and NLO disconnected diagrams



Charged currents

$$\Delta\hat{\Pi}_{\text{charged}}\left(q^2\right) = \frac{1}{m^3L^3}\left(-\frac{13}{24}\Omega_{2,2} + \frac{20}{9}\Omega_{2,3} - \frac{15}{64}\Omega_{4,1} + \frac{7}{24}\Omega_{4,2} + \frac{4}{9}\Omega_{4,3}\right)$$
$$+ \frac{c_1}{m^2L^2\pi}\left(-\frac{8}{3}\Omega_{-1,3} + \Omega_{1,2} + \frac{8}{3}\Omega_{1,3} + \frac{1}{8}\Omega_{3,1}\right)$$
$$+ \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$