Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

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in collaboration with M. Hoferichter

JHEP **1907**, 073 (2019) [arXiv:1905.13198 [hep-ph]]

and with G. Colangelo, M. Hoferichter, and M. Procura

JHEP **04** (2017) 161, [arXiv:1702.07347 [hep-ph]]
Phys. Rev. Lett. **118** (2017) 232001, [arXiv:1701.06554 [hep-ph]]
and work in progress

12th September 2019

Third Plenary Workshop of the Muon $g-2$ Theory Initiative
INT, University of Washington, Seattle

Reminder: BTT Lorentz decomposition

Lorentz decomposition of the HLbL tensor:

→ Bardeen, Tung (1968) and Tarrach (1975)

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- scalar functions Π_i free of kinematic singularities
  $\Rightarrow$ dispersion relation in the Mandelstam variables

Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

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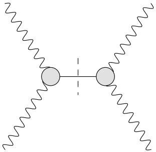
one-pion intermediate state



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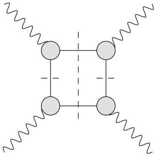
two-pion intermediate state in both channels



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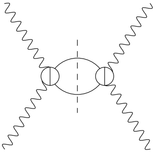
two-pion intermediate state in first channel



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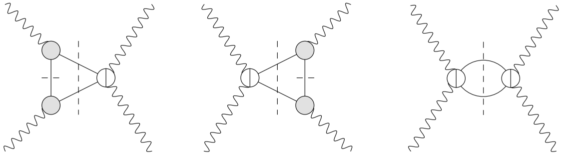
higher intermediate states

Resonance contributions to HLbL?

- unitarity: resonances unstable, not asymptotic states
  $\Rightarrow$ do not show up in unitarity relation
- analyticity: resonances are poles on unphysical Riemann sheets of partial-wave amplitudes
  $\Rightarrow$ describe in terms of multi-particle intermediate states that generate the branch cut
- here: resonant $\pi\pi$ contributions in S-wave (f_0) and D-wave (f_2)
- resonance model-independently encoded in $\pi\pi$-scattering phase shifts

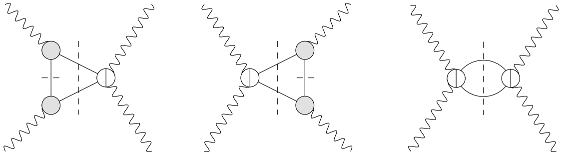
Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

$$\Pi_i^{\pi\pi} = \frac{1}{2}\left(\frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{t'-t} + \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{u'-u} \right.$$
$$+ \text{fixed-}t$$
$$\left. + \text{fixed-}u\right)$$

Rescattering contribution



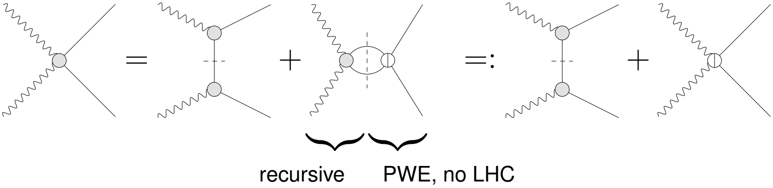
- expansion into partial waves
- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^*\gamma^{(*)} \to \pi\pi$:

$$\mathrm{Im}_{\pi\pi} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) \propto \sigma_\pi(s) h_{J,\lambda_1\lambda_2}(s) h^*_{J,\lambda_3\lambda_4}(s)$$

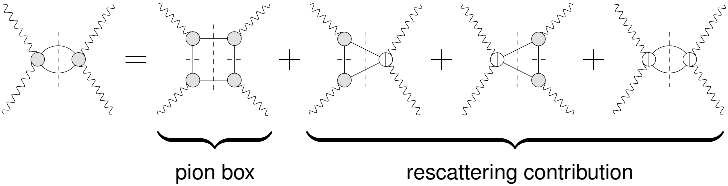
- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box

Topologies in the rescattering contribution

Our S-wave solution for $\gamma^*\gamma^* \to \pi\pi$:



Two-pion contributions to HLbL:



The subprocess

Omnès solution of unitarity relation for $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves:

$$h_i(s) = \Delta_i(s) + \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{K_{ij}(s,s') \sin\delta_0(s') \Delta_j(s')}{|\Omega_0(s')|}$$

- $\Delta_i(s)$: inhomogeneity due to left-hand cut
- $\Omega_0(s)$: Omnès function with $\pi\pi$ S-wave phase shifts $\delta_0(s)$ as input
- $K_{ij}(s,s')$: integration kernels
- S-waves: kernels emerge from a 2×2 system for $h_{0,++}$ and $h_{0,00}$ and two scalar functions $A_{1,2}$

S-wave rescattering contribution

- pion-pole approximation to left-hand cut
  $\Rightarrow q^2$-dependence given by F_π^V
- phase shifts based on modified inverse-amplitude method ($f_0(500)$ parameters accurately reproduced)
- result for S-waves: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Extension to D-waves → JHEP **1907**, 073 (2019)

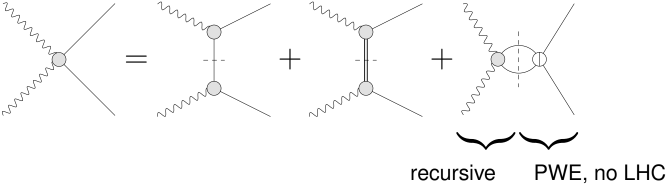
- D-waves describe $f_2(1270)$ resonance in terms of $\pi\pi$ rescattering
- inclusion of higher left-hand cuts (ρ, ω resonances) necessary to reproduce observed $f_2(1270)$ resonance peak in on-shell $\gamma\gamma \to \pi\pi$
- NWA for vector resonance LHC with $V\pi\gamma$ interaction

$$\mathcal{L} = eC_V \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} \partial_\lambda \pi V_\sigma$$

- coupling C_V related to decay width $\Gamma(V \to \pi\gamma)$
- off-shell behaviour described by resonance transition form factors $F_{V\pi}(q^2)$

Topologies in the Omnès solution

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ with higher left-hand cuts provides the following:



Modified Omnès representation

→ García-Martín, Moussallam 2010

$$h_i(s) = N_i(s) + \frac{\Omega(s)}{\pi}\left\{ \int_{-\infty}^{0} ds' \frac{K_{ij}(s,s')\mathrm{Im}h_j(s')}{\Omega(s')} + \int_{4M_\pi^2}^{\infty} ds' \frac{K_{ij}(s,s')\sin\delta(s')N_j(s')}{|\Omega(s')|} \right\}$$

- $N_i(s)$: only Born term as inhomogeneity
- higher left-hand cuts in first dispersion integral: only imaginary part required
- $K_{ij}(s,s')$: integration kernels from the full 5×5 D-wave Roy–Steiner system, diagonalisable by basis change

Modified Omnès representation

→ García-Martín, Moussallam 2010

- sum rules for subtraction constants almost fulfilled
  ⇒ unsubtracted DR with small adjustment of LHC couplings to account for higher intermediate states

  → also done in Danilkin, Vanderhaeghen 2017

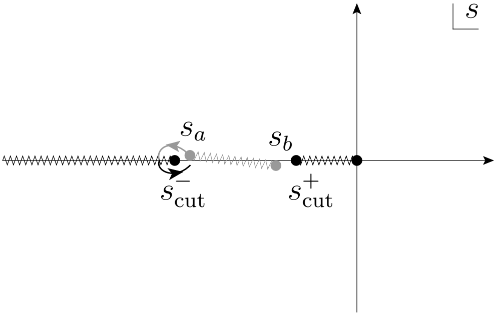
- assumption on asymptotic behaviour:

$$\frac{h(s) - N(s)}{\Omega(s)} \asymp \frac{1}{s}$$

- bad high-energy behaviour of real part of resonance LHC explains the need for subtraction in standard Omnès representation

Anomalous thresholds for large space-like q_i^2

Left-hand cut structure of resonance partial waves:

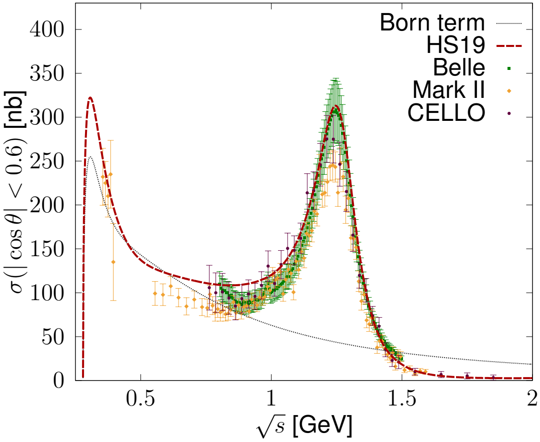


- two logarithmic branch cuts $(-\infty, s_{\text{cut}}^-]$, $[s_{\text{cut}}^+, 0]$
- square-root branch cut on second sheet, but extends into the physical sheet for $q_1^2 q_2^2 > (M_R^2 - M_\pi^2)^2$

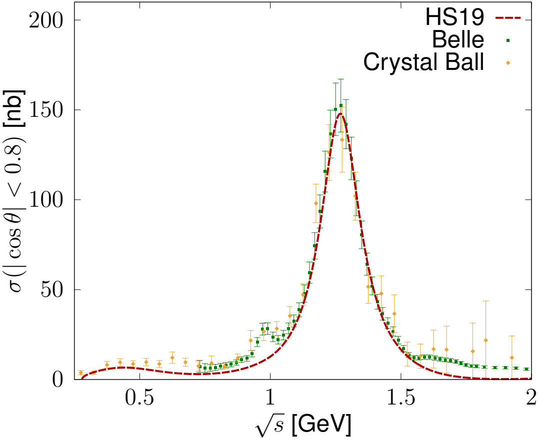
Anomalous thresholds for large space-like q_i^2

- deformation of integration contour for $q_1^2 q_2^2 > (M_R^2 - M_\pi^2)^2$
- anomalous singularity s_a behaves for some D-wave contributions like $(s_a - s)^{-7/2}$
- contour integral around s_a does not vanish and makes result finite
- cancellation performed analytically, avoiding numerical instabilities

On-shell results: $\gamma\gamma \to \pi^+\pi^-$



On-shell results: $\gamma\gamma \to \pi^0\pi^0$



Doubly-virtual results

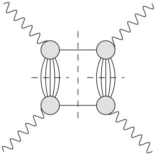
- all technical issues with D-waves solved
- D-wave solution expressed in terms of $V\pi\gamma^*$ transition form factors
- asymptotic ω TFF behaviour $\sim 1/Q^4$

  $\rightarrow$ Farrar, Jackson 1975
- dispersive representation for space-like ω/ρ TFFs via $\pi^0 \rightarrow \gamma^*\gamma^*$

  $\rightarrow$ with M. Hoferichter, B.-L. Hoid, B. Kubis, work in progress

Contribution to a_μ^{HLbL}

- resonance box is UV divergent



- modified Omnès representation cures UV behaviour for sum of resonance LHC and rescattering
- compute D-wave contribution to a_μ^{HLbL} in one go
- numerics in progress...

Conclusion and outlook

- precise prediction for S-wave $\pi\pi$-rescattering contribution with pion-pole left-hand cut:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

- technical problems for D-wave reconstruction solved: inclusion of heavier LHCs, anomalous thresholds, asymptotic behaviour
- upcoming BESIII data allow extraction of presently unknown space-like TFF $\rightarrow$ talk by Ch. Redmer
- D-wave contribution to a_μ work in progress
- compare to narrow-width approximation of $f_2(1270)$

“Dispersion relations are always true!”

Arkady Vainshtein, 11th September 2019