Dispersive constraints on the two-pion contribution to hadronic vacuum polarisation

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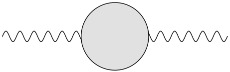
Outline

1. Unitarity and analyticity
2. Dispersion relation for the pion vector form factor
3. Fit results and contribution to the muon $g-2$
4. Summary

Hadronic vacuum polarisation (HVP)

Photon HVP function:



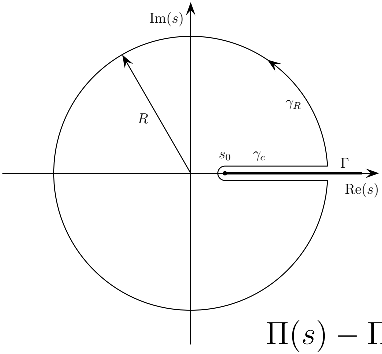
$$= i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$$

Unitarity of the S-matrix implies the optical theorem:

$$\mathrm{Im}\Pi(s) = \frac{s}{e(s)^2}\sigma(e^+e^- \to \mathrm{hadrons})$$

Dispersion relation

Causality implies analyticity:



Cauchy integral formula:

$$\Pi(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\Pi(s')}{s'-s} ds'$$

Deform integration path:

$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\Pi(s')}{(s'-s-i\epsilon)s'} ds'$$

HVP contribution to $(g-2)_\mu$

$$a_\mu^{\text{HVP}} = \frac{m_\mu^2}{12\pi^3} \int_{s_{\text{thr}}}^{\infty} ds \, \frac{\hat{K}(s)}{s} \, \sigma(e^+e^- \to \text{hadrons})$$

- basic principles: unitarity and analyticity
- direct relation to experiment: total hadronic cross section $\sigma(e^+e^- \to \text{hadrons})$
- dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE, SND)

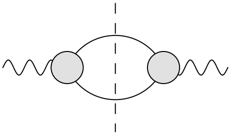
Two-pion contribution to HVP

- $\pi\pi$ contribution amounts to more than 70% of HVP contribution
- responsible for a similar fraction of HVP uncertainty

Unitarity and analyticity

Implications of unitarity (two-pion intermediate states):

1. $\pi\pi$ contribution to HVP—pion VFF
2. pion VFF—$\pi\pi$ scattering
3. $\pi\pi$ scattering—$\pi\pi$ scattering



: $\sigma(e^+e^- \to \pi^+\pi^-) \propto |F_\pi^V(s)|^2$

analyticity $\Rightarrow$ usual DR for HVP contribution

Unitarity and analyticity

Implications of unitarity (two-pion intermediate states):

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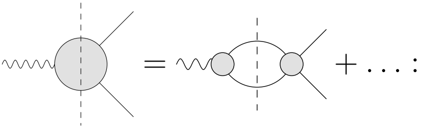
2. pion VFF—$\pi\pi$ scattering
3. $\pi\pi$ scattering—$\pi\pi$ scattering



$$F_\pi^V(s) = |F_\pi^V(s)| e^{i\delta_1^1(s)+\ldots}$$

analyticity $\Rightarrow$ DR for pion VFF, Omnès solution

Unitarity and analyticity

Implications of unitarity (two-pion intermediate states):

1. $\pi\pi$ contribution to HVP—pion VFF
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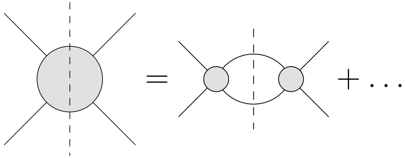
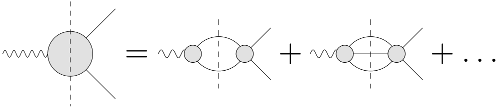


analyticity, crossing, PW expansion ⇒ Roy equations

Two-pion contribution to HVP

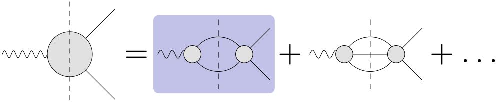
- VFF itself fulfils a unitarity relation:



- use the constraints of analyticity and unitarity to better understand uncertainties in HVP $\pi\pi$ channel

  $\rightarrow$ de Trocóniz, Yndouráin, 2001, 2004; Leutwyler, Colangelo 2002, 2003; Ananthanarayan et al. 2013, 2016

Dispersive representation of pion VFF

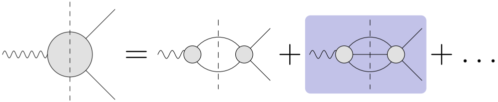


$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- Omnès function with elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input:

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

Dispersive representation of pion VFF



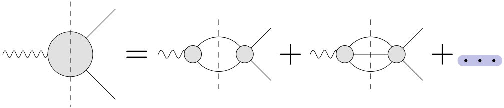
$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- isospin-breaking 3π intermediate state: negligible apart from ω resonance (ρ–ω interference effect)

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\text{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4 ,$$

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

Dispersive representation of pion VFF



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- heavier intermediate states: 4π (mainly $\pi^0\omega$), $\bar{K}K, \ldots$
- described in terms of a conformal polynomial with cut starting at $\pi^0\omega$ threshold

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k(z^k(s) - z^k(0))$$

- correct P-wave threshold behaviour imposed

Input and systematic uncertainties

- elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ from Roy-equation analysis, including uncertainties

  $\rightarrow$ Ananthanarayan et al., 2001; Caprini et al., 2012

- high-energy continuation of phase shift above validity of Roy equations
- ω width
- systematics in conformal polynomial: order N, one mapping parameter

Free fit parameters

- value of the elastic $\pi\pi$-scattering P-wave phase shift δ_1^1 at two points (0.8 GeV and 1.15 GeV): number of free parameters dictated by Roy equations
- ρ–ω mixing parameter ϵ_ω
- ω mass
- energy rescaling for the experimental input, which allows for a calibration uncertainty
- $N-1$ coefficients in the conformal polynomial

VFF fit to the following data

- time-like cross section data from high-statistics e^+e^- experiments SND, CMD-2, BaBar, KLOE
- space-like VFF data from NA7
- Eidelman–Łukaszuk bound on inelastic phase:

  → Eidelman, Łukaszuk, 2004
- iterative fit routine including full experimental covariance matrices and avoiding D'Agostini bias

  → D'Agostini, 1994; Ball et al. (NNPDF) 2010

VFF fit results

	χ^2/dof	M_ω [MeV]	$10^3 \times \xi_j$	$\delta_1^1(s_0)$ [°]	$\delta_1^1(s_1)$ [°]	$10^3 \times \epsilon_\omega$
SND	$51.9/37 = 1.40$	$781.49(32)(2)$	$0.0(6)(0)$	$110.5(5)(8)$	$165.7(0.3)(2.4)$	$2.03(5)(2)$
CMD-2	$87.4/74 = 1.18$	$781.98(29)(1)$	$0.0(6)(0)$	$110.5(5)(8)$	$166.4(0.4)(2.4)$	$1.88(6)(2)$
BaBar	$299.1/262 = 1.14$	$781.86(14)(1)$	$0.0(2)(0)$	$110.4(3)(7)$	$165.7(0.2)(2.5)$	$2.04(3)(2)$
KLOE″	$222.5/185 = 1.20$	$781.81(16)(3)$	$\begin{cases} 0.5(2)(0) \\ -0.3(2)(0) \\ -0.2(3)(0) \end{cases}$	$110.3(2)(6)$	$165.6(0.1)(2.4)$	$1.98(4)(1)$
Energy scan	$152.5/119 = 1.28$	$781.75(22)(1)$		$110.4(3)(8)$	$166.0(0.2)(2.4)$	$1.97(4)(2)$
All e^+e^-	$731.6/582 = 1.26$	$781.68(9)(4)$		$110.4(1)(7)$	$165.8(0.1)(2.4)$	$2.02(2)(3)$
All e^+e^-, NA7	$776.2/627 = 1.24$	$781.68(9)(3)$		$110.4(1)(7)$	$165.7(0.1)(2.4)$	$2.02(2)(3)$

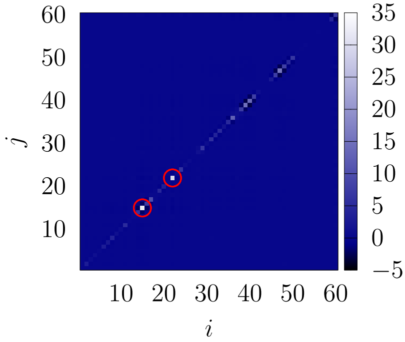
- 1st error: fit uncertainty; 2nd error: systematics
- fit uncertainty inflated by $\sqrt{\chi^2/\mathrm{dof}}$

VFF fit results

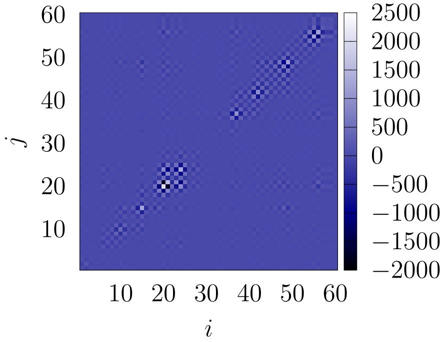
- good fits to all experiments possible (p-value around 3% to 14%) with a few caveats:
  - either M_ω or energy recalibration has to be fit (practically identical results)
  - two outliers in KLOE08 set (> 30 units in χ^2)
  - BESIII covariance matrix cannot be used
- well-known discrepancy between BaBar and KLOE
  - fits to single data sets
  - combinations and error inflation by $\sqrt{\chi^2/\mathrm{dof}}$
- inelastic effects dominate uncertainty for $(g-2)_\mu$

VFF fit results

KLOE08 bin contributions to χ^2



BESIII bin contributions to χ^2



VFF fit results

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ω mass

fit result for ω mass:

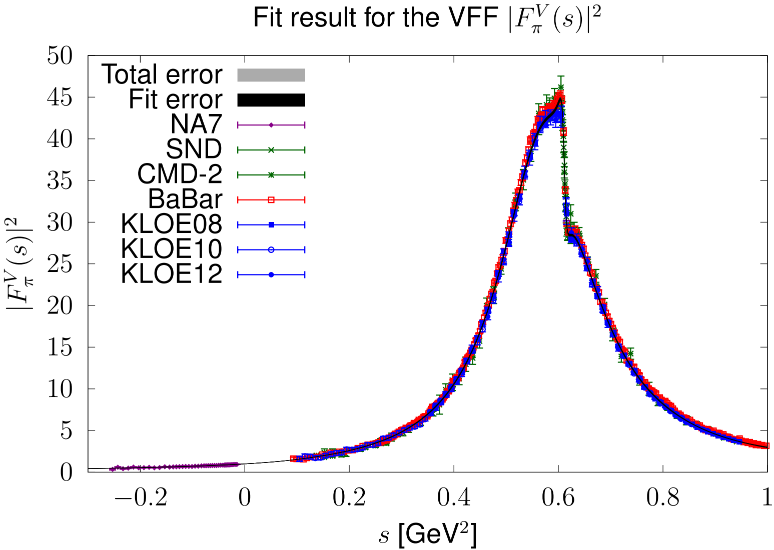
combined fit: $M_\omega = 781.69(9)(3)$ MeV

fits to single experiments: $M_\omega = 781.49 \ldots 782.05$ MeV

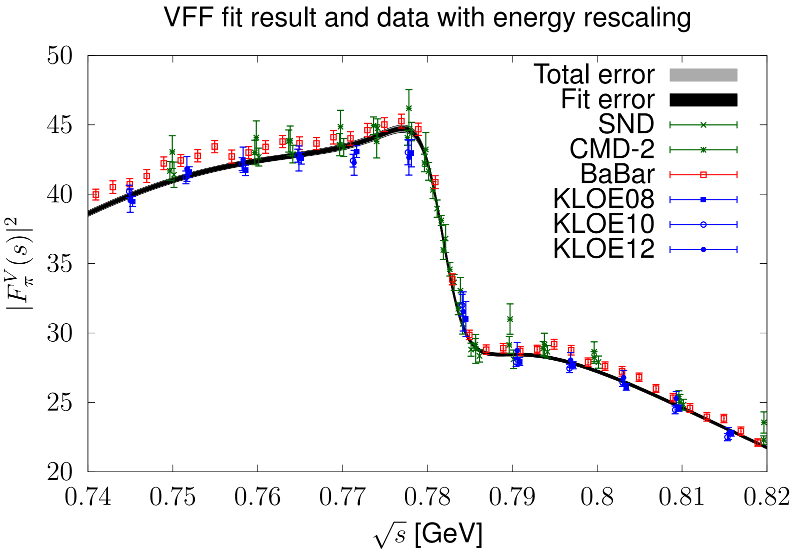
compare to PDG value (dominated by 3π channel):

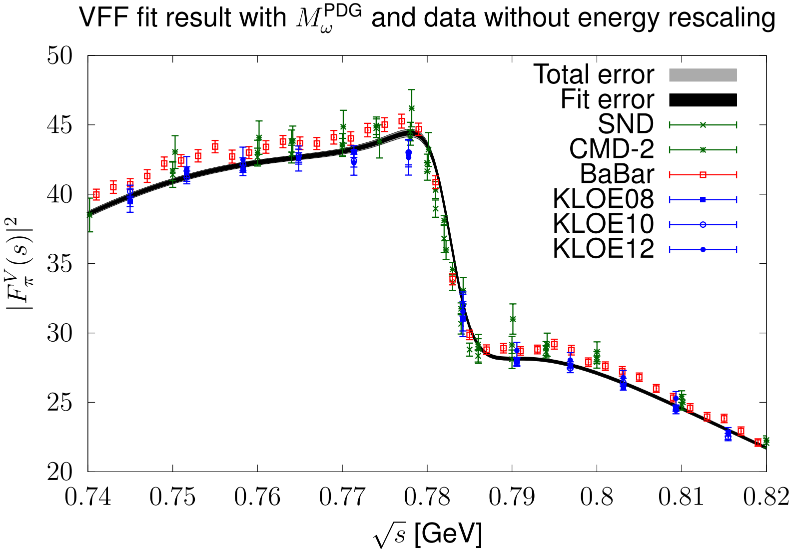
$$M_\omega^{\mathrm{PDG}} = 782.65(12)\,\text{MeV}$$

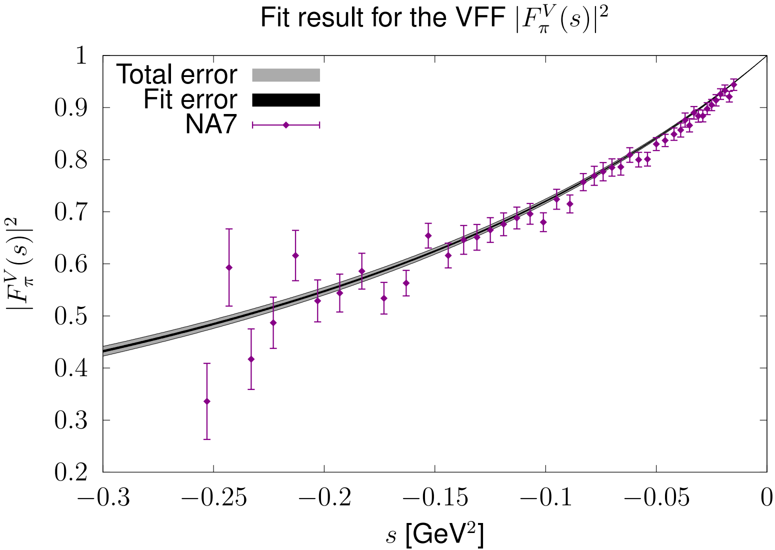
discrepancy can only be partially explained by additional radiative channels (without affecting results for $a_\mu^{\mathrm{HVP},\pi\pi}$) $\rightarrow$ thanks to Bastian for this suggestion

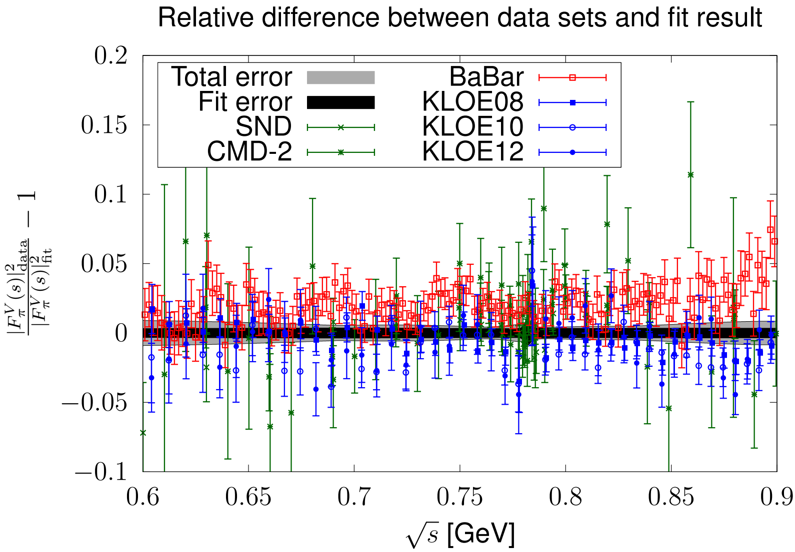


Fit result for the VFF $|F_\pi^V(s)|^2$
Total error
Fit error
NA7
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$|F_\pi^V(s)|^2$
50
45
40
35
30
25
20
15
10
5
0
−0.2
0
0.2
0.4
0.6
0.8
1
s [GeV^2]


VFF fit result and data with energy rescaling
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$|F_\pi^V(s)|^2$
$\sqrt{s}$ [GeV]
50
45
40
35
30
25
20
0.74
0.75
0.76
0.77
0.78
0.79
0.8
0.81
0.82


VFF fit result with M_ω^{PDG} and data without energy rescaling
50
45
40
35
30
25
20
0.74
0.75
0.76
0.77
0.78
0.79
0.8
0.81
0.82
$|F_\pi^V(s)|^2$
$\sqrt{s}$ [GeV]
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12


Fit result for the VFF $|F_\pi^V(s)|^2$
Total error
Fit error
NA7
$|F_\pi^V(s)|^2$
s [GeV^2]
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
−0.3
−0.25
−0.2
−0.15
−0.1
−0.05
0


Relative difference between data sets and fit result
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0.6
0.65
0.7
0.75
0.8
0.85
0.9
$\frac{|F_\pi^V(s)|^2_{\rm data}}{|F_\pi^V(s)|^2_{\rm fit}} - 1$
$\sqrt{s}$ [GeV]

Contribution to $(g-2)_\mu$

- low-energy $\pi\pi$ contribution:

$$a_\mu^{\mathrm{HVP},\pi\pi}|_{\leq 0.63\,\mathrm{GeV}} = 132.8(0.4)(1.0) \times 10^{-10}$$

$\Rightarrow$ compare to $131.1(1.0)$ $\rightarrow$ KNT18

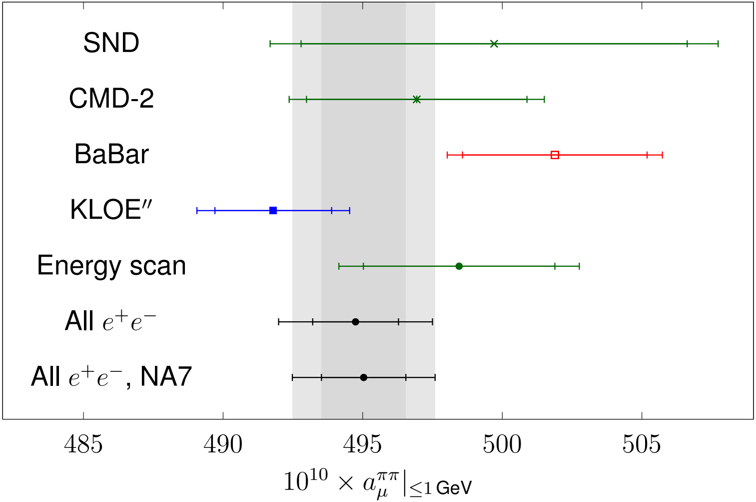
$132.9(8)$ $\rightarrow$ Ananthanarayan et al., 2018

$133.4(5)(4)$ $\rightarrow$ DHMZ19

- $\pi\pi$ contribution up to 1 GeV:

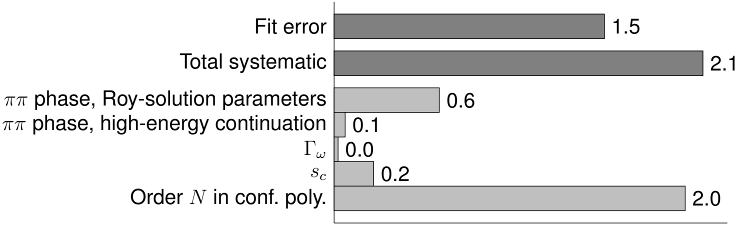
$$a_\mu^{\mathrm{HVP},\pi\pi}|_{\leq 1\,\mathrm{GeV}} = 495.0(1.5)(2.1) \times 10^{-10}$$

Result for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV

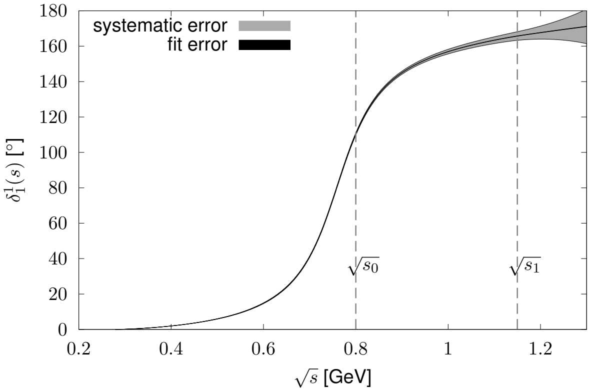


Error budget

uncertainties on $a_\mu^{\mathrm{HVP},\pi\pi}|_{\leq 1\,\mathrm{GeV}}$ in combined fit to all experiments:



Improved determination of $\delta_1^1(s)$



$$\delta_1^1(s_0) = 110.4(1)(7)^\circ = 110.4(7)^\circ$$

$$\delta_1^1(s_1) = 165.7(0.1)(2.4)^\circ = 165.7(2.4)^\circ$$

Determination of the pion charge radius

$$F_\pi^V(s) = 1 + \frac{1}{6}\langle r_\pi^2\rangle s + \mathcal{O}(s^2)$$

DR for F_π^V implies sum rule for charge radius:

$$\langle r_\pi^2\rangle = \frac{6}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{\mathrm{Im} F_\pi^V(s)}{s^2} = 0.429(4)\,\text{fm}^2$$

together with $\langle r_\pi^2\rangle = 0.432(4) \rightarrow$ Ananthanarayan et al., 2017

triggered a revision of the PDG value:

PDG 2018: $\langle r_\pi^2\rangle = 0.452(11)\,\text{fm}^2$

PDG 2019: $\langle r_\pi^2\rangle = 0.434(5)\,\text{fm}^2$

(model-dependent $eN \rightarrow e\pi N$ now excluded)

Overview

Summary

- precise dispersive determination of pion VFF
- comprehensive analysis of uncertainties in $\pi\pi$ channel
- valuable to corroborate uncertainties of direct integration methods
- precise prediction for low-energy region, but valid up to 1 GeV (inelasticities must be taken into account):

$$a_\mu^{\mathrm{HVP},\pi\pi}|_{\leq 1\,\mathrm{GeV}} = 495.0(1.5)(2.1) \times 10^{-10}$$

- side-products: improved determination of $\pi\pi$ P-wave phase shift; pion charge radius