π^0-TFF and π^0-pole in HLbL from lattice QCD

Andreas Nyffeler

PRISMA Cluster of Excellence, Institut für Kernphysik,
Helmholtz-Institut Mainz
Johannes Gutenberg Universität Mainz, Germany
nyffeler@uni-mainz.de

INT Workshop INT-19-74W: Hadronic contributions to $(g-2)_\mu$
3rd Plenary Workshop of the Muon $g-2$ Theory Initiative
University of Washington, Seattle, USA, September 9-13, 2019

References

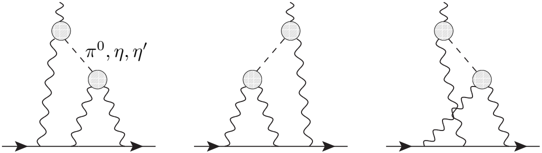
- Lattice calculation of the pion transition form factor $\pi^0 \to \gamma^*\gamma^*$
  A. Gérardin, H. Meyer, AN, Phys. Rev. D94, 074507 (2016)
  arXiv:1607.08174 [hep-lat]

- Lattice calculation of the pion transition form factor with $N_f = 2+1$ Wilson quarks
  A. Gérardin, H. Meyer, AN, Phys. Rev. D100, 034520 (2019)
  arXiv:1903.09471 [hep-lat]

Study of pion transition form factor (TFF) and pion-pole in HLbL might also help to better control long-distance behavior / finite volume effects of full HLbL calculation on same lattice ensembles.

Pion-pole contribution to a_μ^{HLbL} in dispersive framework

Pion-pole prescription from Knecht + AN '02, Colangelo et al. '14, '15, Pauk + Vanderhaeghen '14



$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right)$$

α_e is the fine-structure constant and [Jegerlehner + AN '09]

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1, Q_2, \tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1, Q_2, \tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

3-dim. integration over lengths $Q_i = |(Q_i)_\mu|, i = 1, 2$ of the two Euclidean momenta and angle θ between them $Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$ with $\tau = \cos\theta$.
$w_{1,2}(Q_1, Q_2, \tau)$ are model-independent weight functions which are concentrated at small momenta below $1\ \mathrm{GeV}$ [AN '16].
TFF $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ from data-driven dispersive approach or lattice QCD.

Pion transition form factor from lattice QCD

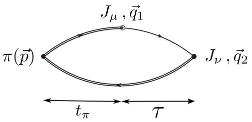
Pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ important by itself: yields insights into dynamics of QCD at low and high energies: chiral anomaly (decay $\pi^0 \to \gamma\gamma$, tests of ChPT), Brodsky-Lepage (pion distribution amplitude), OPE, pion-pole in HLbL, ...

Exploratory lattice studies of TFF at rather large pion mass and single lattice spacing by Dudek + Edwards '06; Cohen *et al.* '08; Lin + Cohen '12; Shintani *et al.* '09; Feng *et al.* '11. Or interested more in low-energy region: $\pi^0 \to \gamma\gamma$, Feng *et al.* '12.

In Euclidean space-time [Ji + Jung '01; Cohen *et al.* '08; Feng *et al.* '12]:

$$\begin{aligned} M_{\mu\nu}^E(p, q_1) &= -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3 z\, e^{-i\vec{q}_1 \vec{z}} \langle 0 | T \left\{ J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) \right\} | \pi(p) \rangle \\ &= \epsilon_{\mu\nu\alpha\beta}\, q_{1\,\alpha}\, q_{2\,\beta}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \end{aligned}$$

- Analytical continuation: $q_1 = (\omega_1, \vec{q}_1)$
- We must keep $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles

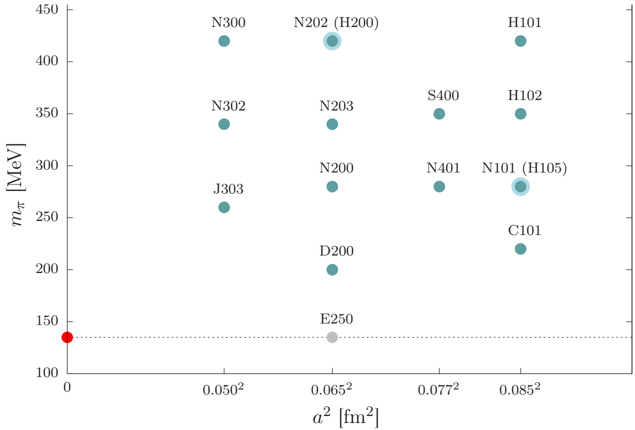


The main object to compute is the Euclidean three-point correlation function:

$$C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x}, \vec{z}} \left\langle T \left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0) \right\} \right\rangle e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1 \vec{z}}$$

Lattice setup for our analysis

CLS $N_f = 2+1$ ensembles:



- 15 ensembles of $\mathcal{O}(a)$-improved Wilson-Clover fermions
- Pion masses in range $200 - 420$ MeV (E250 with $m_{\pi,\mathrm{phys}}$ not yet used)
- 4 lattice spacings: $a = (0.050, 0.065, 0.077, 0.085)$ fm

Improvements compared to our earlier work from 2016

- Full $\mathcal{O}(a)$-improvement of vector currents (Gérardin, Harris, Meyer '19) $\Rightarrow$ continuum extrapolation $\sim a^2$

  Use two different discretizations of vector currents: local and conserved (combined continuum extrapolation).

- Ensembles with different volumes to study finite-size effects $\Rightarrow$ negligible at our level of precision.
- Hypercubic artefacts (breaking of spatial rotational invariance on lattice): small, can increase statistics by averaging over all equivalent combinations of $(\vec{q}_1^2, \vec{q}_2^2)$.
- Disconneced contributions: studied 5 ensembles, effect at the level of a few percent in TFF, estimated effect on pion-pole contribution (thanks to Konstantin Ottnad for providing correlation functions).
- Added moving frame to get larger kinematical reach.

Kinematic reach in the photon virtualities

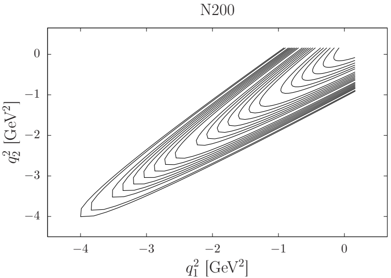
$$q_1^2 = \omega_1^2 - \vec{q}_1^{\,2}$$

$$q_2^2 = (E_\pi - \omega_1)^2 - (\vec{p} - \vec{q}_1)^2$$

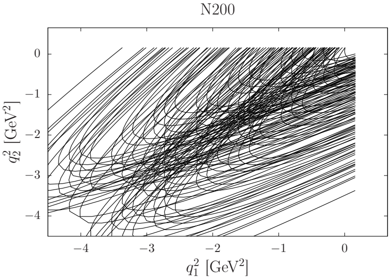
- ω_1 is a free parameter: $q_1 = (\omega_1, \vec{q}_1)$
- Discrete spatial momenta on finite lattice: $\vec{q}_1 = (2\pi/L)\vec{n}, \quad \vec{n} \in \mathbb{Z}^3$

Example: CLS ensemble N200 ($48^3 \times 128$ lattice, $a = 0.065$ fm, $m_\pi = 284$ MeV)

Pion rest frame ($\vec{p} = \vec{0}$)



Moving frame ($\vec{p} = (2\pi/L)\vec{z}$)



- Access only to subsets of mostly spacelike photon momenta in (q_1^2, q_2^2)-plane.
- Computed all spatial momenta $\vec{q}_1$ to cover photon virtualities up to $Q_{1,2}^2 \sim 3\ \mathrm{GeV}^2$ (double-virtual) and $\sim 1.5\ \mathrm{GeV}^2$ (single-virtual).
- On the lattice it is easier to get many points for the double-virtual TFF ! In contrast to experiments, where there are no data yet for the double-virtual case.

Extrapolation of lattice data for TFF to the physical point

Based on analytical properties of TFF, assume modified double z-expansion for space-like momenta (model-independent) [Boyd *et al.* '96; Bourrely *et al.* '09]:

$$P(Q_1^2, Q_2^2)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) =$$
$$\sum_{n,m=0}^{N} c_{nm} \left(z_1^n - (-1)^{N+n+1} \frac{n}{N+1}\, z_1^{N+1} \right) \left(z_2^m - (-1)^{N+m+1} \frac{m}{N+1}\, z_2^{N+1} \right)$$

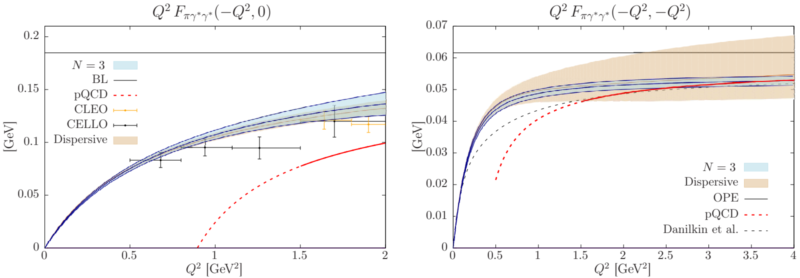
Expansion with $c_{nm} = c_{mn}$ (Bose symmetry) in the conformal variables:

$$z_k = \frac{\sqrt{t_c + Q_k^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_k^2} + \sqrt{t_c - t_0}}\,, \quad k = 1, 2, \quad t_0 = t_c \left(1 - \sqrt{1 + Q_{\mathrm{max}}^2 / t_c} \right), \quad t_c = 4 m_\pi^2$$

- Map branch cut starting at t_c onto unit circle $|z_k| = 1$.
- Choice of t_0 reduces maximum value of $|z_k|$ in range $[0, Q_{\mathrm{max}}^2]$. For $Q_{\mathrm{max}}^2 = 4\ \mathrm{GeV}^2$ get $|z_{\mathrm{max}}| = 0.46$ and expect quick convergence.
- Choice $P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2}$ ensures that TFF falls off like $1/Q^2$ in all directions in (Q_1^2, Q_2^2) plane (Brodsky-Lepage, OPE).
- Imaginary part of TFF behaves like $(q^2 - t_c)^{3/2}$ near threshold (P-wave). Implemented by imposing $\left[\mathrm{d}\mathcal{F}_{\pi^0\gamma^*\gamma^*} / \mathrm{d}z_k \right]_{z_k = -1} = 0, k = 1, 2$.
- Extrapolation of coefficients c_{nm} to physical quark mass (with $\widetilde{y} \equiv m_\pi^2 / (16\pi^2 f_\pi^2)$) and to continuum (for two discretizations of currents) with fit ansatz:

$$c_{nm}(\widetilde{y}, a) = c_{nm}(\widetilde{y}^{\mathrm{phys}}, 0) + \gamma_{nm}\, (\widetilde{y} - \widetilde{y}_{\mathrm{phys}}) + \delta_{nm}^d \left(\frac{a}{a_{\beta=3.55}} \right)^2, \; d = 1, 2$$

Final result for pion TFF $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ at physical point



- Results of fit with double z-expansion for $N = 3$ ($\chi^2/\mathrm{d.o.f.} = 1.1$, uncorrelated global fit; for $N = 1, 2$ we get $\chi^2/\mathrm{d.o.f.} = 1.5, 1.2$). Tested fast convergence of z-expansion with increasing N for mock-data from LMD+V as toy model (precision below 1% already for $N = 3$).
- TFF and its error available on grid in (Q_1^2, Q_2^2)-plane for $0 \leq Q_i^2 \leq 4.975\ \mathrm{GeV}^2$ with step-size $0.025\ \mathrm{GeV}^2$ in file TFF.dat from arXiv:1903.09471 [hep-lat].
- Horizontal black lines: predictions from Brodsky-Lepage (single-virtual) and OPE (double-virtual). Do not impose prefactor as constraint.
- Prediction at large Q^2 with perturbative QCD includes higher twist and NLO corrections and assumes asymptotic pion distribution amplitude.
- Fits of lattice data with simple resonance models lead to bad $\chi^2/\mathrm{d.o.f.} = 4.8$ (VMD), 1.5 (LMD).

Normalization of the TFF and the decay width $\Gamma(\pi^0 \to \gamma\gamma)$

- Tension of 1.1 σ [1.7 σ] between measurements by PrimEx '11 [Prim-Ex II '18 (preliminary)] and ChPT at NNLO (Moussallam + Kampf '09 [K+M]):

$$\Gamma(\pi^0 \to \gamma\gamma)^{\text{exp}} = 7.82(22) \quad (2.8\%) \quad [7.80(13) \quad (1.7\%)] \ \mathrm{eV}$$

$$\Gamma(\pi^0 \to \gamma\gamma)^{\text{ChPT}} = 8.09(11) \quad (1.4\%) \ \mathrm{eV}$$

  Other earlier work at NLO by Goity *et al.* '02, Ananthanarayan + Moussallam '02 get similar values and errors. Ioffe + Oganesian '07 (QCD sum rules) obtain lower central value: $\Gamma(\pi^0 \to \gamma\gamma) = 7.93(12)\ \mathrm{eV}$

- Relation of width to normalization of TFF (α_e: fine-structure constant):

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = \frac{\pi \alpha_e^2 m_\pi^3}{4} \left[\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)\right]^2$$

Normalization of the TFF and the decay width $\Gamma(\pi^0 \to \gamma\gamma)$ (continued)

- Normalization of TFF in chiral limit at LO (WZW):

$$\alpha \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F}$$

  F = pion decay constant in chiral limit.

- Chiral logarithms in $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$ are absent at NLO, once one expresses F by decay constant at physical pion mass (Donoghue *et al.* '85+'88(E); Bijnens *et al.* '88). Chiral log's negligible at NNLO [K+M].

- Motivates extrapolation of normalization of TFF on lattice using ansatz:

$$f_\pi \, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \widetilde{\alpha} + \gamma \, m_\pi^2 + \delta_d \left(\frac{a}{a_{\beta=3.55}} \right)^2 , \quad C_7^{\mathrm{Wr}} = -\frac{3}{64} \gamma$$

  f_π is decay constant on lattice with given m_π.
  Can obtain LEC C_7^{Wr} in odd-intrinsic-parity sector of ChPT at $\mathcal{O}(p^6)$ (Bijnens *et al.* '02) by varying pion mass on lattice !

Low-energy constants $C_{7,8}^{\mathrm{Wr}}$ in ChPT in odd-parity sector from lattice

- Fit with z-expansion for $|Q_i| < 1\ \mathrm{GeV}$ ($\chi^2/\mathrm{d.o.f.} = 1.1$ already for $N = 1$):

$$\begin{aligned} \alpha &= 0.264(8)(4)\ \mathrm{GeV}^{-1} \\ C_7^{\mathrm{Wr}} &= 0.16(18) \times 10^{-3}\ \mathrm{GeV}^{-2} \end{aligned}$$

- For comparison: $\alpha^{\mathrm{PrimEx[PrimEx-II]}} = 0.276(4)\ [0.275(2)]\ \mathrm{GeV}^{-1}$.

- Our value of C_7^{Wr} with its uncertainty is compatible with conflicting estimates in literature:

$$\begin{aligned} |C_7^{\mathrm{Wr}}| &< 0.06 \times 10^{-3}\ \mathrm{GeV}^{-2} \quad [\mathrm{K+M}] \\ C_7^{\mathrm{Wr}} &= 0.35(7) \times 10^{-3}\ \mathrm{GeV}^{-2} \quad [\mathrm{LMD(+P)}] \end{aligned}$$

  LMD(+P): resonance estimate with LMD+P model (Moussallam '95, Knecht + AN '01, Kampf + Novotny '11).

- Following the same procedure as K+M we get with $\Gamma(\eta \to \gamma\gamma)$ from PDG 2018 the other relevant LEC and $\Gamma(\pi^0 \to \gamma\gamma)$:

$$\begin{aligned} C_8^{\mathrm{Wr}} &= 0.56(17) \times 10^{-3}\ \mathrm{GeV}^{-2} \\ \Gamma(\pi^0 \to \gamma\gamma) &= 8.07(10)\ \mathrm{eV} \end{aligned}$$

- In K+M: $C_8^{\mathrm{Wr}} = 0.58(20) \times 10^{-3}\ \mathrm{GeV}^{-2}$.
- Lattice can at the moment not resolve tension between PrimEx measurement and ChPT prediction for decay width.

Pion-pole contribution to HLbL from lattice QCD

Lattice result from double z-expansion:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.7 \pm 3.4 \pm 0.9 \pm 0.5) \times 10^{-11} = (59.7 \pm 3.6) \times 10^{-11} \quad \text{(6\% precision)}$$

- First error statistical: includes lattice spacing uncertainty (1% error in $a \Rightarrow$ 2% error in $a_\mu^{\mathrm{HLbL};\pi^0}$), renormalization of vector currrents (negligible), extrapolation to physical point (could be improved by including ensemble at physical pion mass).
- In contrast to many phenomenological evaluations, our lattice calculation of $a_\mu^{\mathrm{HLbL};\pi^0}$ is more accurate than twice the lattice determination of the normalization $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$ (3.5% uncertainty).
- Second error systematics: includes effect of the truncation of z-expansion.
- Third error: estimate of disconnected contribution $\Delta a_\mu^{\mathrm{HLbL};\pi^0;\mathrm{disc}} = -1.0(0.3) \times 10^{-11}$. Use conservative 50% uncertainty.
- Result confirmed by using fit of lattice data with LMD+V model or a Canterbury approximant (generalization of Padé approximant to 2 variables).

Pion-pole contribution to HLbL from lattice QCD (continued)

Lattice combined with published PrimEx '11 normalization of TFF:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (62.3 \pm 2.0 \pm 0.9 \pm 0.5) \times 10^{-11} = (62.3 \pm 2.3) \times 10^{-11} \quad (3.7\%\ \text{precision})$$

- Statistical error smaller because of precise result from PrimEx '11 for normalization $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$ with 1.4%.
- Result agrees perfectly with determinations using dispersion relations (DR; Hoferichter *et al.* '18) and Canterbury approximants (CA; Masjuan + Sanchez-Puertas '17) that use PrimEx normalization:

$$a_\mu^{\mathrm{HLbL};\pi^0}(\mathrm{DR}) = 62.6^{+3.0}_{-2.5} \times 10^{-11}$$

$$a_\mu^{\mathrm{HLbL};\pi^0}(\mathrm{CA}) = (63.6 \pm 2.7) \times 10^{-11}$$

Conclusions and Outlook

- Calculation of double-virtual π^0 transition form factor (TFF) $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ for $0 \leq Q_i^2 \leq 5\ \mathrm{GeV}^2$ from first principles with lattice QCD. Extrapolated to physical point using double z-expansion.
- Good agreement with experimental data for single-virtual TFF for $Q^2 \leq 2\ \mathrm{GeV}^2$ and with data-driven theoretical approaches like dispersion relations or Canterbury approximants. Peculiarity on lattice: smaller uncertainties for double-virtual TFF than for single-virtual case.
- Normalization on lattice $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = 0.264(8)(4)\ \mathrm{GeV}^{-1}$, lower than PrimEx [PrimEx-II]: $0.276(4)\ [0.275(2)]\ \mathrm{GeV}^{-1}$.
- From m_π^2 dependence of TFF extracted low-energy constant in ChPT:

$$C_7^{\mathrm{Wr}} = 0.16(18) \times 10^{-3}\ \mathrm{GeV}^{-2}$$

  Inconclusive with respect to resonance estimates of C_7^{Wr}.
- Pion-pole contribution to HLbL:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.7 \pm 3.6) \times 10^{-11} \quad \text{(pure lattice calculation, 6\% precision)}$$
$$a_\mu^{\mathrm{HLbL};\pi^0} = (62.3 \pm 2.3) \times 10^{-11} \quad \text{(PrimEx normalization, 3.7\% precision)}$$

- Outlook: η, η' TFFs much more difficult to obtain on lattice, disconnected contributions dominate.

Backup slides

Shape of integrand for ensemble D200 ($a = 0.065$ fm, $m_\pi = 200$ MeV)

$$M^E_{\mu\nu} = \frac{2E_\pi}{Z_\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1\tau}\, e^{-E_\pi\tau}$$

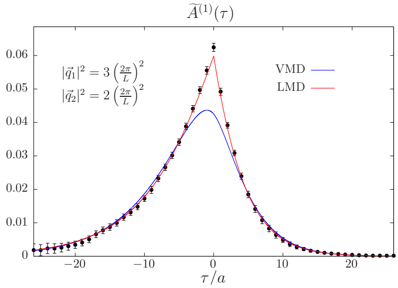
$$A_{\mu\nu}(\tau) = \lim_{t_\pi\to\infty} C_{\mu\nu}(\tau, t_\pi)e^{E_\pi t_\pi}$$

$$\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \equiv P_{\mu\nu}\omega_1 + Q_{\mu\nu}$$

$$\widetilde{A}_{\mu\nu}(\tau) = -iQ^E_{\mu\nu}\,\widetilde{A}^{(1)}(\tau) + P^E_{\mu\nu}\,\frac{\mathrm{d}\widetilde{A}^{(1)}}{\mathrm{d}\tau}(\tau)$$

$$P^E_{\mu\nu} = iP_{\mu\nu}, \quad Q^E_{\mu\nu} = (-i)^{n_0}Q_{\mu\nu}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi\tau} & \tau < 0 \end{cases}$$



Cusp at $\tau = 0$ related to OPE (coefficient $\beta \neq 0$ in LMD model)

Finite time extent of the lattice.

Signal deteriorates at large $|\tau|$.

→ Fit the data at large $\tau > \tau_c$ using vector meson dominance (VMD) model or lowest meson dominance (LMD) model:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

From difference between models estimate systematic error.

Take only points in TFF where at least 80% of integral from lattice data.

→ Introduces a cut-off $\tau_c \gtrsim 1.5$ fm.

CLS $N_f = 2 + 1$ ensembles

id	$L^3 \times T$	a [fm]	m_π [MeV]	$m_\pi L$	#confs
H101	$32^3 \times 96$	0.08636	416(6)	5.8	1000
H102	$32^3 \times 96$		354(5)	5.0	1900
H105*	$32^3 \times 96$		281(4)	3.9	2800
N101	$48^3 \times 128$		280(4)	5.9	1600
C101	$48^3 \times 96$		224(3)	4.7	2200
S400	$32^3 \times 96$	0.07634	349(5)	4.3	1700
N401	$48^3 \times 128$		286(4)	5.3	950
H200*	$32^3 \times 96$	0.06426	419(6)	4.4	2000
N202	$48^3 \times 128$		411(5)	6.4	900
N203	$48^3 \times 128$		346(5)	5.4	1500
N200	$48^3 \times 128$		284(3)	4.4	1700
D200	$64^3 \times 128$		200(3)	4.2	1100
N300	$48^3 \times 128$	0.04981	422(5)	5.1	1200
N302	$48^3 \times 128$		343(5)	4.2	1100
J303	$64^3 \times 192$		258(3)	4.2	650

- Ensembles with asterisk * not used in final analysis, but used to control finite size effects.