White Paper Summary: Lattice QCD calculations of the HVP

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Hadronic contributions to $(g\text{-}2)_\mu$
Third Plenary Workshop of the Muon g-2 Theory Initiative
Institute for Nuclear Theory, University of Washington
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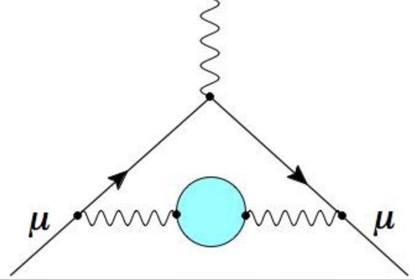
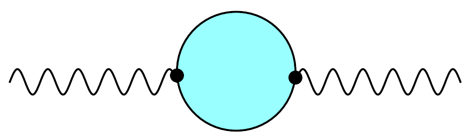


Hadronic vacuum polarization



$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

$$\Pi_{\mu\nu} = \int d^4x e^{iqx} \langle j_\mu(x) j_\nu(0) \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2)$$

Leading order HVP correction: $$a_\mu^{\rm HVP,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2 \omega(q^2)\, \hat{\Pi}(q^2)$$

- Use optical theorem and dispersion relation to rewrite the integral in terms of the hadronic *e+e-* cross section:

$$a_\mu^{\rm HVP,LO} = \frac{m_\mu^2}{12\pi^3} \int ds \frac{\hat{K}(s)}{s} \sigma_{\rm exp}(s)$$

- This talk: discuss $a_\mu^{\rm HVP,LO}$ calculated in lattice QCD

Lattice HVP WP authors

Tom Blum, Mattia Bruno, Christine Davies, Michele Della Morte, Davide Giusti, Steven Gottlieb, Vera Gülpers, Gregorio Herdoíza, Taku Izubuchi, Christoph Lehner, Laurent Lellouch, Marina Marinkovic, Aaron S. Meyer, Kohtaroh Miura, Antonin Portelli, Silvano Simula, Ruth Van de Water, Georg von Hippel, Hartmut Wittig

Lattice HVP WP organization

I. Introduction
   A. The hadronic vacuum polarization
   B. Calculating and integrating $\Pi(q^2)$ to obtain a_μ
   C. Time moments
   D. Coordinate-space representation
   E. Common issues
II. Strategies
   A. Connected light-quark contribution $a_\mu^{\rm HLO}(ud)$
      1. Statistical errors
      2. Finite volume effects and long-distance two-pion contributions
      3. Discretization and scale setting
      4. Chiral extrapolation/interpolation
   B. Connected strange and charm contributions $a_\mu^{\rm HLO}(s), a_\mu^{\rm HLO}(c), a_\mu^{\rm HLO}(b)$
   C. Disconnected term [$a_\mu^{\rm HLO}$discussion]
   D. Strong and em IB contributions $\delta a_\mu^{\rm HLO}$
III. Comparisons
   A. Comparison of total LO-HVP contribution
   B. Flavor-by-flavor comparison
   C. Toward lattice QCD consensus and permil-level precision
IV. Connections
   A. HVP from lattice QCD and the MUonE experiment
   B. HVP from tau decays
   C. Hadronic corrections to the running of α and $\sin^2\theta_W$
V. Summary and conclusions
   A. Current status
   B. Lessons learned
   C. Expected progress in the next (2?) years

Outline

- Introduction
- Methods for a_μ^{HLO} with lattice QCD
- Charm and Strange contributions
- Noise reduction methods for light quark contributions
- Finite Volume corrections
- Lattice scale
- Continuum extrapolation
- Light quark connected a_μ^{HVP} $(m_u = m_d)$
- QED and Strong Isospin Breaking corrections
- Disconnected a_μ^{HVP}
- Comparisons
- Summary and outlook

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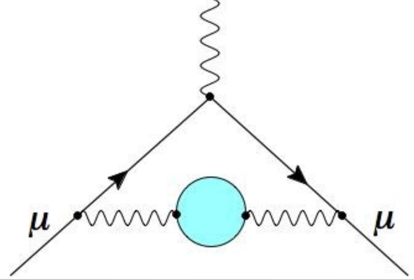
Reviews by:
K. Miura @ Lattice 2018
V. Gülpers @ Lattice 2019



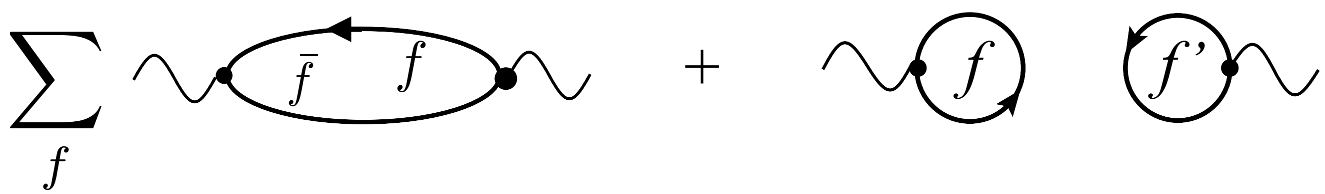
Lattice HVP: Introduction

Calculate $a_\mu^{\rm HVP}$ in Lattice QCD:

$$a_\mu^{\rm HLO} \equiv a_\mu^{\rm HVP,LO} = \sum_f a_{\mu,f}^{\rm HVP,LO} + a_{\mu,{\rm disc}}^{\rm HVP,LO}$$

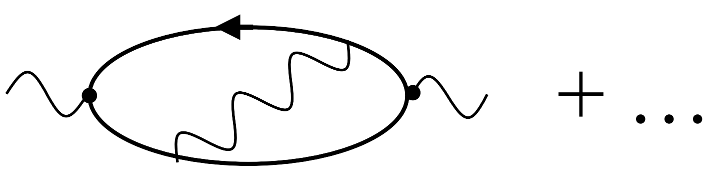
- Separate into connected for each quark flavor + disconnected contributions (gluon and sea-quark background not shown in diagrams)

  Note: almost always $m_u = m_d$

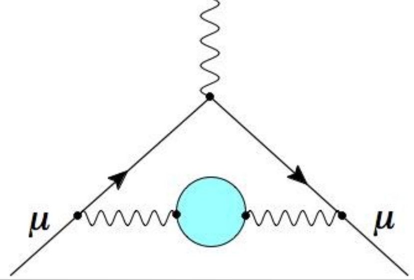


f= ud, s, c, b

- need to add QED and strong isospin breaking ($\sim m_u - m_d$) corrections:

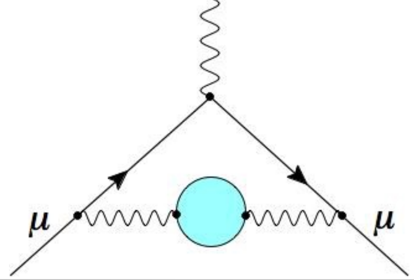


- either perturbatively on isospin symmetric QCD background
- or by using QCD + QED ensembles with $m_u \neq m_d$



Lattice HVP: Introduction

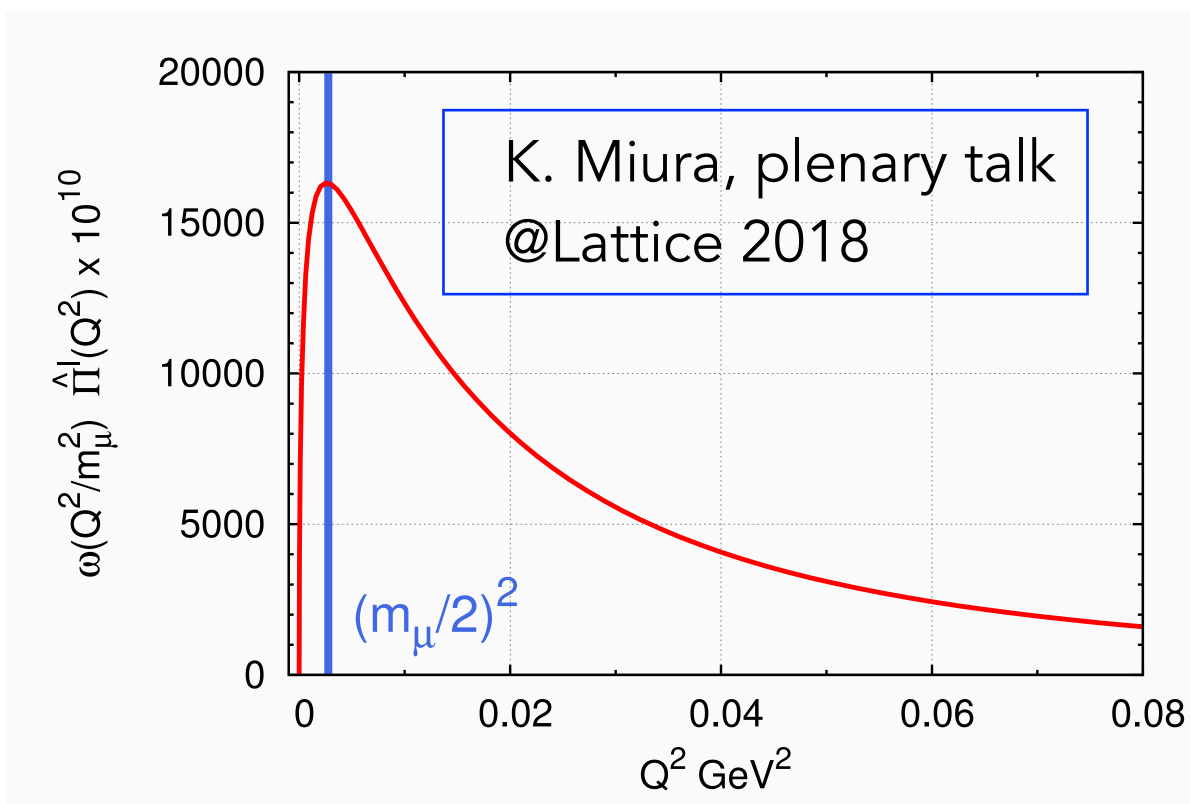
- Target: < 0.5% total error
- light-quark connected contribution, $a_{\mu,ud}^{\mathrm{HLO}}$, is ~90% of total, with 1-3% error
- ``heavy” flavor contributions, $a_{\mu,s}^{\mathrm{HLO}}$, $a_{\mu,c}^{\mathrm{HLO}}$, $a_{\mu,b}^{\mathrm{HLO}}$ are ~8%, 2%, 0.05% of total a_{μ}^{HLO}, can be calculated with sufficient precision
- disconnected contribution is ~2% of total a_{μ}^{HLO}, contributes ~0.3-1% error to a_{μ}^{HLO}
- Challenges:
  - ✓ needs ensembles with (light sea) quark masses at their physical values
  - finite volume corrections, continuum extrapolation: guided by EFT
  - include QED and strong isospin breaking corrections ($m_u \neq m_d$)
  - growth of statistical errors at large Euclidean times
    - ➠ noise reduction methods
      include guidance from EFT
    - ➠ include two-pion channels into analysis

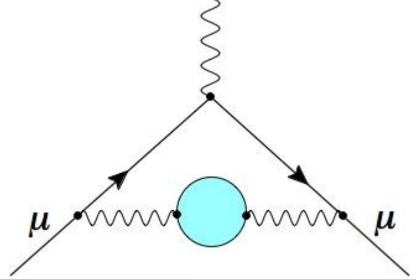


Methods

Leading order HVP correction: $$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2\, \omega(q^2)\, \hat{\Pi}(q^2)$$

- Calculate $a_\mu^{\rm HLO}$ in Lattice QCD:
  - ✦ Calculate $\hat{\Pi}(q^2)$ and evaluate the integral
    [Blum,PRL 03, Lautrup et al, 71]
    + use Padé approximants to parameterize function at low q^2.
    [Aubin, Blum, Golterman, Peris, PRD12]

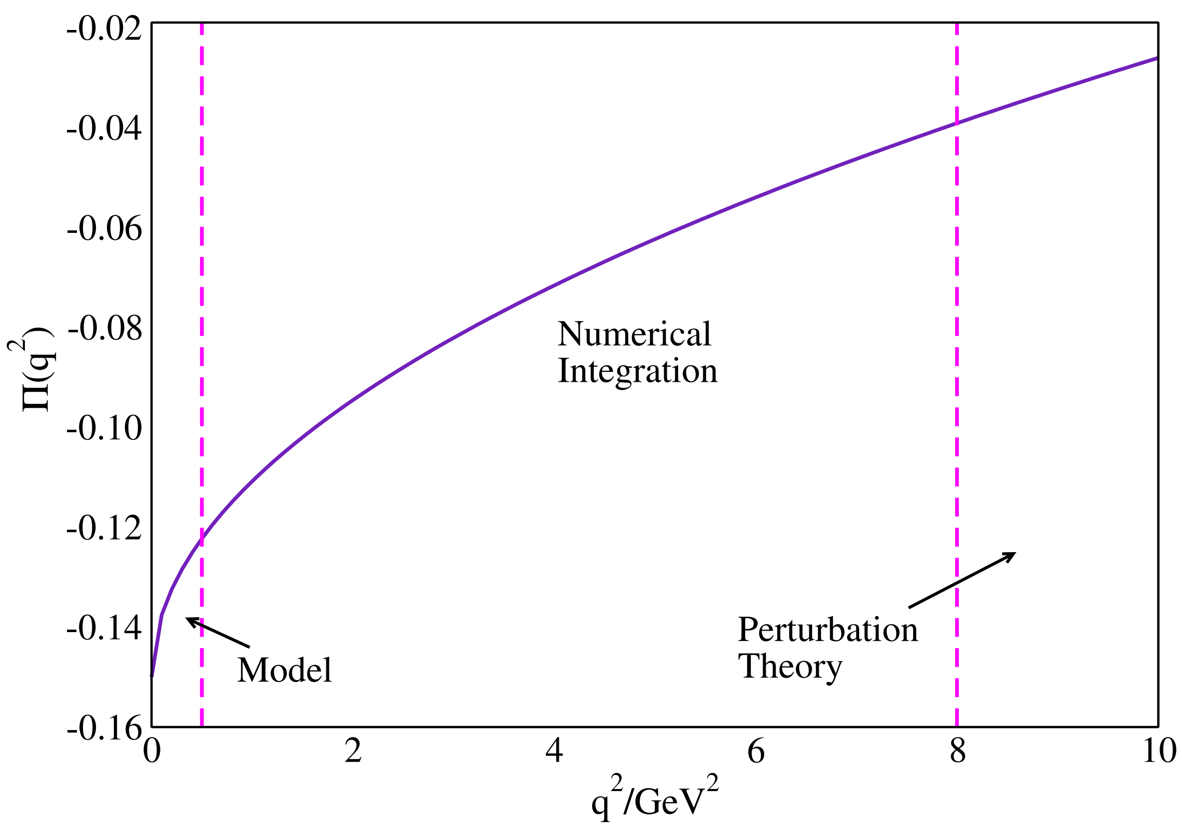




Methods

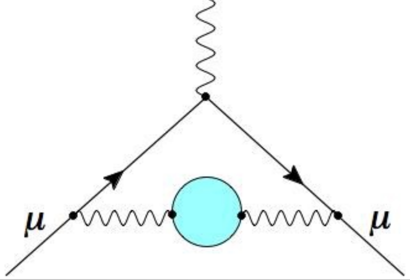
Leading order HVP correction: $$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2\, \omega(q^2)\, \hat{\Pi}(q^2)$$

- Calculate $a_\mu^{\rm HLO}$ in Lattice QCD:
  - ✦ Hybrid method
    [Blum, Golterman, Maltman, Peris, PRD14]



- in low-q² region use Padé or conformal polynomials, … (or MUonE results)
- in intermediate q² region integrate lattice data
- match to PT in high-q² region

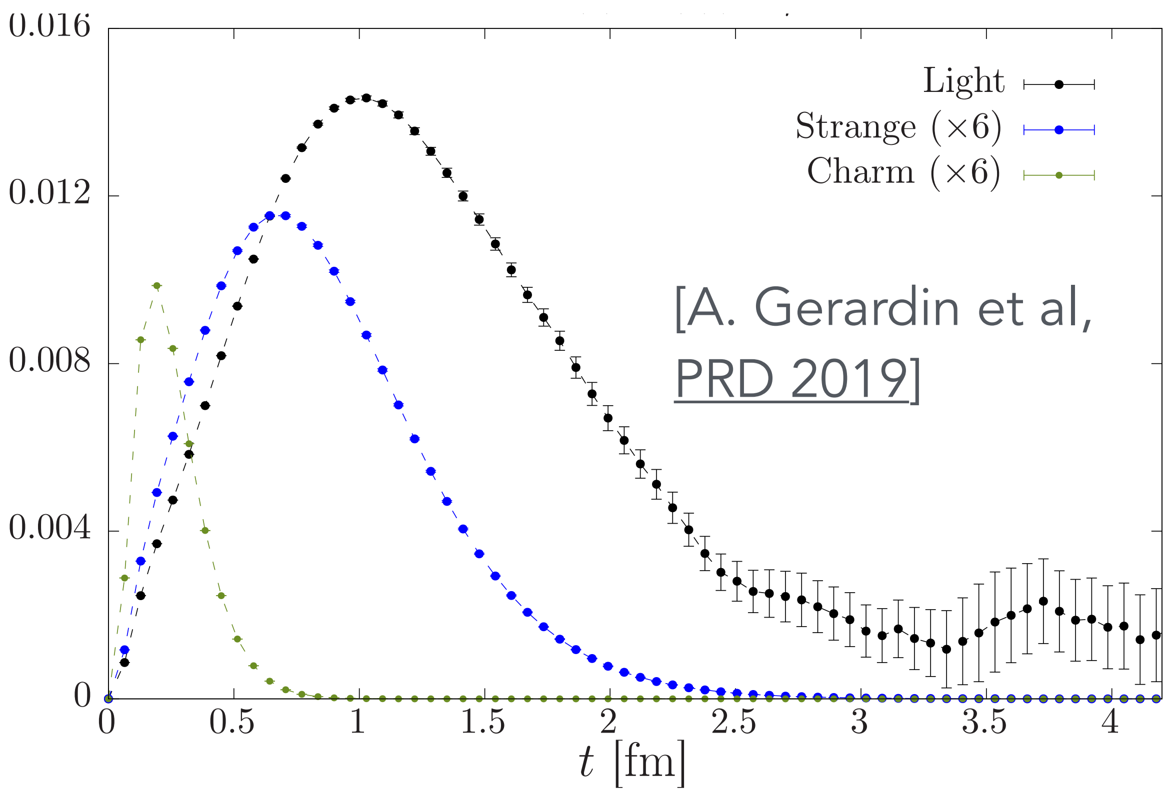
see also Marinkovic talk in MUonE session



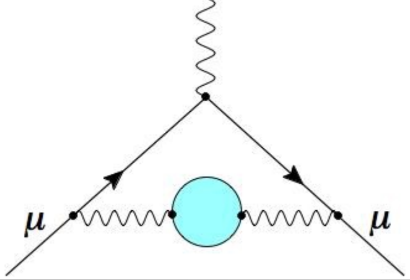
Methods

Leading order HVP correction: $$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2\, \omega(q^2)\, \hat{\Pi}(q^2)$$

- Calculate $a_\mu^{\rm HLO}$ in Lattice QCD:
  - ✦ Time-momentum representation: [Bernecker & Meyer, EPJ 12]
    reorder the integrations with $G(t) = \frac{1}{3}\sum_{i,x}\langle j_i(x,t)\, j_i(0,0)\rangle$
    $$a_\mu^{\rm HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dt\, \tilde{\omega}(t)\, G(t)$$



- Need to extend $G(t)$ for $t > T$ using spectral representation
- noise reduction methods to control growth of statistical errors at large t needed for light-quark contribution



Methods

Leading order HVP correction: $$a_\mu^{\mathrm{HLO}} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2\, \omega(q^2)\, \hat{\Pi}(q^2)$$

- Calculate a_μ^{HLO} in Lattice QCD:
  - ✦ Time-moments: Taylor expand $\hat{\Pi}(q^2) = \sum_k q^{2k} \Pi_k$

    Compute the Taylor coefficients from time moments

    $G_{2n} = a \sum_t t^{2n} G(t)$: $\quad \Pi_k = (-1)^{k+1} \dfrac{G_{2k+2}}{(2k+2)!}$

    and obtain $\hat{\Pi}(q^2)$ from $[n,n]$ and $[n,n\text{-}1]$ Padé approximants

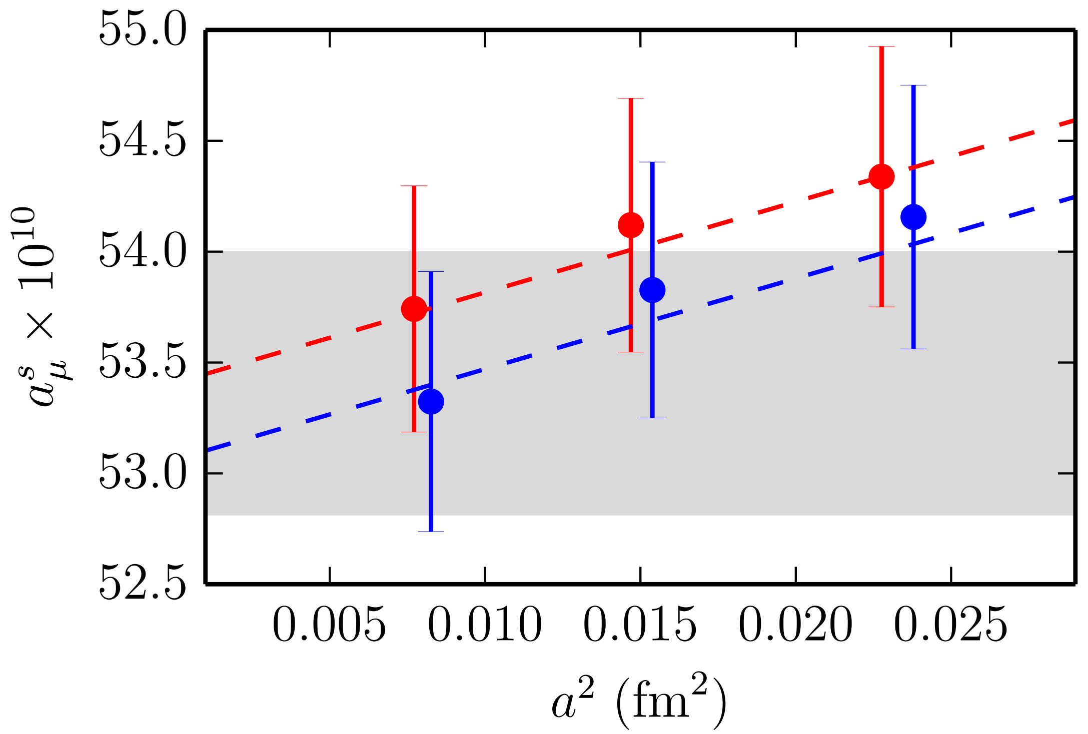
    [HPQCD (Chakraborty et al), PRD 14]

    Can apply corrections (finite volume, discretization) to the Taylor coefficients **before** constructing a_μ
  - ✦ Note: The time-moments method yields results that are numerically equivalent to the time-momentum representation.

charm, strange connected a_μ

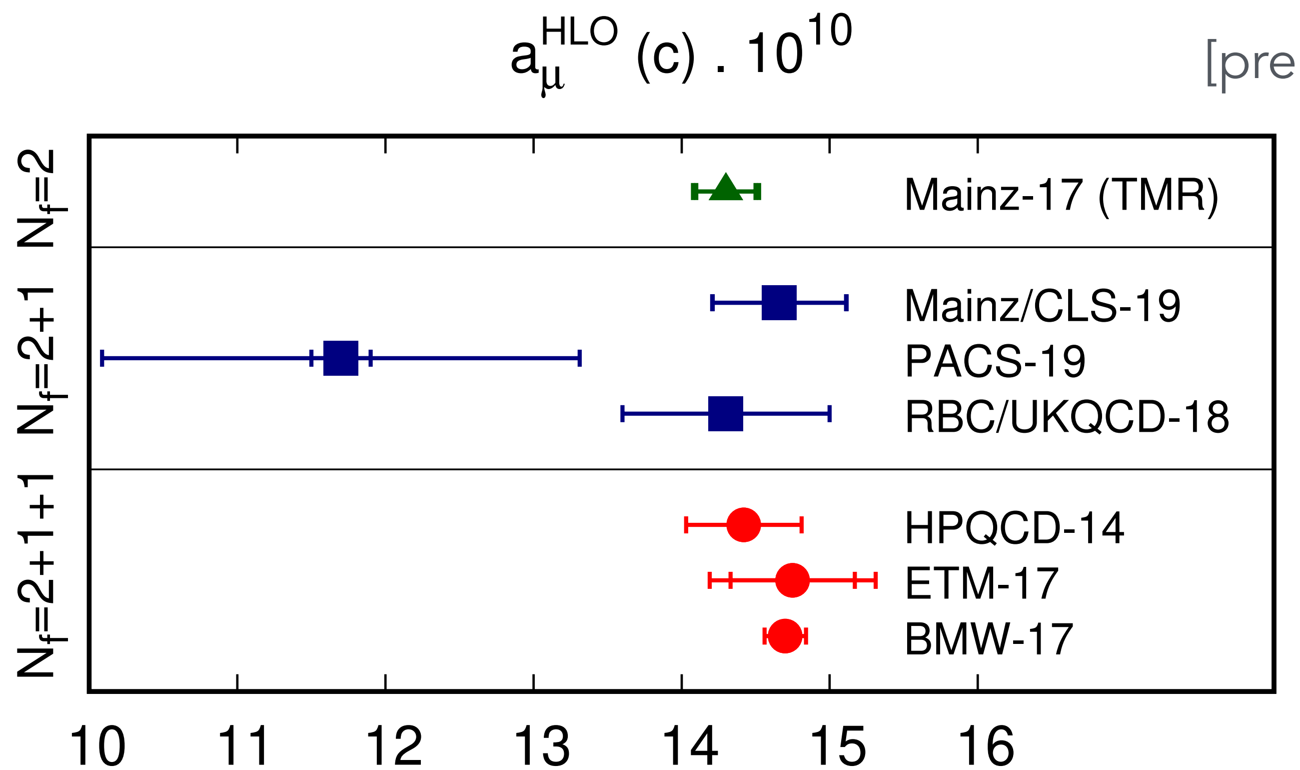
- long-distance noise not a major source of error
- FV corrections smaller
- discretization effects (especially for charm) a more significant source of error, but controllable with improved actions and small lattice spacings

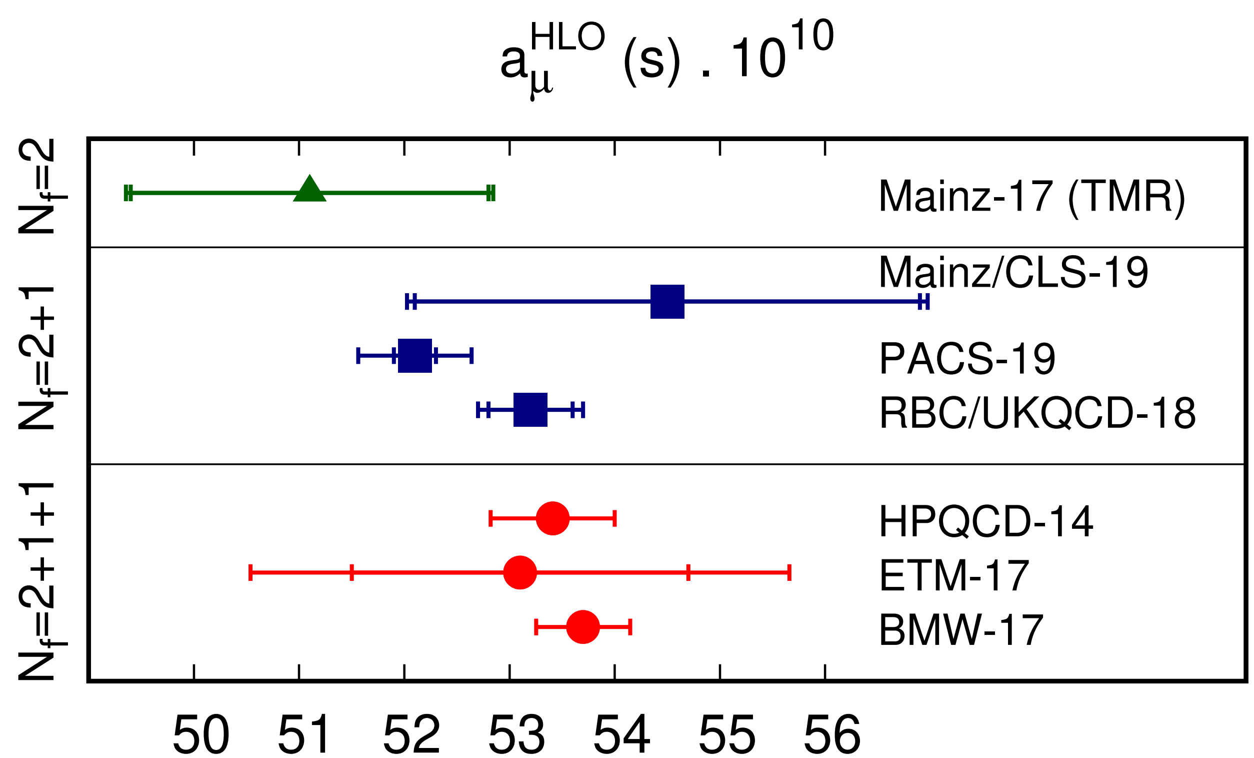
[HPQCD (Chakraborty et al), PRD 14]

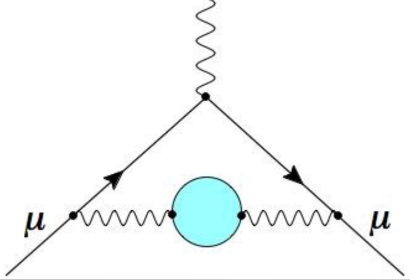


charm, strange connected a_μ: Comparison

[prepared by K. Miura for WP]





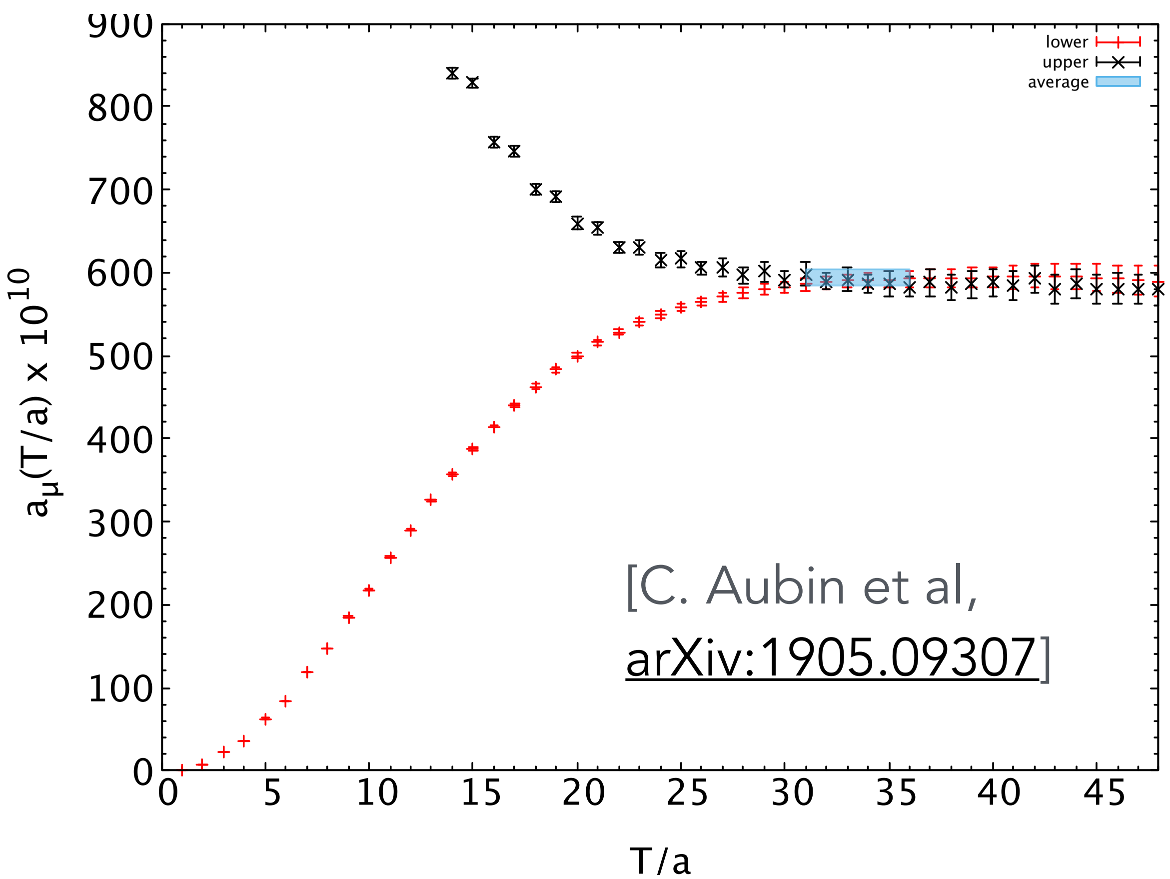


Noise Reduction Methods

$$G(t) = \frac{1}{3}\sum_{i,x}\langle j_i(x,t)\, j_i(0,0)\rangle$$

- Start with spectral decomposition: $G(t) = \sum_{n=0}^{\infty} A_n^2\, e^{-E_n t}$
  - ✦ bounding method: [Borsanyi et al, PRL 2018, Blum et al, PRL 2018]

    for $t > t_c$: $0 \leq G(t_c)\, e^{-E_{t_c}(t-t_c)} \leq G(t) \leq G(t_c)\, e^{-E_0(t-t_c)}$

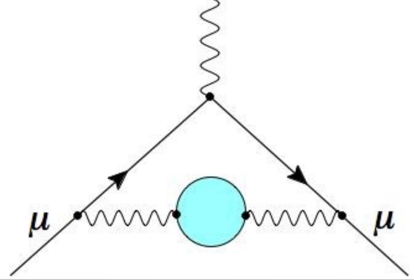


[C. Aubin et al, arXiv:1905.09307]

E_{t_c}: effective mass of G at t_c

E_0: ground state energy

replace $G(t > t_c)$ with upper and lower bound, vary t_c



Noise Reduction Methods

$$G(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x,t)\, j_i(0,0) \rangle$$

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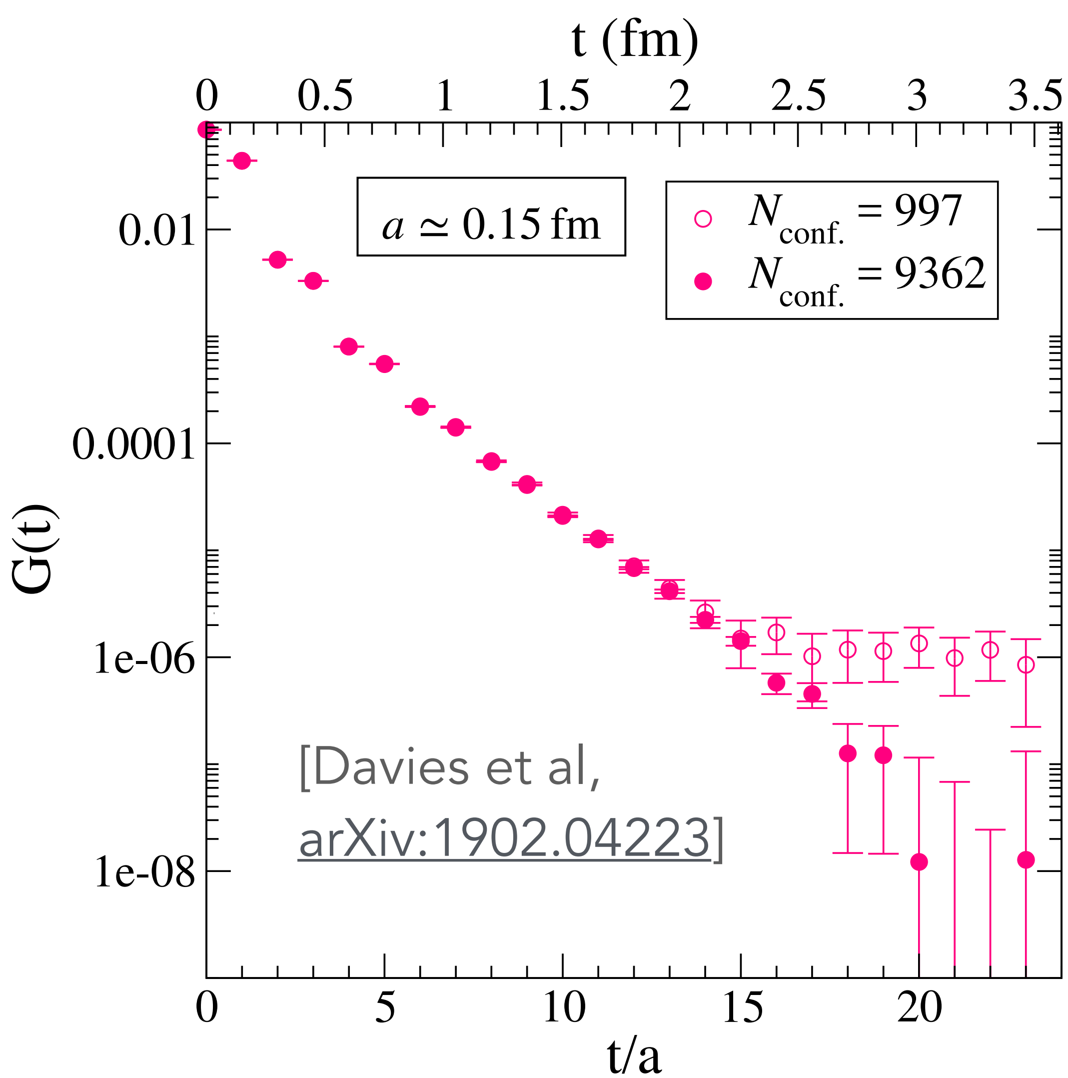
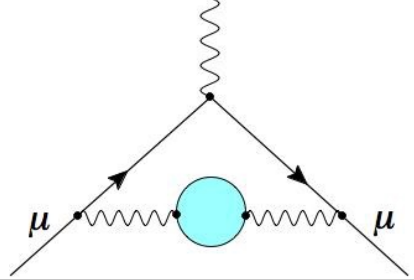
  - ✦fit method: [Chakraborty et al, PRD 2017]



-perform multi-exponential fits to $G(t)$ in range $t_{\min} \leq t \leq t_{\max}$

- replace $G(t)$ with fit for $t \geq t^* \simeq 2 - 2.5\text{fm}$

-tests of fit method using high statistics data and EFT guidance



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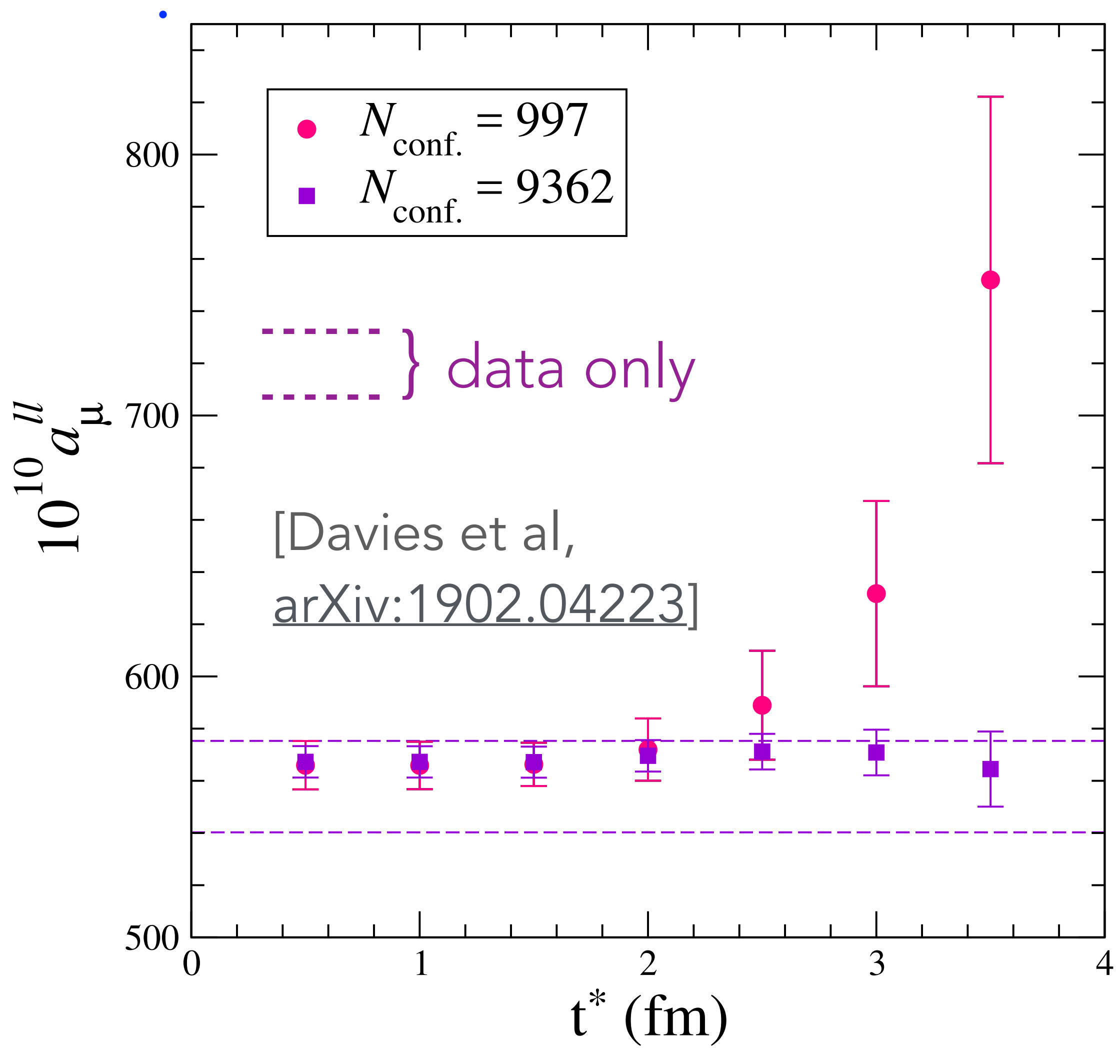

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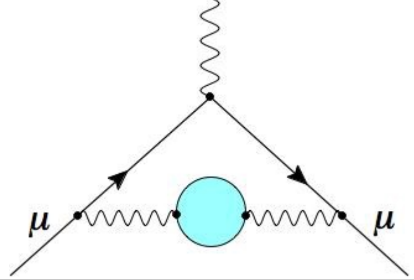
-consistent with bounding method

-can add contributions from two-pion states to reconstruct G(t) at large t



Noise Reduction Methods

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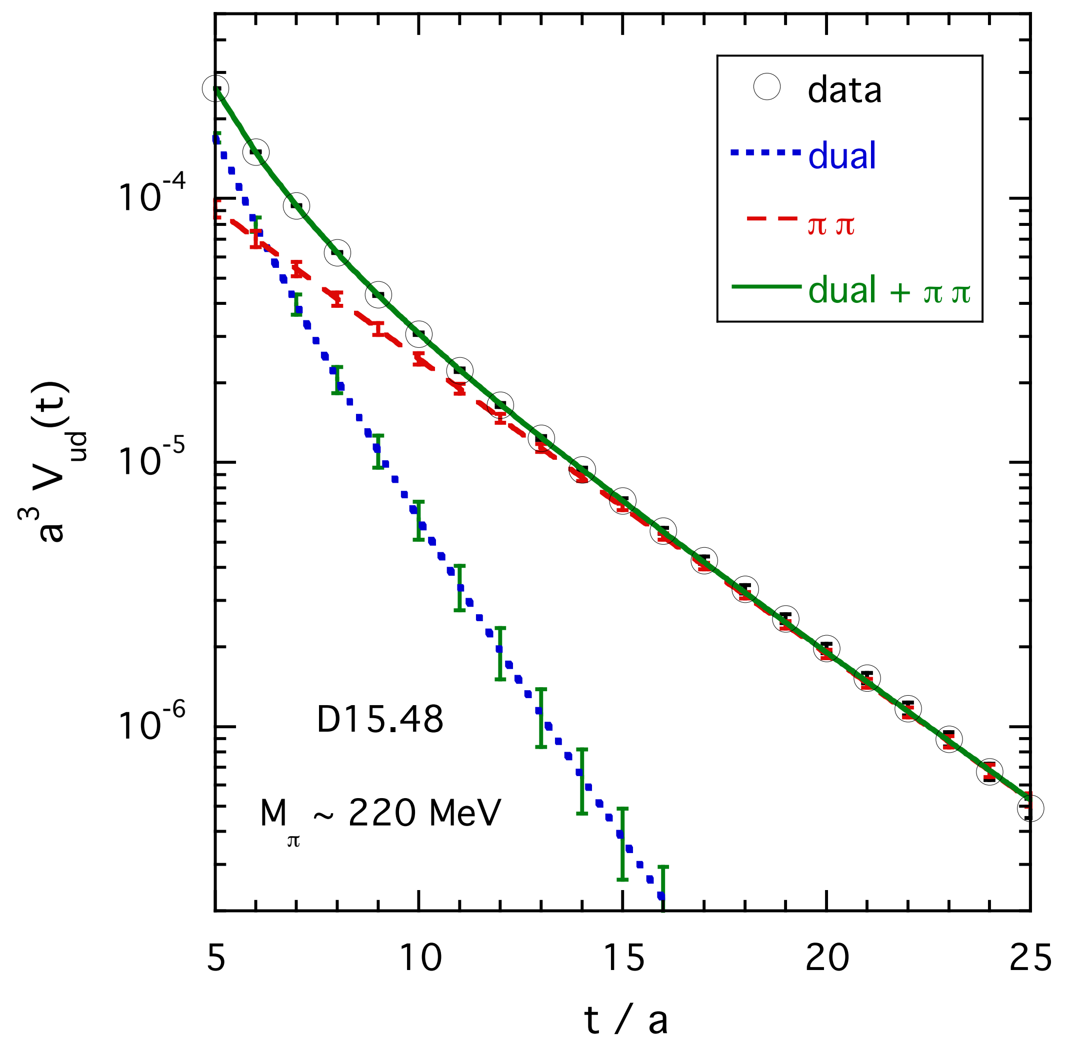
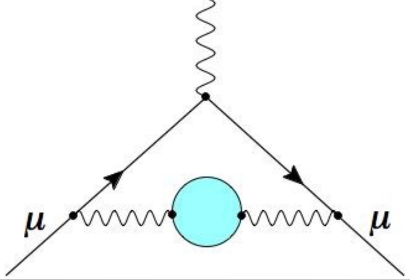
- Start with spectral decomposition: $G(t) = \sum_{n=0}^{\infty} A_n^2\, e^{-E_n t}$
  - ✦ include resonant two-pion states into representation of correlation function

    [D. Giusti et al, PRD 2018]





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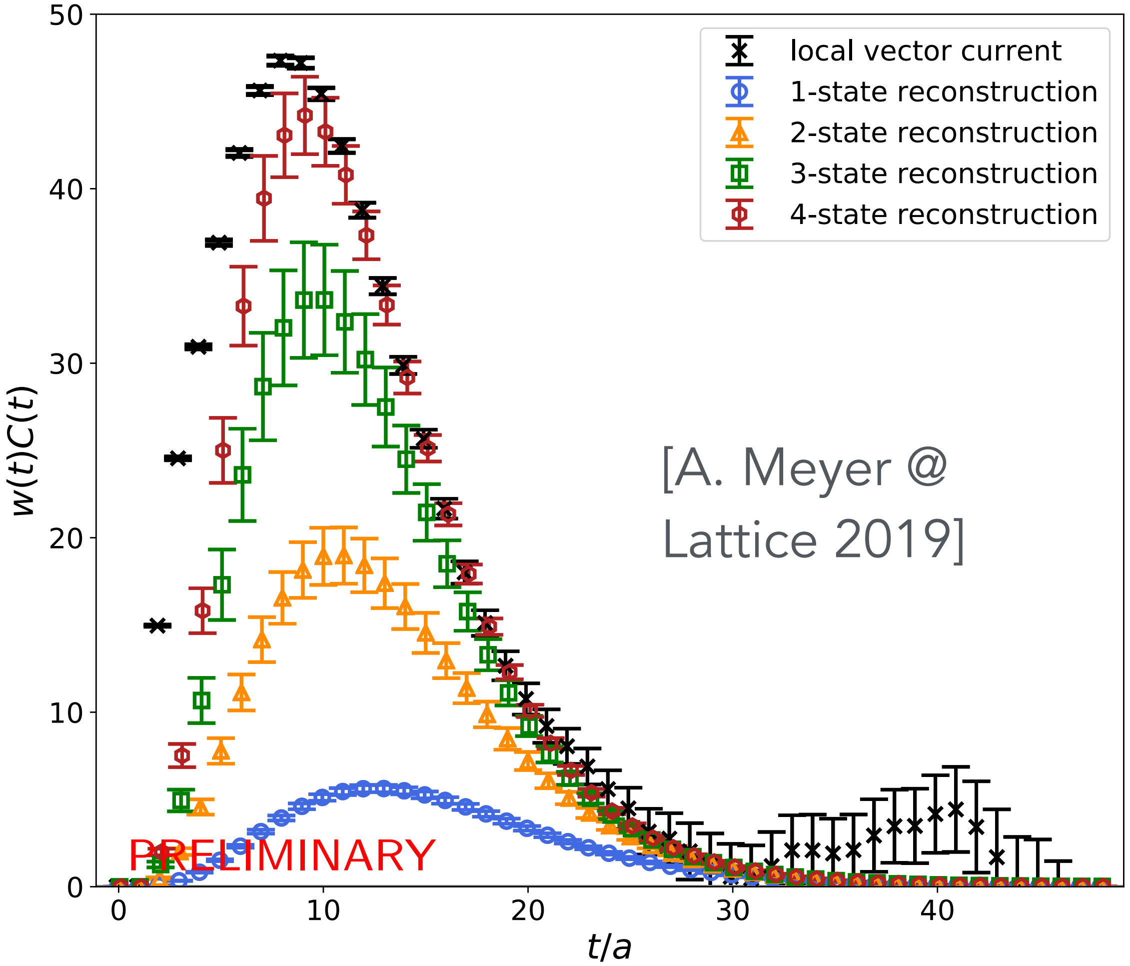
- Start with spectral decomposition: $G(t) = \sum_{n=0}^{\infty} A_n^2\, e^{-E_n t}$
  - ✦ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states
  - ✦ use to reconstruct $G(t > t_c)$
  - ✦ can be used to improve bounding method:

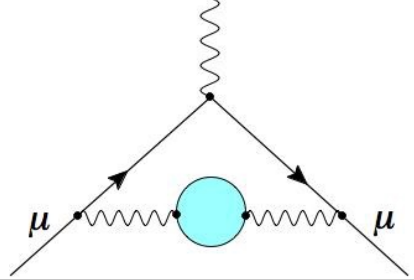
$$G(t) \to G(t) - \sum_{n=0}^{N} A_n^2\, e^{-E_n t}$$

use E_{N+1} in upper bound

See also:
A. Gerardin et al, PRD 2019



[A. Meyer @ Lattice 2019]



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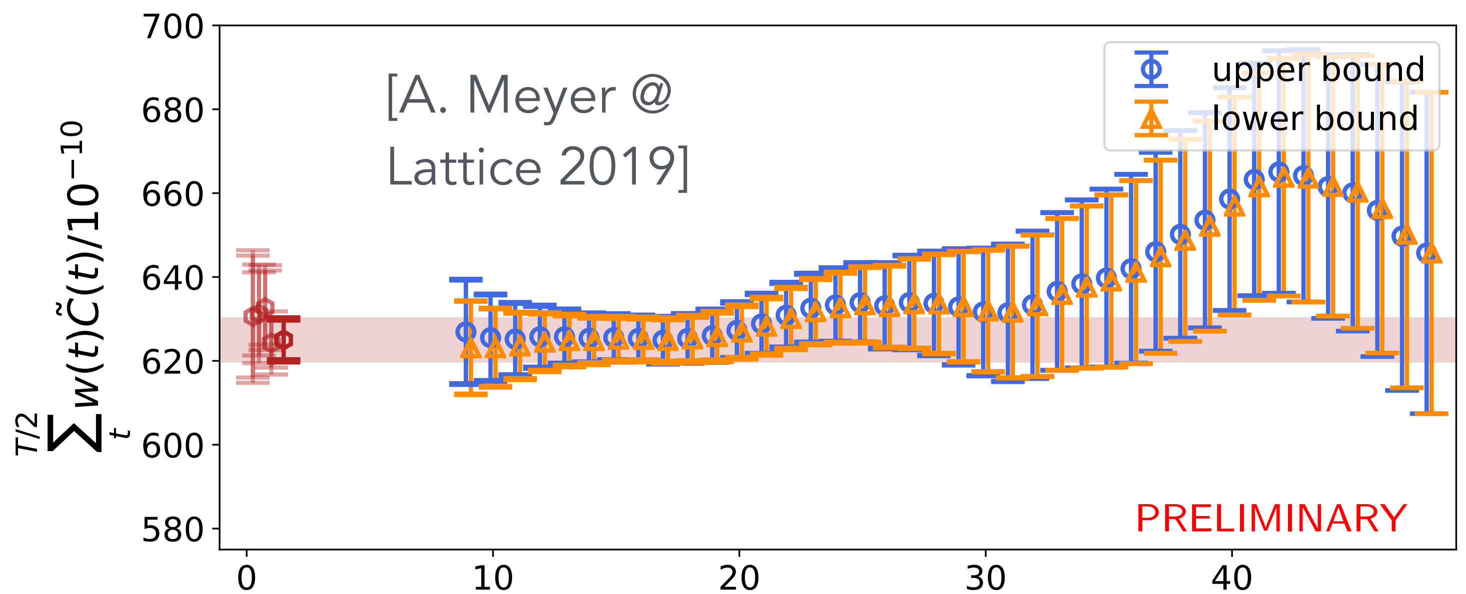


with N = 4

See also:
A. Gerardin et al, PRD 2019

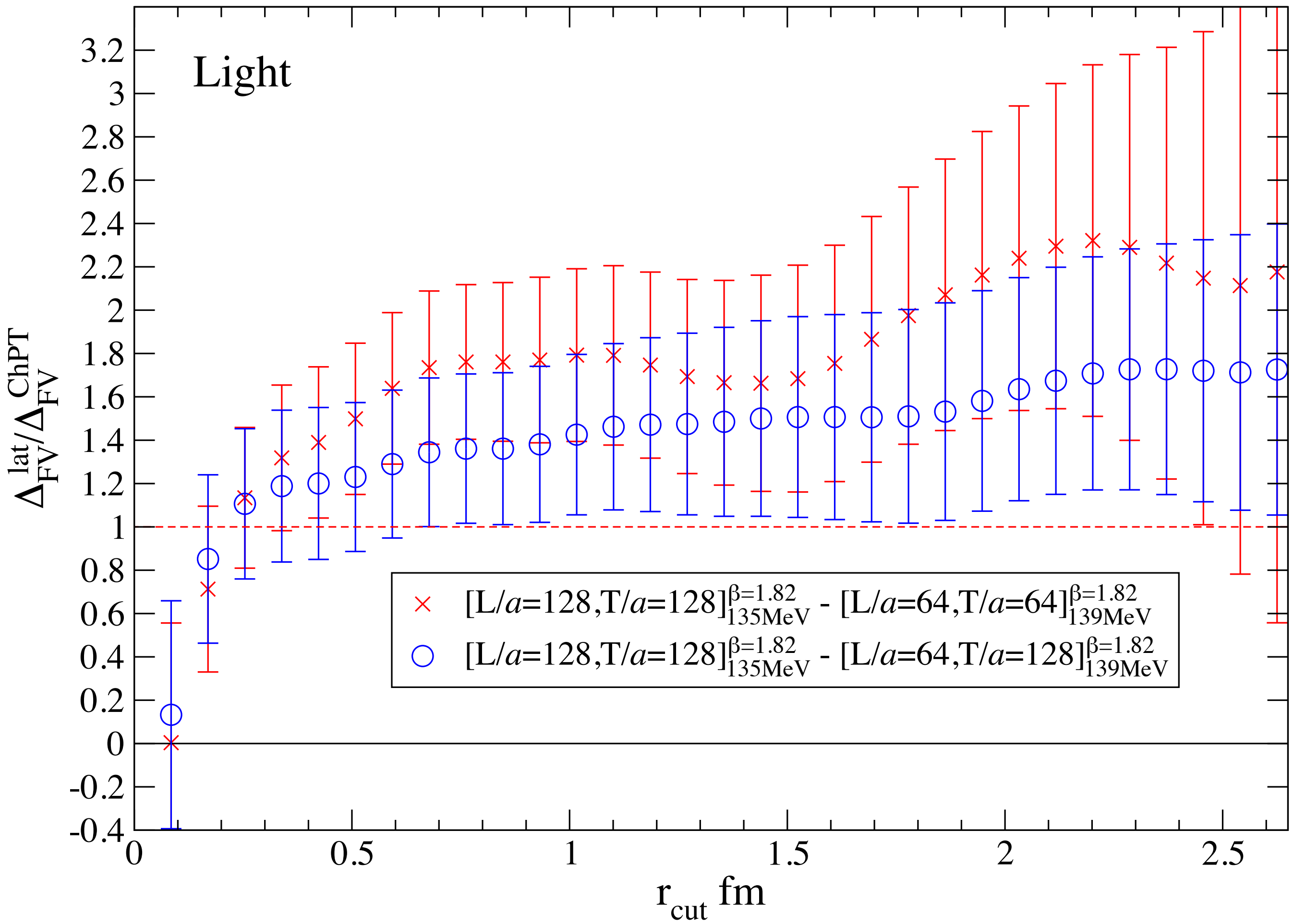
Finite Volume (FV) Corrections

- Finite Volume affects long-distance physics, driven by lightest states in the system: two-pion states (again)
- expected size (based on NLO ChPT) ~2-3% on typical lattice volumes
- hard to calculate precisely by brute force:

FV corrections appear to be larger than expected by NLO ChPT, but errors are large.

See also:
A. Gerardin et al, PRD 2019,
D. Giusti et al, PRD 2018,
Della Morte et al, JHEP 2017 ,…

[Shintani & Kuramashi, 2019 PRD]



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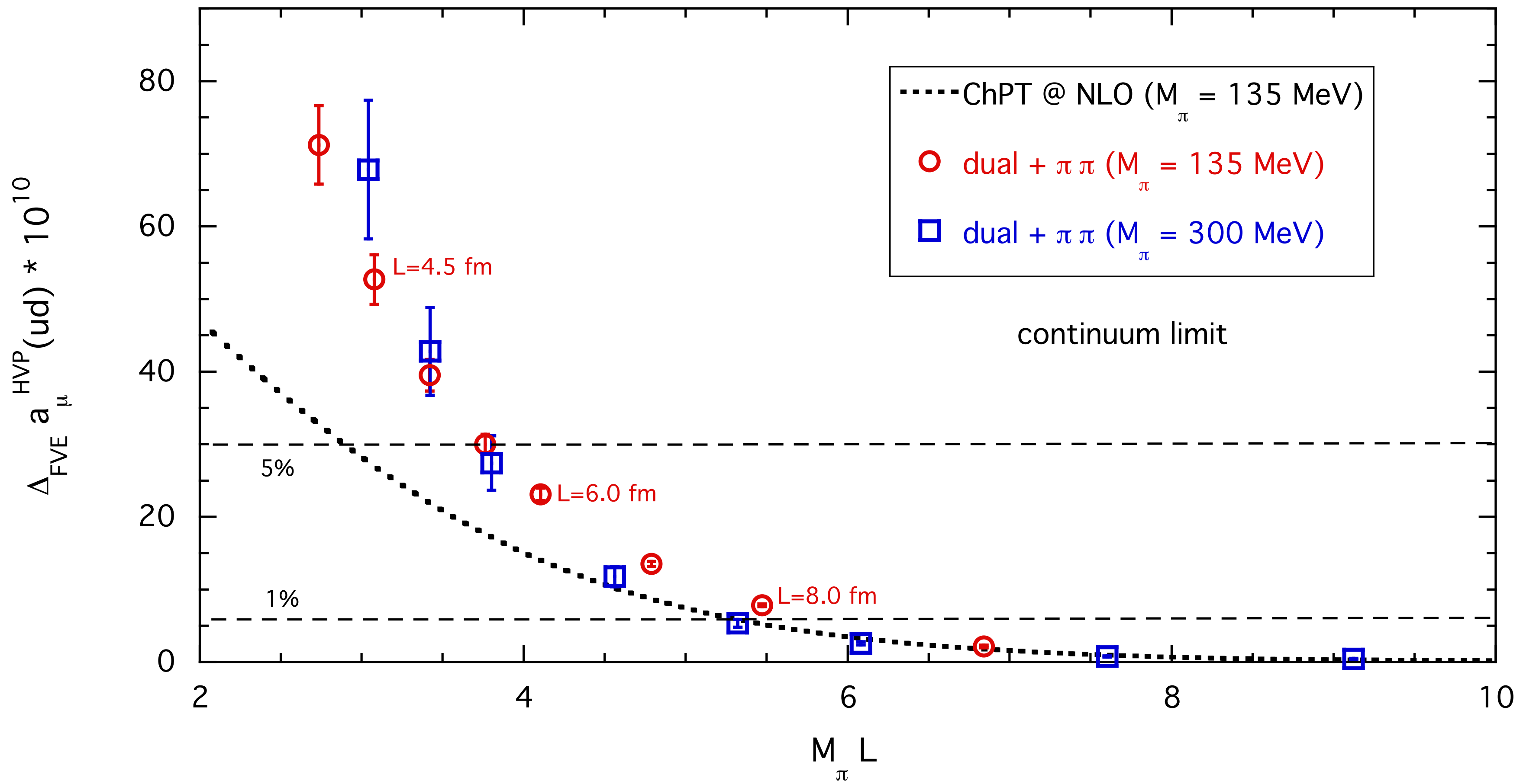
- use theory guidance:
  include resonant two-pion states [D. Giusti et al, PRD 2018]



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- use theory guidance:
  include resonant two-pion states [D. Giusti et al, PRD 2018], ChPT (NLO + NNLO) [Bijnens & Relefors, JHEP 2017, C. Aubin et al, arXiv:1905.09307, …], Gounaris-Sakurai parameterization of timelike form factor [H. Meyer, 2011 PRL, …], modified chiral theory which includes $\rho - \gamma - \pi\pi$ interactions [Chakraborty et al, 1601.03071], Hamiltonian approach [Hansen & Patella, arXiv:1904.10010], …
  together with spectral reconstruction (if possible) [A. Gerardin et al, PRD 2019, Lehner @ Lattice 2019,…]
- staggered fermions:
  taste-breaking effects ➠ pion mass splittings (at finite lattice spacing)
  ➠ affect FV corrections

Scale Setting

- a_μ is dimensionless, but depends on the lattice indirectly, through masses in lattice units in the Kernel. In particular, am_μ:

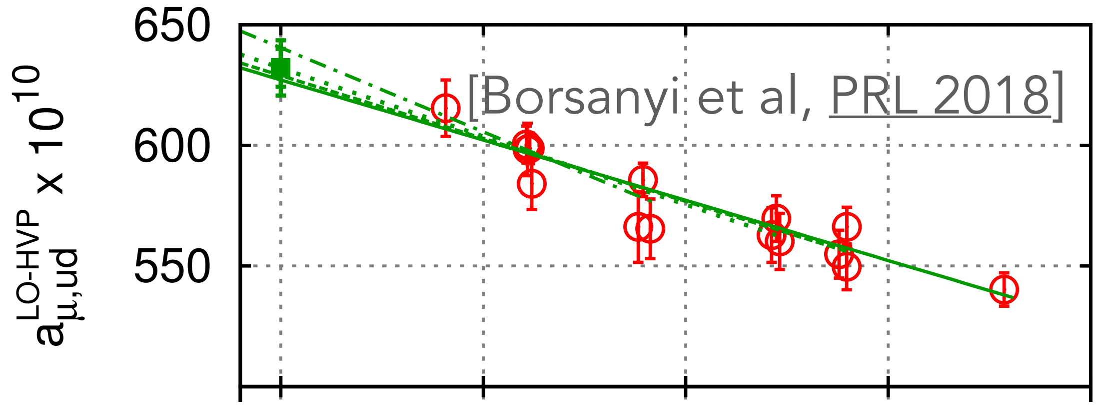
$$\frac{\delta a_\mu^{\rm hvp}}{a_\mu^{\rm hvp}} = \underbrace{\frac{1}{a_\mu^{\rm hvp}} \left| a \frac{da_\mu^{\rm hvp}}{da} \right|}_{\approx\, 1.8} \frac{\delta a}{a}$$

[H. Wittig @ 1st Muon g-2 Theory Initiative workshop]

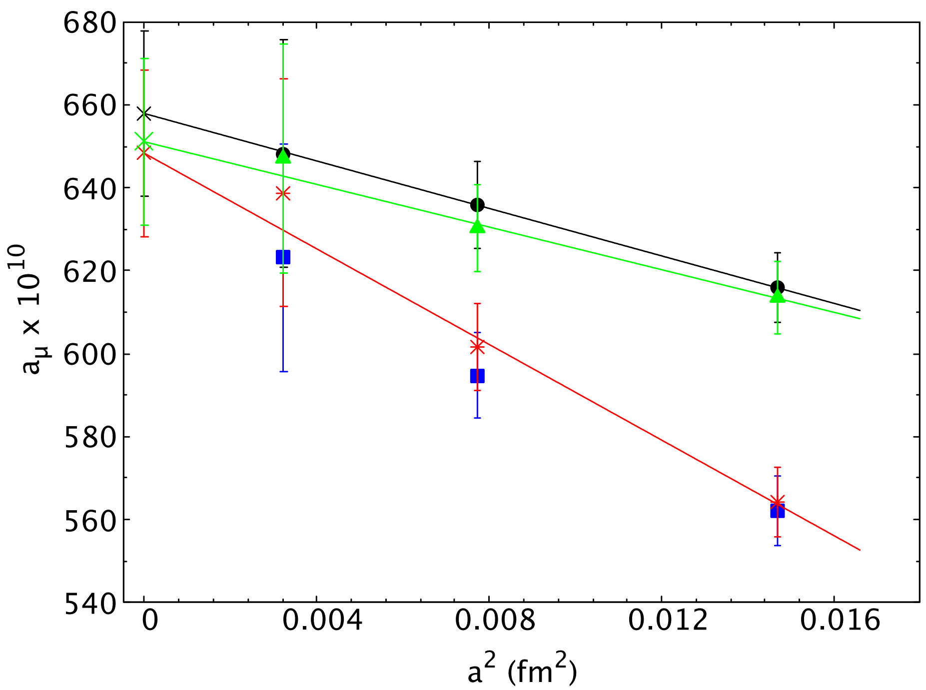
- need a good physical quantity to determine lattice spacing to high precision (< 0.2%). Currently in use:
  - f_π — depends on V_{ud} and requires radiative QED corrections
  - Ω baryon mass (RBC/UKQCD)

Continuum extrapolation

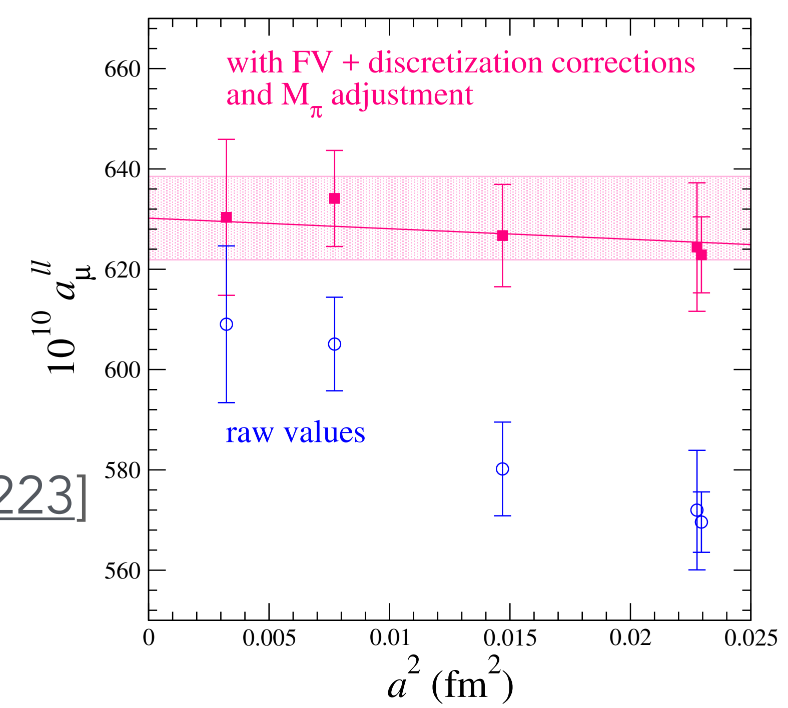
- ….performed by every lattice group.
- Having more than 3 lattice spacings is desirable.
- Observed dependence depends on the details of the actions and current used, and on what corrections are added before extrapolation.



C. Aubin et al, arXiv:1905.09307



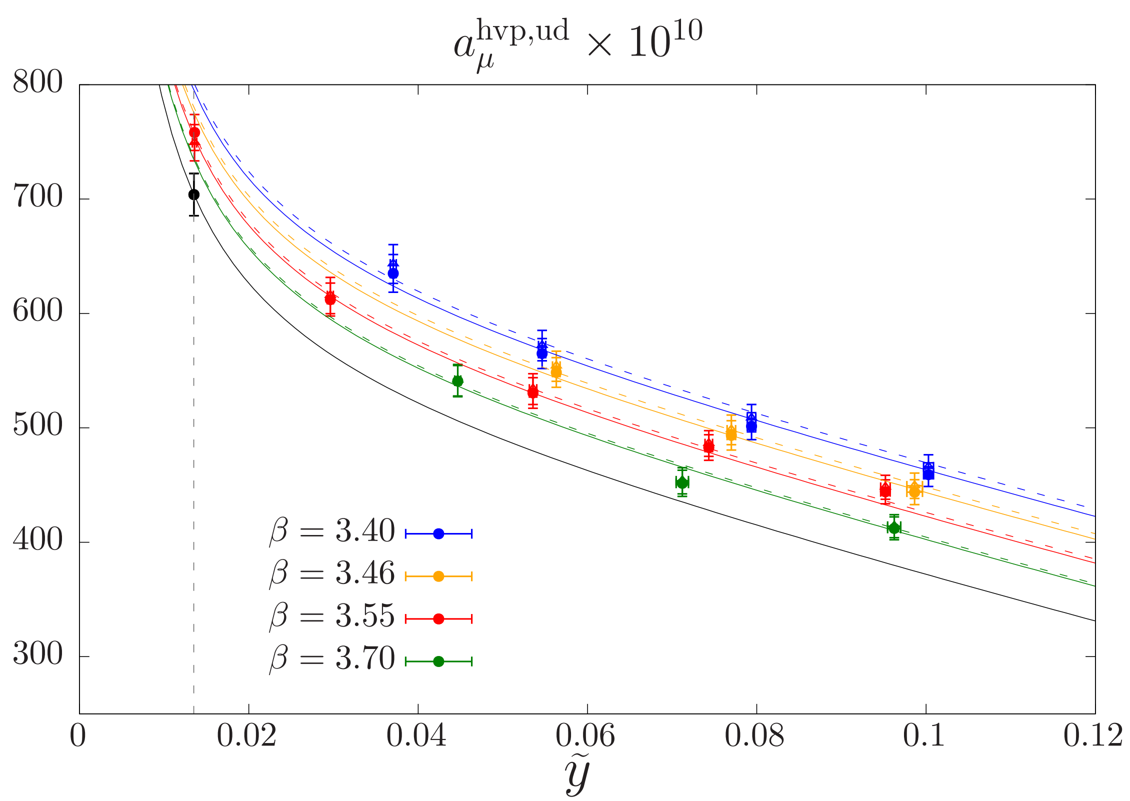
[Davies et al, arXiv:1902.04223]



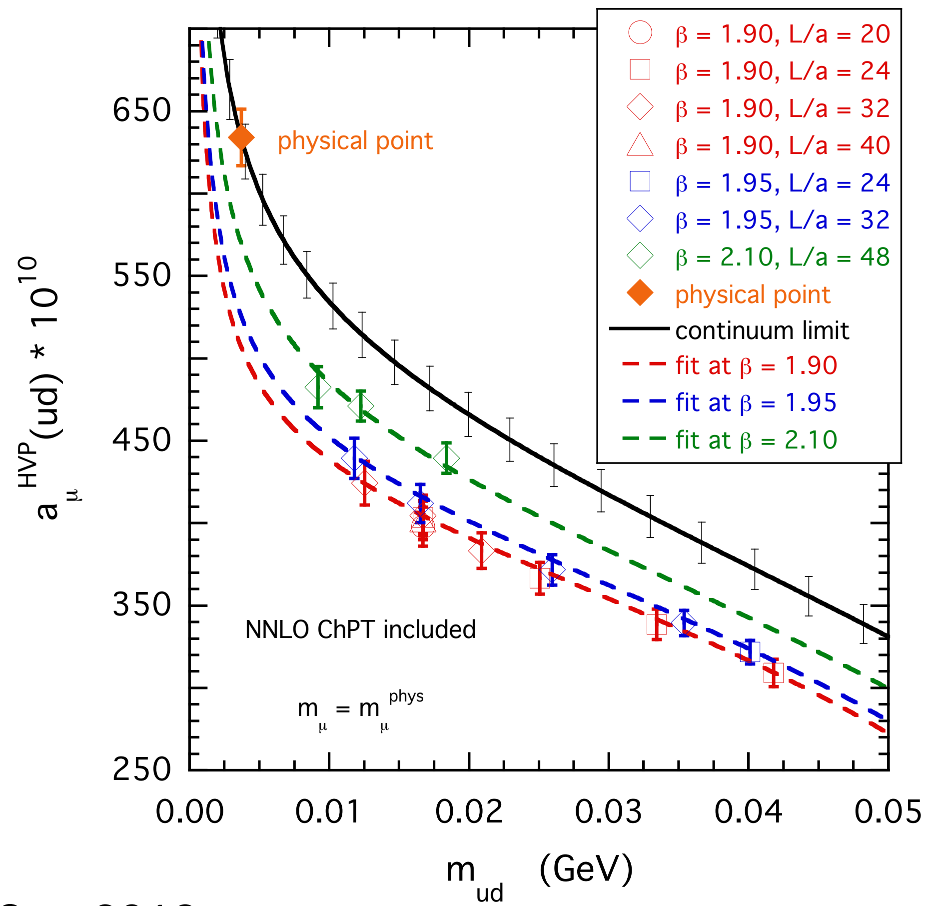
Combined continuum and chiral extrapolation

- if not using only physical mass ensembles.
- Having more than 3 lattice spacings is desirable.
- Observed dependence depends on the details of the actions and current used, and, on what corrections are added before extrapolation.

A. Gerardin et al, PRD 2019,



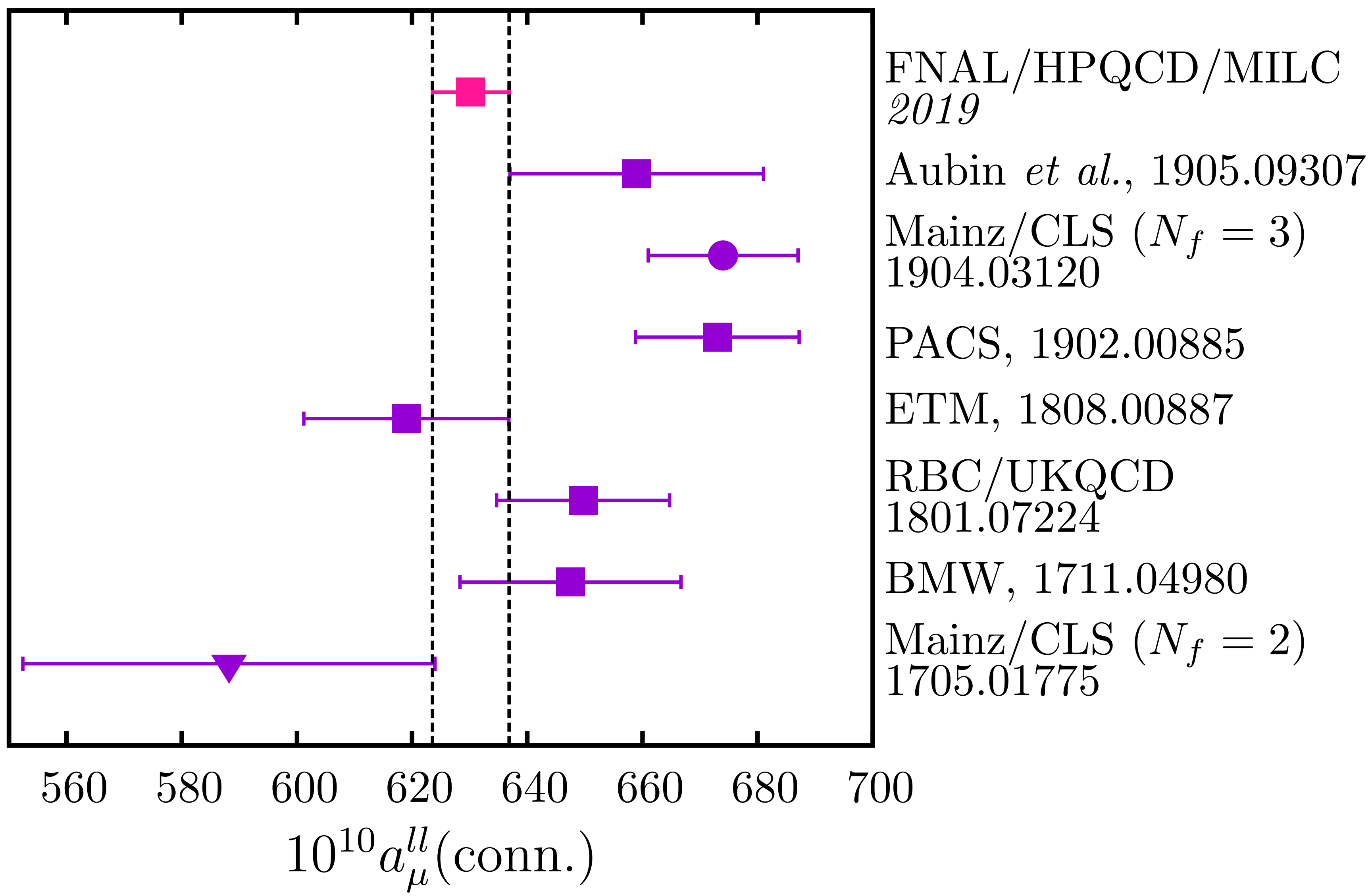
D. Giusti et al, PRD 2018,



Light-quark connected a_μ: Comparison

at $m_u = m_d$ and $m_{\pi^0} \simeq 135\,\mathrm{MeV}$

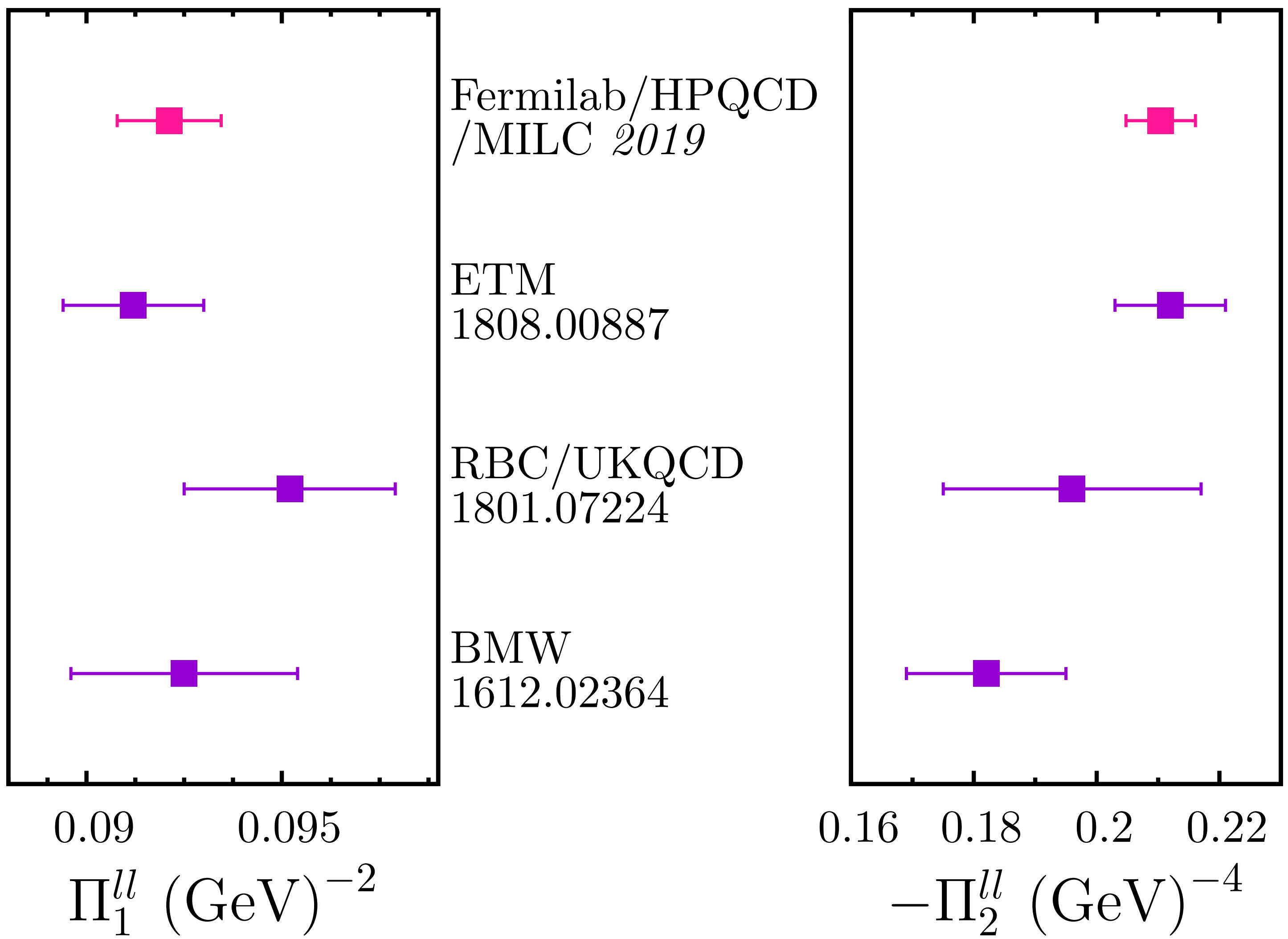
[Davies et al, arXiv:1902.04223]



Light-quark connected Π_1, Π_2: Comparison

at $m_u = m_d$ and $m_{\pi^0} \simeq 135\,\mathrm{MeV}$

[Davies et al, arXiv:1902.04223]



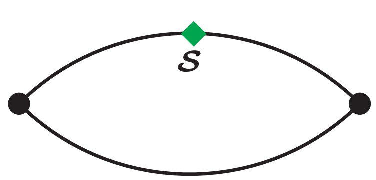
QED + Strong IB corrections

- need to be considered together, since QED effects affect mass splittings, and QED (α) and SIB $(m_d - m_u)/\Lambda$ effects are similar in size
- start with QCD only + isospin $(m_u = m_d)$ with $m_{\pi^0} \simeq 135\,\mathrm{MeV}$
- can obtain strong IB corrections from
  - looking at the difference between $m_d - m_u \neq 0$ and $m_u = m_d$
    [Chakraborty et al, 2018 PRL]
  - perturbative expansion:

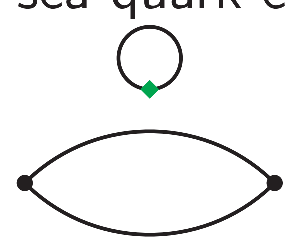
V. Gülpers @ Lattice 2019

- perturbative expansion in $\mathbf{\Delta m = (m_u - m_d)}$
  [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle \mathbf{O} \rangle_{\mathbf{m_u \neq m_d}} = \langle \mathbf{O} \rangle_{\mathbf{m_u = m_d}} + \mathbf{\Delta m} \frac{\partial}{\partial \mathbf{m}} \langle \mathbf{O} \rangle \Big|_{\mathbf{m_u = m_d}} + \mathcal{O}\left(\mathbf{\Delta m^2}\right)$$



sea quark effects:



- ETMC [D. Giusti *et al*, arXiv:1901.10462]
  $\delta a_\mu = \mathbf{6.0(2.3) \times 10^{-10}}$
- RBC/UKQCD [T. Blum, VG *et al*, Phys.Rev.Lett. 121 (2018) no.2, 022003]
  $\delta a_\mu = \mathbf{10.6(4.3)_S \times 10^{-10}}$
  + work in progress
  [C. Lehner, Mon 14:20]

QED + Strong IB corrections

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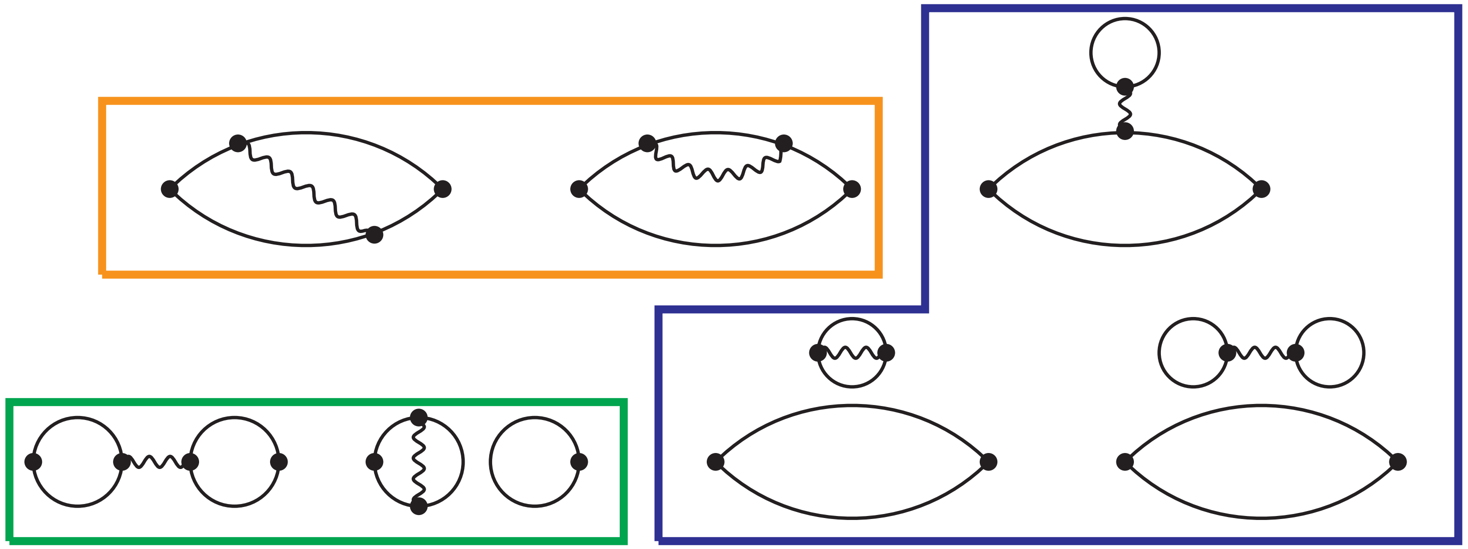
- perturbative expansion of the path integral in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]



quark-connected

quark-disconnected

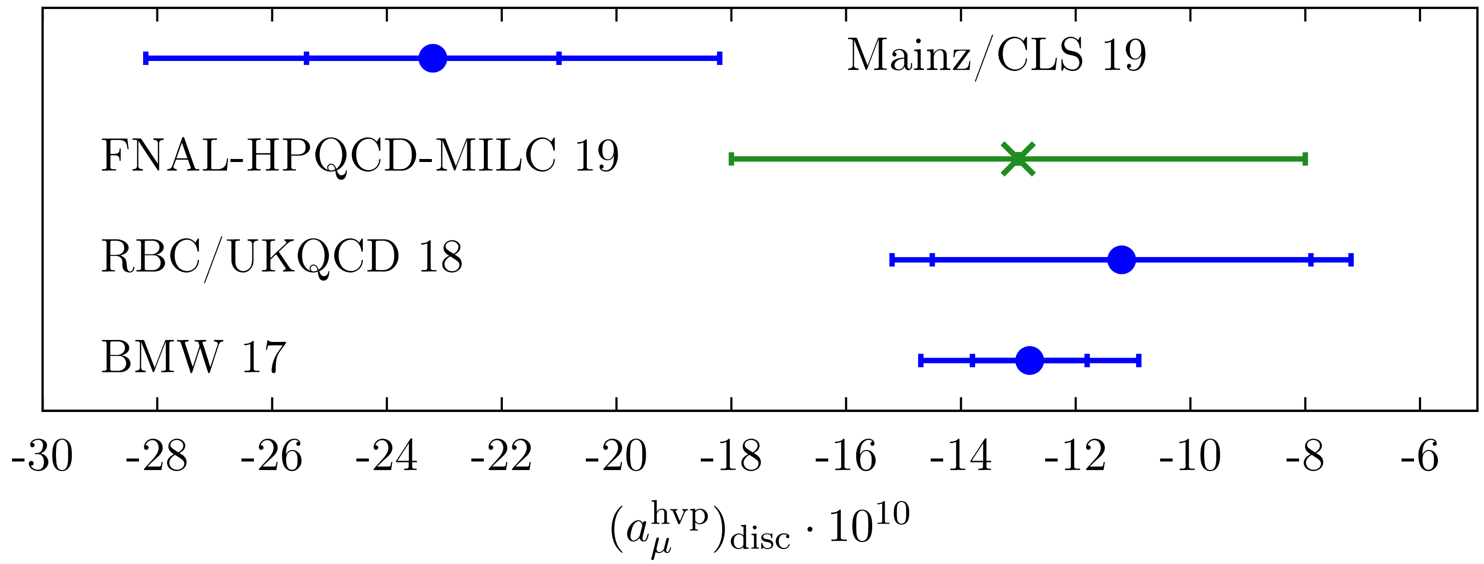
sea-quark effects

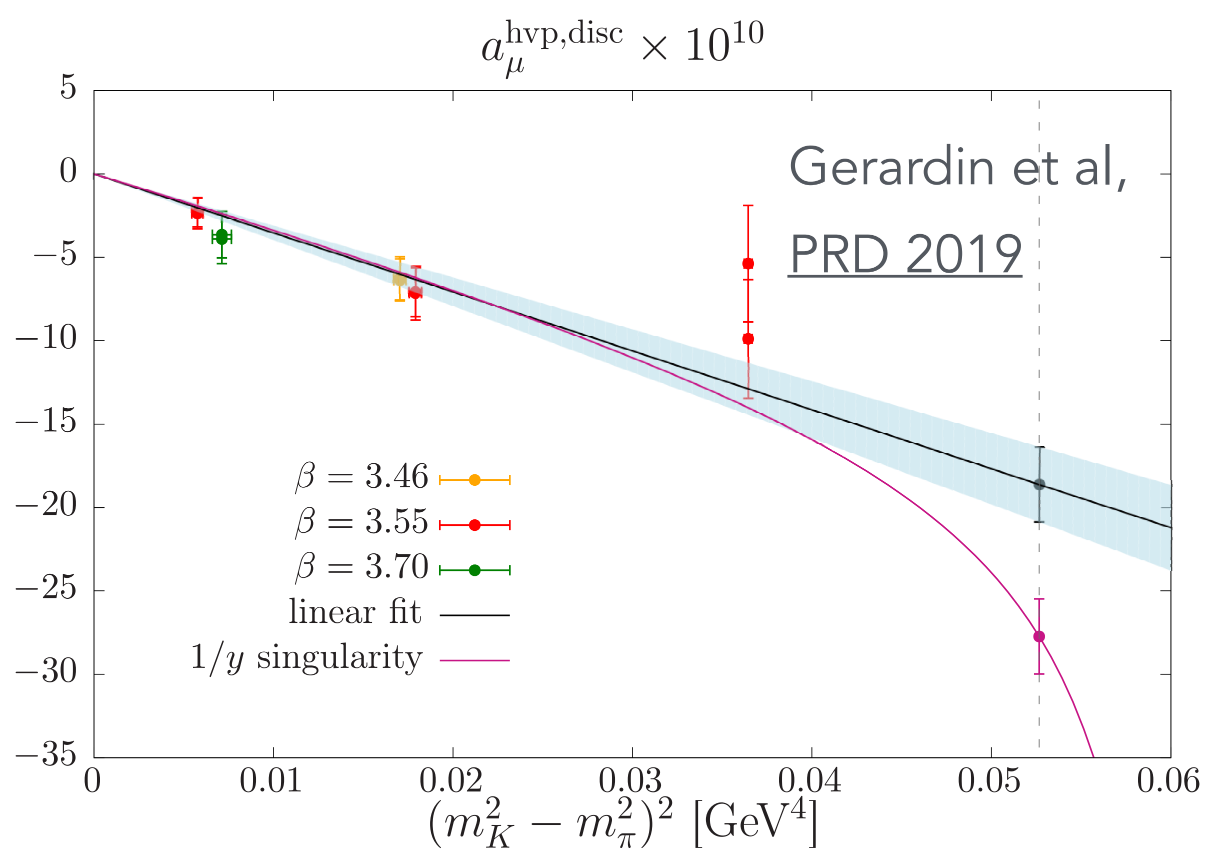
- Finite Volume corrections for QED on the lattice
  $\rightarrow$ $\mathbf{1/(m_\pi L)^3}$ for QED corrections to HVP in $\mathrm{QED_L}$ [N. Hermansson Truedsson, Mon 16:50]
  [J. Bijnens *et al*, arXiv:1903.10591], [D.Giusti *et al*, JHEP 1710 (2017) 157]
  $\rightarrow$ negligible for required precision

- work in progress by RBC/UKQCD, ETM, BMW, Mainz, Fermilab-HPQCD-MILC

Disconnected Contribution, $a_{\mu,\text{disc}}^{\text{HLO}}$

[prepared by K. Miura for WP]



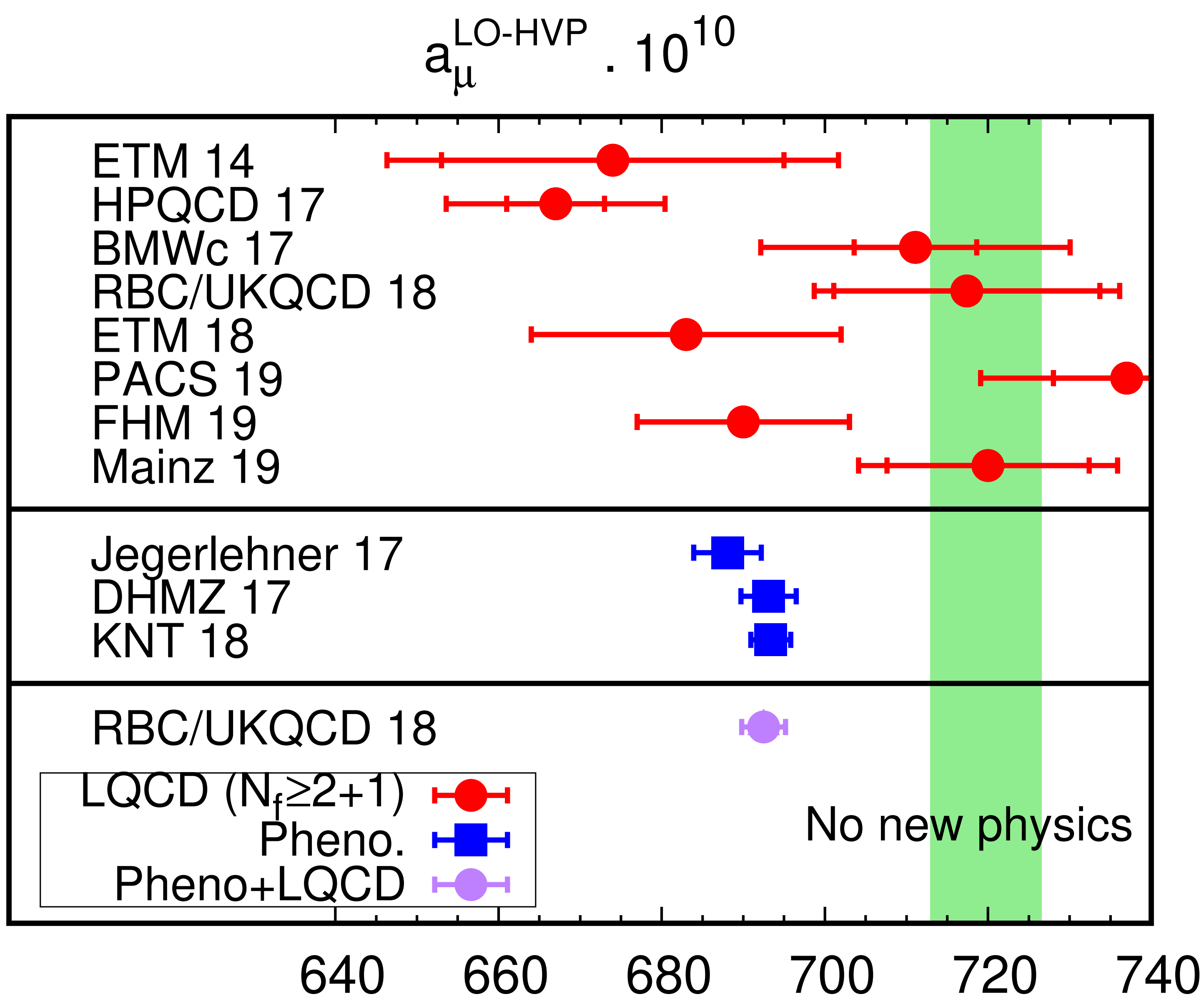


Mainz lattice data at unphysical mass are consistent with BMW and RBC/UKQCD results.

Fermilab/HPQCD/MILC work in progress

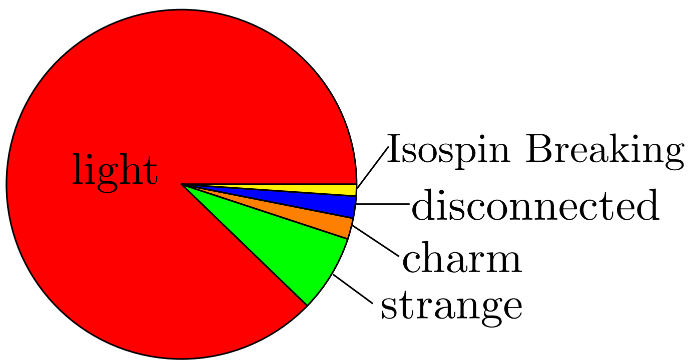
Complete $a_\mu^{\mathrm{HVP,LO}}$: Comparison

[prepared by K. Miura for WP]

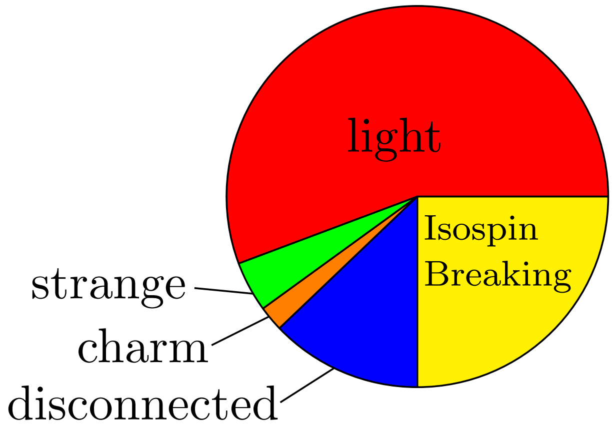


[V. Gülpers, plenary talk @ Lattice 2019]

contribution to $\mathbf{a}_\mu^{\mathrm{hvp}}$



contribution to $\mathbf{\Delta a}_\mu^{\mathrm{hvp}} \approx \mathbf{2.5\%}$



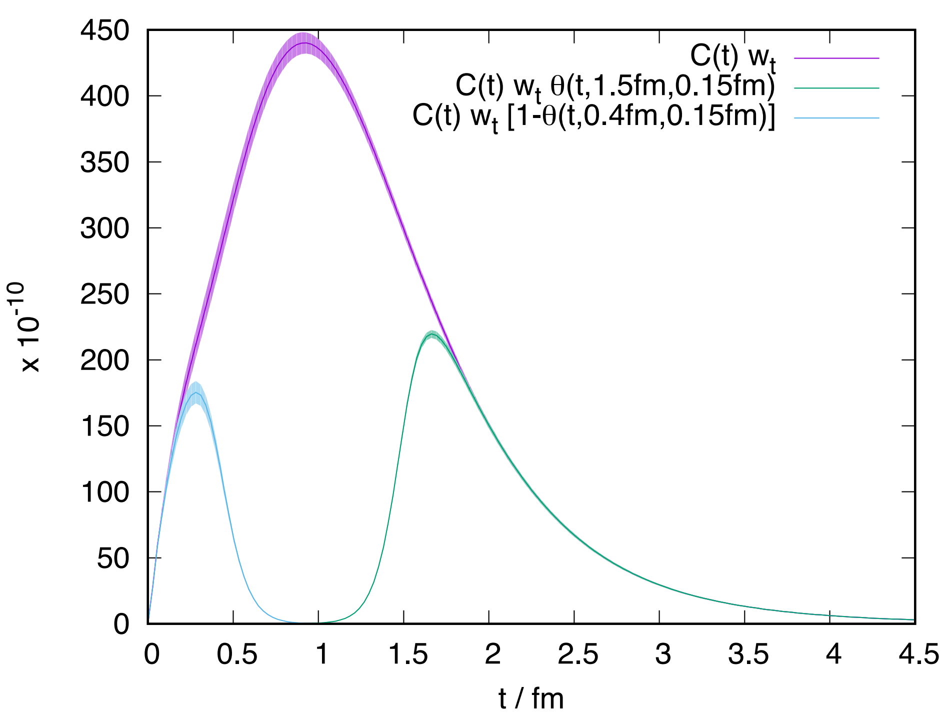
Another Hybrid Method: Windows

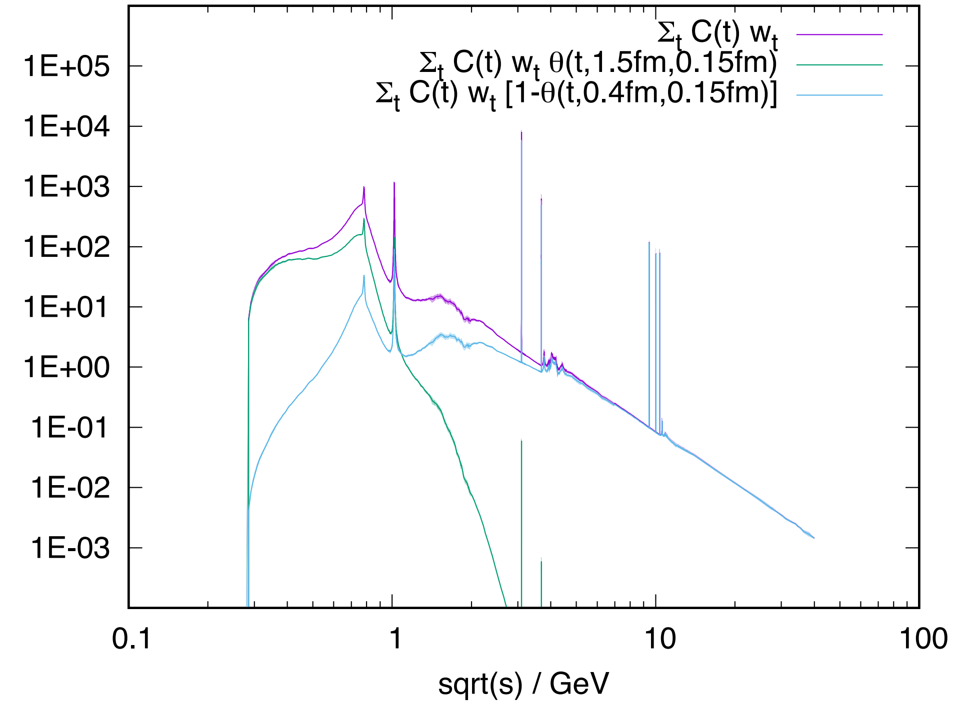
Hybrid method: combine LQCD with R-ratio data

C. Lehner @ HVP KEK 2018 (from T. Blum et al, arXiv:1801.07224)

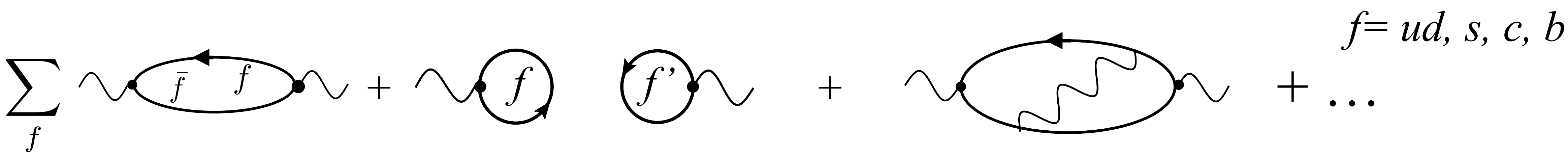
Direct LQCD calculations of HVP are still less precise than dispersive methods. But comparisons between R-ratio and lattice data are already useful.

- Convert R-ratio data to Euclidean correlation function (via the dispersive integral).
- Compare lattice/R-ratio data (after adding all the corrections and extrapolating to continuum, infinite volume).
- Use R-ratio data where LQCD errors are large and vice versa.



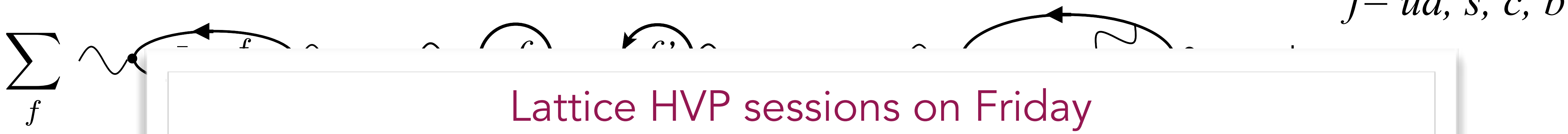


Summary and Outlook



- ☆ light-quark contribution to $a_\mu^{\rm HLO}$ is the biggest source of uncertainty in lattice QCD calculations.
  progress in the last few years ➠ moving towards 1% uncertainty
- ☆ advanced methods (spectral reconstruction) for controlling long-distance noise, better understanding of FV effects
  challenge: check consistency between different methods
- ☆ results for subleading corrections (disconnected, SIB, QED) now from more than one group, more are in progress
  ➠ still need to improve precision
- ☆ Looking forward to the detailed discussions to map out how to add comparisons, improve precision

Summary and Outlook



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Lattice HVP sessions on Friday

Status/update talks:

Davide Giusti — ETMC
Antoine Gerardin — Mainz:group
Laurent Lellouch — BMWc
Christoph Lehner — RBC/UKQCD
Steve Gottlieb — FNAL/HPQCD/MILC
Tom Blum — Aubin et al

Connections:

Marina Marinkovic — Lattice QCD for MUonE (Tuesday)
Nils Hermandsson-Truedsson — FV effects QED corrections
Mattia Bruno — Tau/Isospin-breaking corrections
Marco Cé — HVP contribution to the running of α and $\sin^2\theta$

Thank you!

Farah Willenbrock