Hadronic Light-by-Light RBC

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Sept 11, 2019

Hadronic contributions to $(g-2)_\mu$, INT Workshop INT-19-74W

Physics-Astronomy Auditorium, University of Washington

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Outline

- **Introduction**

HLbL: QED$_L$ approach

- Connected diagrams:
  Exact photon propagator
  The moment method
- Disconnected diagrams
- Results

HLbL: QED$_\infty$ approach

- QED kernel and subtraction
- QCD @ $m_\pi = 340$ MeV
  (cross check with Mainz)
- QCD @ $m_\pi = 142$ MeV
- Long distance π^0-pole
- Subleading contributions

- Summary and Outlook

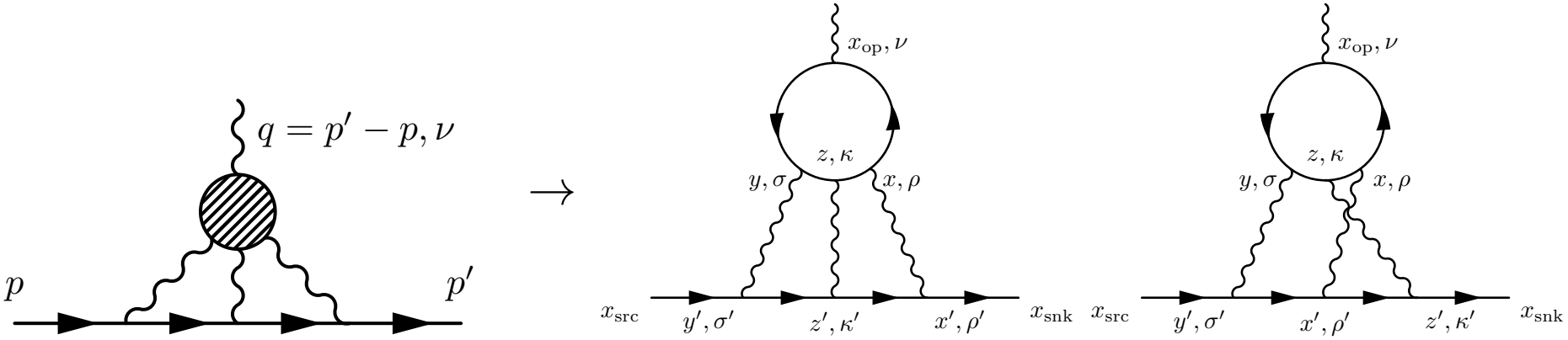
Muon $g-2$: Fermilab E989, J-PARC E34

SM (Model HLbL)	11659181.3 ± 4.0
BNL E821 Exp	11659208.9 ± 6.3
Diff (Exp − SM)	27.6 ± 7.5

3.7σ deviations

New Physics?

HLbL: diagrams



- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional four different permutations of photons not shown.
- The photons can be connected to different quark loops. These are referred to as the disconnected diagrams. They will be discussed later.

Contribution	PdRV(09)	N/JN(09)	J(17)	White Paper
π^0, η, η'-poles	114 ± 13	99 ± 16	95.45 ± 12.40	93.8 ± 4.0
π, K-loop/box	-19 ± 19	-19 ± 13	-20 ± 5	-16.4 ± 0.2
S-wave $\pi\pi$	–	–	–	-8 ± 1
scalars	-7 ± 7	-7 ± 2	-5.98 ± 1.20	$\}\ -2 \pm 3$
tensors	–	–	1.1 ± 0.1	
axials	15 ± 10	22 ± 5	7.55 ± 2.71	8 ± 8
q-loops / SD	2.3	21 ± 3	22.3 ± 5.0	10 ± 10
total	105 ± 26	116 ± 39	100.4 ± 28.2	$85 \pm XX$

HLbL in units of 10^{-11}.

PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N = Nyffeler; J = Jegerlehner

- We will use the unit in 10^{-10} in the rest of the talk.
- The total HLbL contribution is on the order of 10×10^{-10}.

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Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

Thomas Blum,[1,2] Saumitra Chowdhury,[1] Masashi Hayakawa,[3,4] and Taku Izubuchi[5,2]
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(Received 18 July 2014; published 7 January 2015)

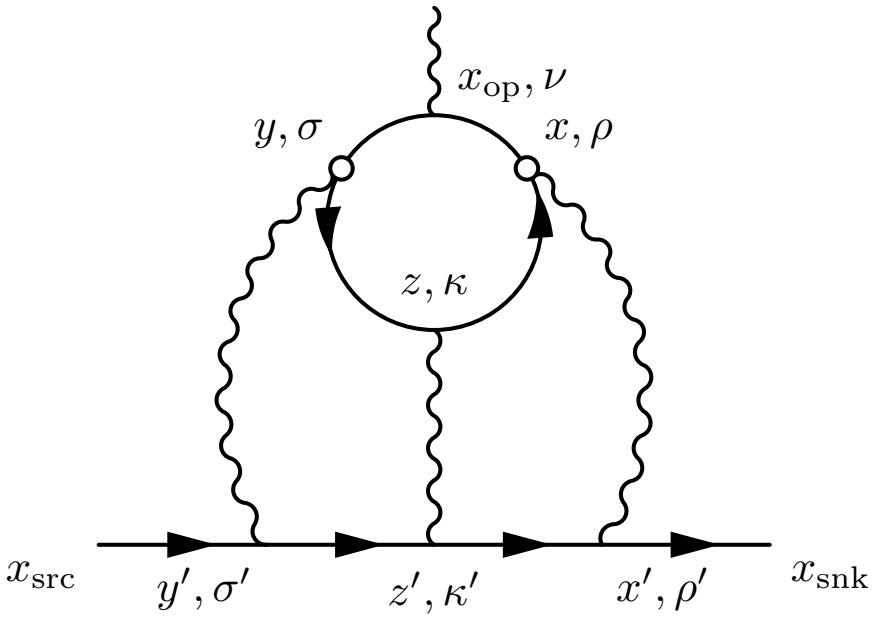
The most compelling possibility for a new law of nature beyond the four fundamental forces comprising the standard model of high-energy physics is the discrepancy between measurements and calculations of the muon anomalous magnetic moment. Until now a key part of the calculation, the hadronic light-by-light contribution, has only been accessible from models of QCD, the quantum description of the strong force, whose accuracy at the required level may be questioned. A first principles calculation with systematically improvable errors is needed, along with the upcoming experiments, to decisively settle the matter. For the first time, the form factor that yields the light-by-light scattering contribution to the muon anomalous magnetic moment is computed in such a framework, lattice QCD + QED and QED. A nonperturbative treatment of QED is used and checked against perturbation theory. The hadronic contribution is calculated for unphysical quark and muon masses, and only the diagram with a single quark loop is computed for which statistically significant signals are obtained. Initial results are promising, and the prospect for a complete calculation with physical masses and controlled errors is discussed.

DOI: 10.1103/PhysRevLett.114.012001 PACS numbers: 12.38.Gc, 12.20.-m, 12.38.-t, 13.40.Em

Introduction.—The muon anomalous magnetic moment, or anomaly $a_\mu = (g_\mu - 2)/2$, provides one of the most stringent tests of the standard model because it has been measured to great accuracy (0.54 ppm) [1] and calculated to

scattering (HLbL) contribution (Fig. 1), $a_\mu(\mathrm{HLbL})$, from experimental data and a dispersion relation [9,10]. So far, only model calculations have been done. The uncertainty quoted in Table I was estimated by the "Glasgow con-

Exact photon and the moment method



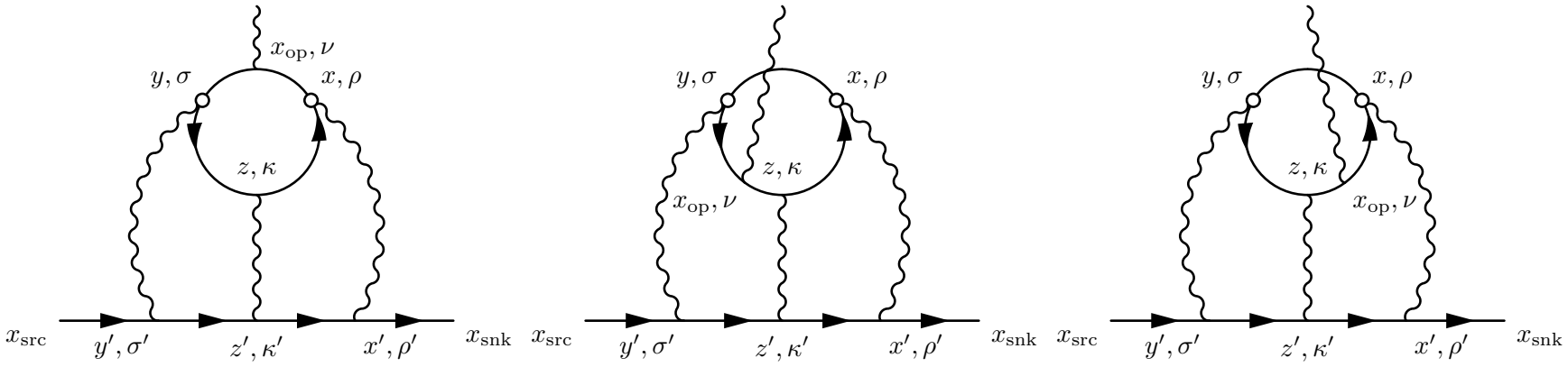
- Two point sources at x, y: randomly sample x and y.
- Importance sampling: focus on small $|x - y|$.
- Complete sampling for $|x - y| \leq 5a$ upto discrete symmetry.

$$\frac{a_\mu}{m_\mu} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_{r=x-y} \sum_{z} \sum_{x_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \bar{u}_{s'}(\vec{0}) i \vec{\mathcal{F}}^C(\vec{0}; x, y, z, x_{\rm op}) u_s(\vec{0})$$

$$\vec{\mu} = \sum_{\vec{x}_{\rm op}} \frac{1}{2} (\vec{x}_{\rm op} - \vec{x}_{\rm ref}) \times \vec{J}(\vec{x}_{\rm op})$$

- Muon is plane wave, $x_{\rm ref} = (x + y)/2$.
- Sum over time component for $x_{\rm op}$.
- Only sum over $r = x - y$.

Reorder the summation



- The three internal vertex attached to the quark loop are equivalent (all permutations are included).
- We can pick the closer two points as the point sources x, y.

$$\sum_{x,y,z} \to \sum_{x,y,z} \begin{cases} 3 & \text{if } |x-y| < |x-z| \text{ and } |x-y| < |y-z| \\ 3/2 & \text{if } |x-y| = |x-z| < |y-z| \\ 3/2 & \text{if } |x-y| = |y-z| < |x-z| \\ 1 & \text{if } |x-y| = |y-z| = |x-z| \\ 0 & \text{others} \end{cases}$$

PHYSICAL REVIEW D **93,** 014503 (2016)

Lattice calculation of hadronic light-by-light contribution to the muon anomalous magnetic moment

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(Received 18 November 2015; published 12 January 2016)

The quark-connected part of the hadronic light-by-light scattering contribution to the muon's anomalous magnetic moment is computed using lattice QCD with chiral fermions. We report several significant algorithmic improvements and demonstrate their effectiveness through specific calculations which show a reduction in statistical errors by more than an order of magnitude. The most realistic of these calculations is performed with a near-physical 171 MeV pion mass on a $(4.6\ \mathrm{fm})^3$ spatial volume using the $32^3 \times 64$ Iwasaki + DSDR gauge ensemble of the RBC/UKQCD Collaboration.

DOI: 10.1103/PhysRevD.93.014503

I. INTRODUCTION

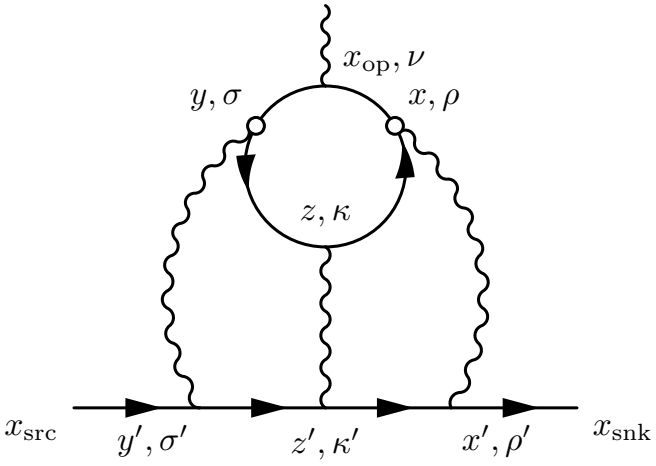
New particles and interactions which occur at a very large energy scale Λ, above the reach of present-day accelerators, may be first discovered through their indirect effects at low energy. A particularly promising low-energy quantity that may reveal such effects is the anomalous moment of the muon. This "anomalous" difference $g_\mu - 2$ between the muon's gyromagnetic ratio g_μ and the Dirac value of 2 for a noninteracting particle can receive contributions from such

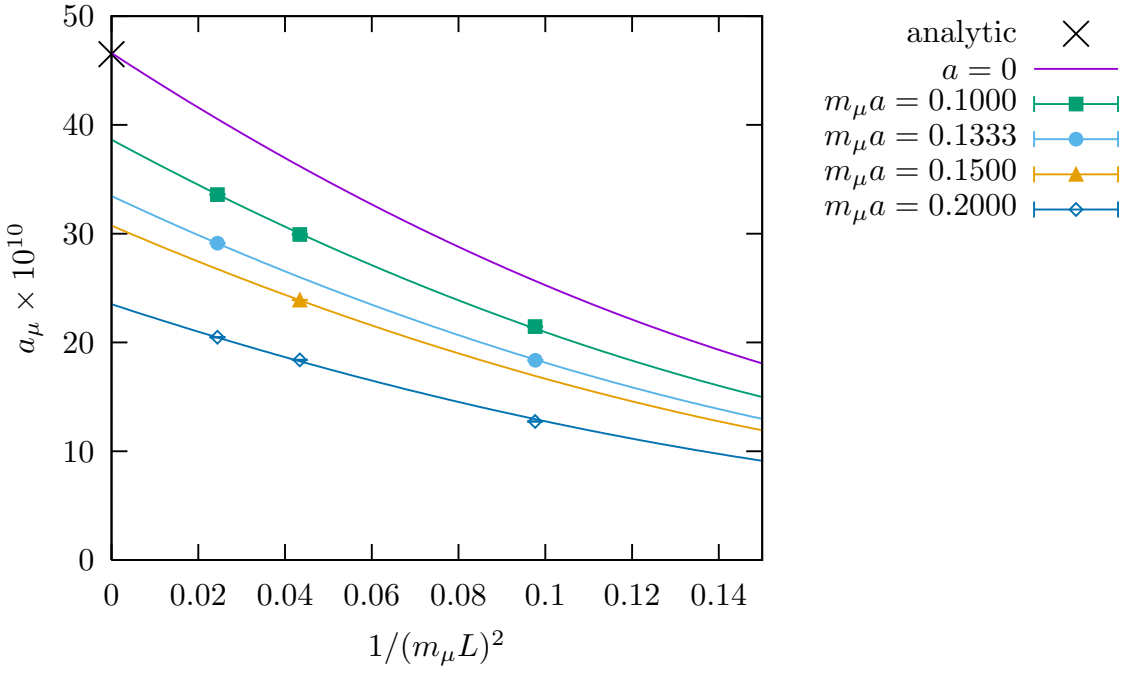
planned at Fermilab (E989) and J-PARC (E34) with a targeted precision as small as 0.14 ppm, and for a reduction in the theoretical errors.

The two components of the theoretical calculation with the largest errors involve couplings to the up, down and strange quarks: the hadronic vacuum polarization (HVP) and hadronic light-by-light scattering (HLbL). These are the first cases in which the effects of the strong interaction enter the determination of $g_\mu - 2$. The HVP effects enter

Muon leptonic LbL

- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.





$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2\, a^2 + c_2'\, a^4) \;\rightarrow\; F_2 = 46.6(2) \times 10^{-10} \quad (19)$$

- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

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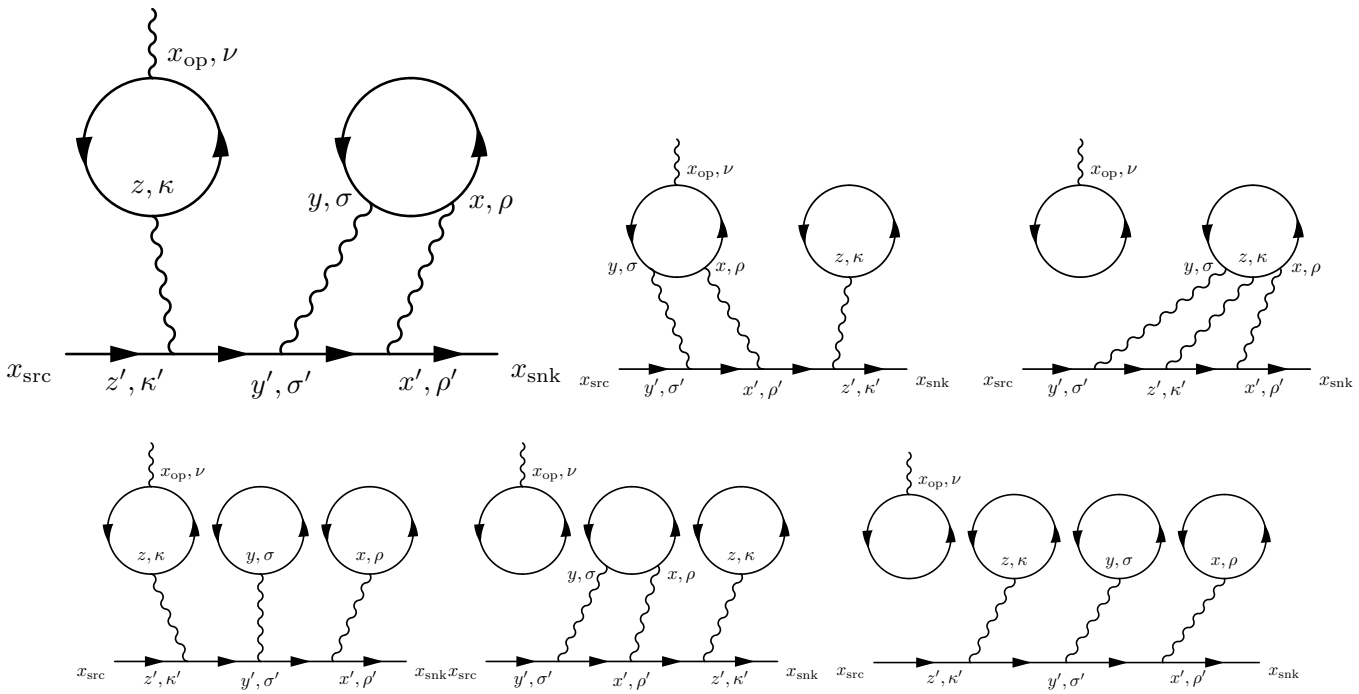
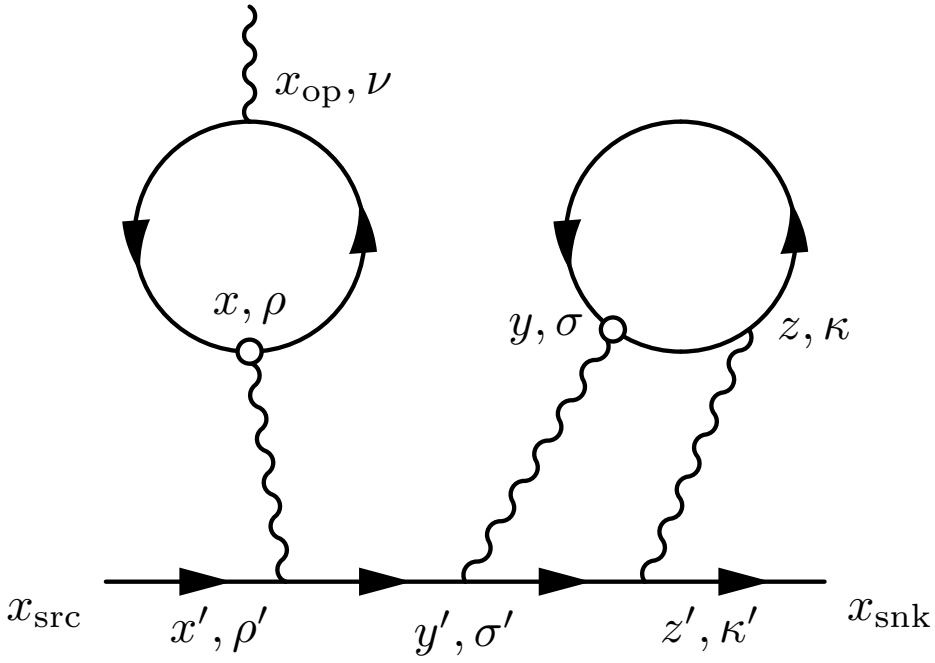
HLbL: disconnected diagrams

- One diagram (the biggest diagram below) do not vanish even in the $\mathrm{SU}(3)$ limit.
- We extend the method and computed this leading disconnected diagram as well.



- Permutations of the three internal photons are not shown.
- **Gluons exchange between and within the quark loops are not drawn.**
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

HLbL: disconnected formula



- Point x is used as the reference point for the moment method.
- We can use two point source photons at x and y, which are chosen randomly. The points x_{op} and z are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = x - y$.

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Connected and Leading Disconnected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment with a Physical Pion Mass

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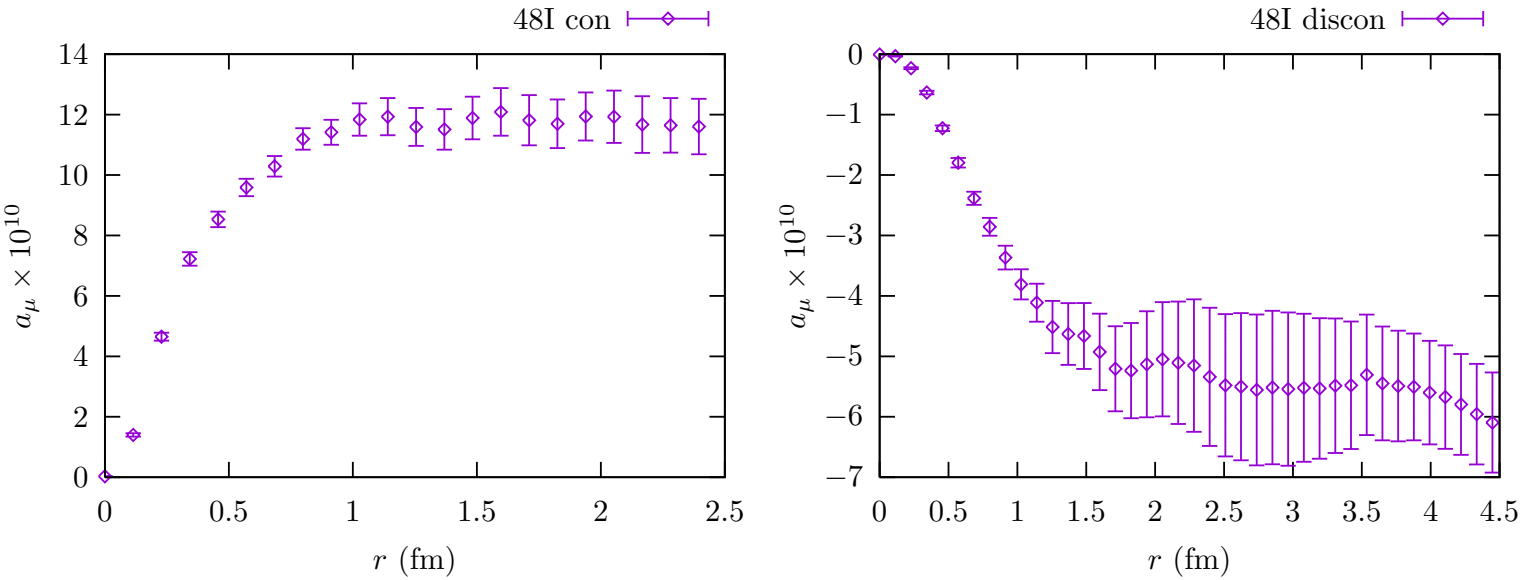
(Received 6 November 2016; published 11 January 2017)

We report a lattice QCD calculation of the hadronic light-by-light contribution to the muon anomalous magnetic moment at a physical pion mass. The calculation includes the connected diagrams and the leading, quark-line-disconnected diagrams. We incorporate algorithmic improvements developed in our previous work. The calculation was performed on the $48^3 \times 96$ ensemble generated with a physical pion mass and a 5.5 fm spatial extent by the RBC and UKQCD Collaborations using the chiral, domain wall fermion formulation. We find $a_\mu^{\mathrm{HLbL}} = 5.35(1.35) \times 10^{-10}$, where the error is statistical only. The finite-volume and finite lattice-spacing errors could be quite large and are the subject of ongoing research. The omitted disconnected graphs, while expected to give a correction of order 10%, also need to be computed.

DOI: 10.1103/PhysRevLett.118.022005

Introduction.—The lattice calculation of the hadronic light-by-light contribution to the muon anomalous magnetic moment is part of the ongoing effort to better understand the approximately 3 standard deviation difference between the extremely accurate BNL E821 experimental result and the current theoretical calculation with similar accuracy constant. However, new, high-energy phenomena that appear at an energy scale Λ can introduce additional structure, leading to new contributions to a_l that are typically suppressed by the ratio $(m_l/\Lambda)^2$, where $l = e$, μ, or τ and m_l is the mass of the corresponding lepton. The muon anomaly may be the best place to search for such

QED$_L$: 48I $m_\pi = 139$ MeV



- Left: **connected diagrams**. Right: leading **disconnected diagrams**.
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.
- We use Lanczos, AMA, and zMobius techniques to speed up the computations.
- 65 configurations are used. They each are separated by 20 MD time units.

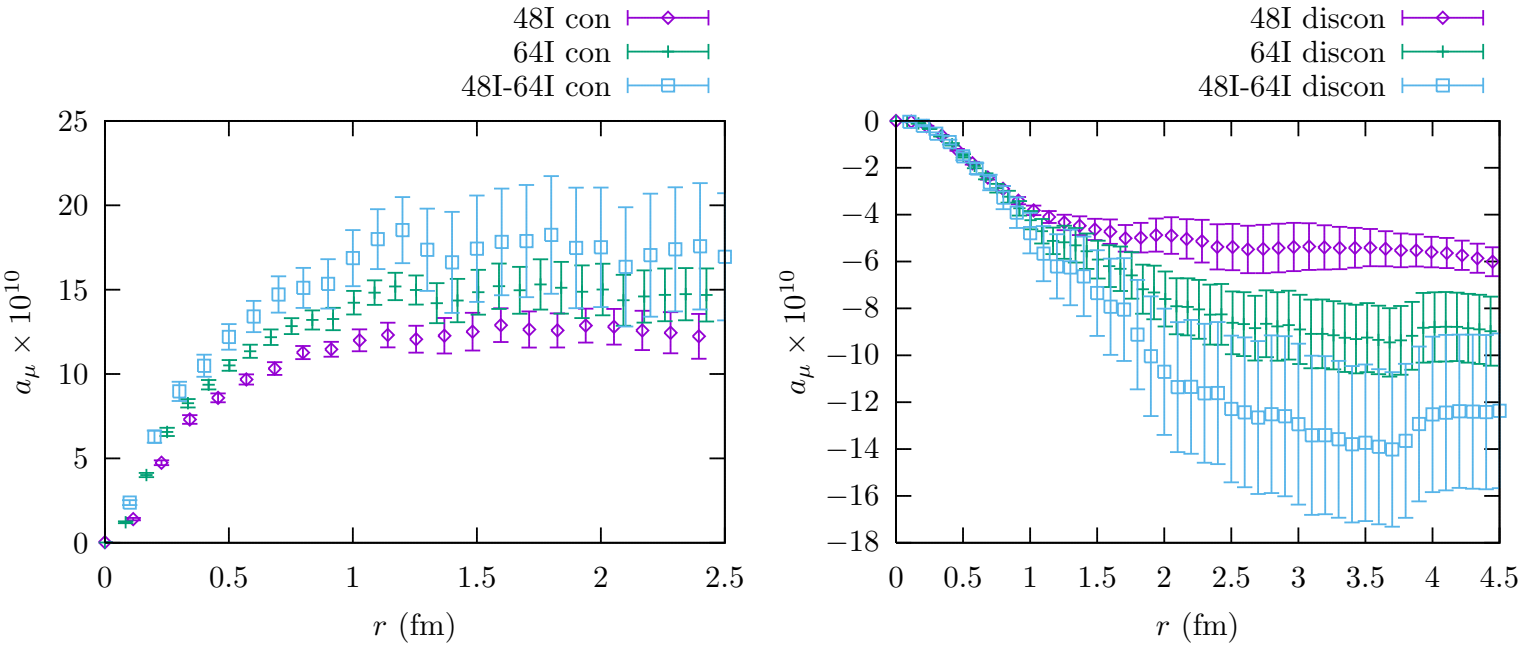
$$a_\mu^{\mathrm{cHLbL}} = (11.60 \pm 0.96) \times 10^{-10} \quad (30)$$

$$a_\mu^{\mathrm{dHLbL}} = (-6.25 \pm 0.80) \times 10^{-10} \quad (31)$$

$$a_\mu^{\mathrm{cHLbL}} = (5.35 \pm 1.35) \times 10^{-10} \quad (32)$$

QED$_L$: 64I $m_\pi = 135$ MeV (prelim)

This is slightly partial quenched calculation performed on the $139\,\mathrm{MeV}$ pion mass ensemble.



- Left: **connected diagrams**. Right: leading **disconnected diagrams**.
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 135\,\mathrm{MeV}$ (corrected from $139\,\mathrm{MeV}$).
- $64^3 \times 128$ lattice, with $a^{-1} = 2.36\,\mathrm{GeV}$, $m_\pi = 135\,\mathrm{MeV}$.

$$a_\mu^{\mathrm{cHLbL}} = (16.94 \pm 3.77) \times 10^{-10} \tag{36}$$

$$a_\mu^{\mathrm{dHLbL}} = (-12.29 \pm 3.34) \times 10^{-10} \tag{37}$$

$$a_\mu^{\mathrm{HLbL}} = (4.66 \pm 4.38) \times 10^{-10} \tag{38}$$

QED$_L$: Hybrid continuum (prelim)

- Very large statistical error (48I: $(6.21 \pm 1.41) \times 10^{-10}$, 48I-64I: $(4.66 \pm 4.38) \times 10^{-10}$).
- Can we do better than this? => Split the a_μ into two parts:

$$a_\mu = a_\mu^{\rm short} + a_\mu^{\rm long} \tag{39}$$

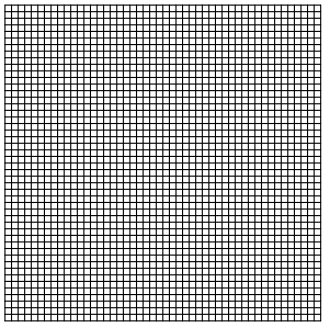
- $a_\mu^{\rm short} = a_\mu(r \leqslant 1\,{\rm fm})$: most of the contribution, small statistical error.
- $a_\mu^{\rm long} = a_\mu(r > 1\,{\rm fm})$: small contribution, large statistical error.

Treat $a_\mu^{\rm short}$ and $a_\mu^{\rm long}$ separately:

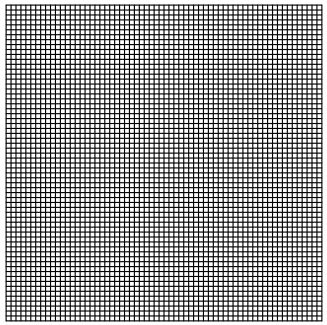
- $a_\mu^{\rm short}$: just like before, continuum extrapolation assuming a^2 scaling.
- $a_\mu^{\rm long}$: simply use the results from 48I, estimate the $\mathcal{O}(a^2)$ error.

$$a_\mu^{\rm cHLbL} = (16.94 \pm 3.77) \times 10^{-10} \Rightarrow (17.35 \pm 1.97_{\rm stat} \pm 0.20_{{\rm sys},a^2}) \times 10^{-10} \tag{40}$$
$$a_\mu^{\rm dHLbL} = (-12.29 \pm 3.34) \times 10^{-10} \Rightarrow (-7.21 \pm 1.03_{\rm stat} \pm 1.02_{{\rm sys},a^2}) \times 10^{-10} \tag{41}$$
$$a_\mu^{\rm HLbL} = (4.66 \pm 4.38) \times 10^{-10} \Rightarrow (5.06 \pm 3.67_{\rm stat} \pm 0.20_{{\rm sys},a^2}) \times 10^{-10} \tag{42}$$

- The above systematic error is the estimated $\mathcal{O}(a^2)$ error. In addition, there can be additional $\mathcal{O}(a^4)$ error.

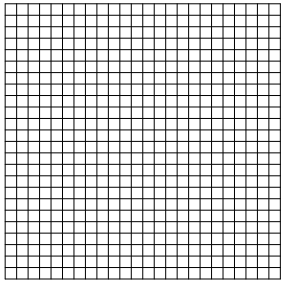


48I: $48^3 \times 96$, $5.5\,\mathrm{fm}$ box

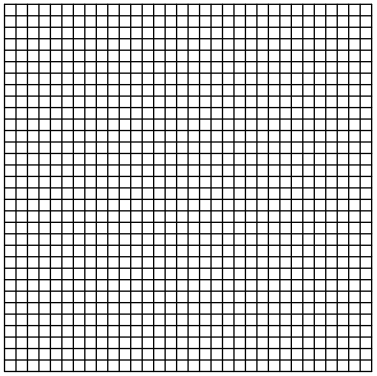


64I: $64^3 \times 128$, $5.5\,\mathrm{fm}$ box

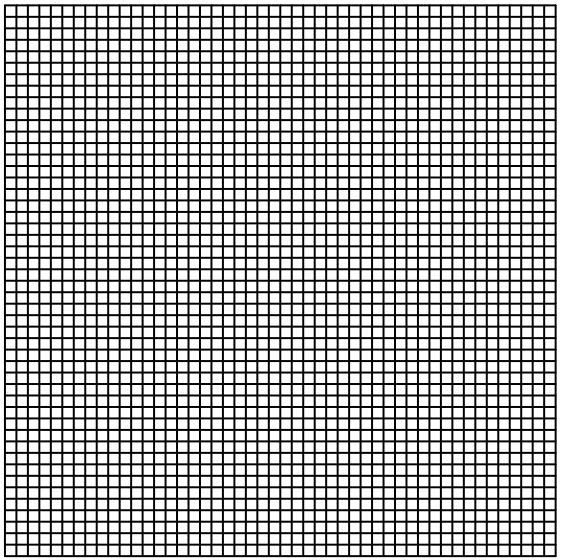
Phys. Rev. D 93, 074505 (2016)



24D: $24^3 \times 64$, $4.8\,\mathrm{fm}$ box



32D: $32^3 \times 64$, $6.4\,\mathrm{fm}$ box



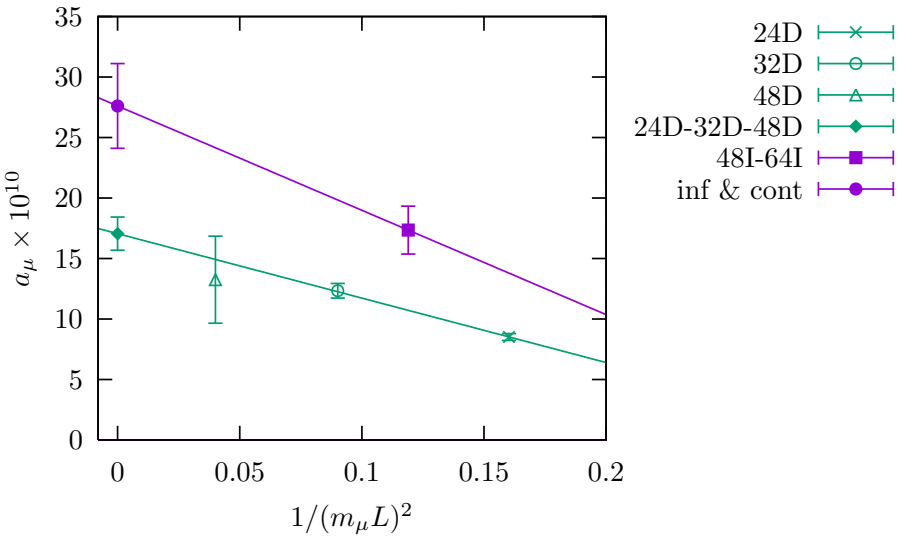
48D: $48^3 \times 64$, $9.6\,\mathrm{fm}$ box

32Dfine: $32^3 \times 64$, $4.8\,\mathrm{fm}$ box

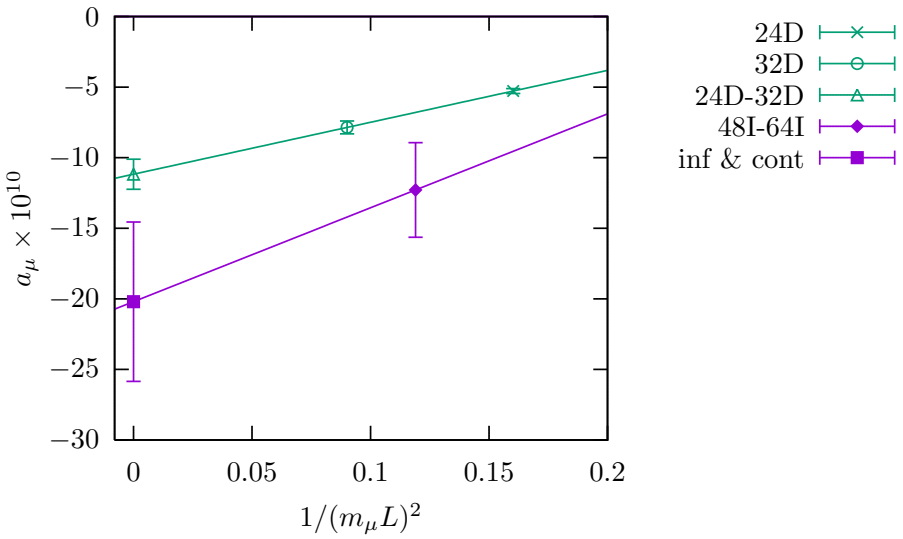
QED$_L$: inf vol & contiuum (prelim)

- MDWF+Iwasaki: continuum limit ($5.4\,\mathrm{fm}$)
- MDWF+DSDR: $a^{-1} = 1.015\,\mathrm{GeV}$: $24^3 \times 64$ ($4.8\,\mathrm{fm}$), $32^3 \times 64$ ($6.4\,\mathrm{fm}$), $48^3 \times 64$ ($9.6\,\mathrm{fm}$).
- MDWF+DSDR: $a^{-1} = 1.371\,\mathrm{GeV}$: $32^3 \times 64$ ($4.6\,\mathrm{fm}$).

$$F_2(a, L) \;=\; F_2\left(1 - \frac{c_1}{(m_\mu L)^2}\right)(1 - c_2\, a^2) \tag{43}$$



Connected diagrams



Disconnected diagrams

$$a_\mu^{\mathrm{cHLbL}} \;=\; (27.61 \pm 3.51_{\mathrm{stat}} \pm 0.32_{\mathrm{sys},a^2}) \times 10^{-10} \tag{44}$$

$$a_\mu^{\mathrm{dHLbL}} \;=\; (-20.20 \pm 5.65_{\mathrm{stat}}) \times 10^{-10} \tag{45}$$

$$a_\mu^{\mathrm{HLbL}} \;=\; (7.41 \pm 6.32_{\mathrm{stat}} \pm 0.32_{\mathrm{sys},a^2}) \times 10^{-10} \tag{46}$$

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- **QED kernel and subtraction**
- QCD @ $m_\pi = 340$ MeV
  (cross check with Mainz)
- QCD @ $m_\pi = 142$ MeV
- Long distance π^0-pole
- Subleading contributions

- Summary and Outlook

Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice

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The anomalous magnetic moment of the muon currently exhibits a discrepancy of about three standard deviations between the experimental value and recent Standard Model predictions. The theoretical uncertainty is dominated by the hadronic vacuum polarization and the hadronic light-by-light (HLbL) scattering contributions, where the latter has so far only been fully evaluated using different models. To pave the way for a lattice calculation of HLbL, we present an expression for the HLbL contribution to $g-2$ that involves a multidimensional integral over a position-space QED kernel function in the continuum and a lattice QCD four-point correlator. We describe our semi-analytic calculation of the kernel and test the approach by evaluating the π^0-pole contribution in the continuum.

PoS(LATTICE2016)16

PHYSICAL REVIEW D **96,** 034515 (2017)

Using infinite-volume, continuum QED and lattice QCD for the hadronic light-by-light contribution to the muon anomalous magnetic moment

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(Received 21 May 2017; published 22 August 2017)

In our previous work, Blum *et al.* [Phys. Rev. Lett. **118**, 022005 (2017)], the connected and leading disconnected hadronic light-by-light contributions to the muon anomalous magnetic moment $(g-2)$ have been computed using lattice QCD ensembles corresponding to physical pion mass generated by the RBC/UKQCD Collaboration. However, the calculation is expected to suffer from a significant finite-volume error that scales like $1/L^2$ where L is the spatial size of the lattice. In this paper, we demonstrate that this problem is cured by treating the muon and photons in infinite-volume, continuum QED, resulting in a weighting function that is precomputed and saved with affordable cost and sufficient accuracy. We present numerical results for the case when the quark loop is replaced by a muon loop, finding the expected exponential approach to the infinite volume limit and consistency with the known analytic result. We have implemented an improved weighting function which reduces both discretization and finite-volume effects arising from the hadronic part of the amplitude.

DOI: 10.1103/PhysRevD.96.034515

I. INTRODUCTION

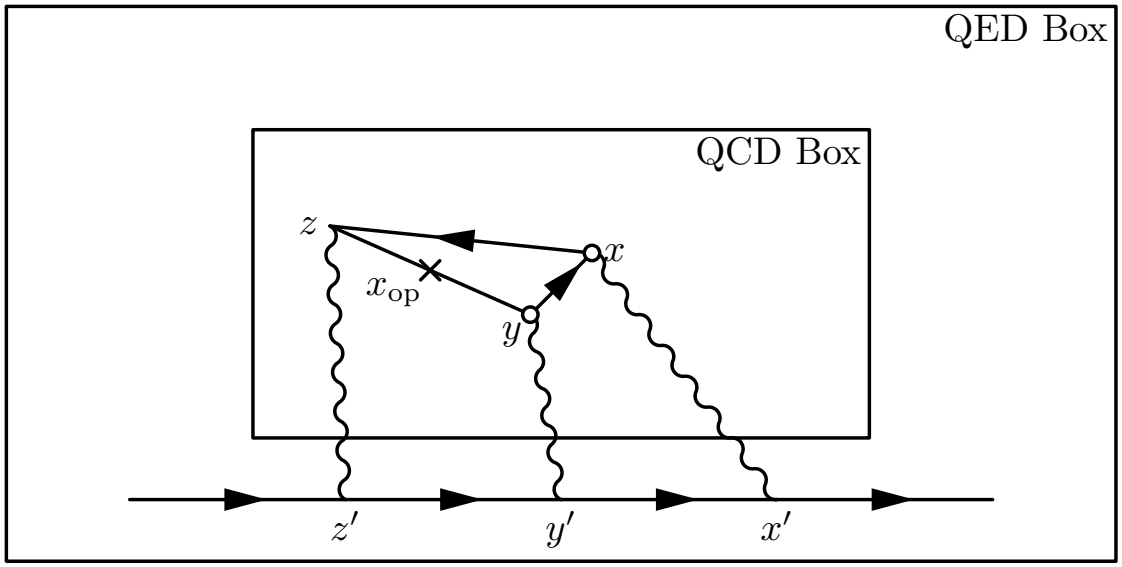
Precision measurements of lepton magnetic dipole moments provide a powerful tool for testing the standard model (SM) of particle physics at high precision. The which outweighs a loss in experimental precision. With the τ being experimentally inaccessible, a_μ is the most promising channel to reveal physics beyond the standard model.

HLbL: QCD box inside QED box

$$\mathcal{F}^C_\nu(x,y,z,x_{\rm op}) \;=\; (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$$

The QED part, $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$ can be evaluated in infinite volume QED box.

The QCD part, $\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ can be evaluated in a finite volume QCD box.



$$i^3\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)+\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x)+\text{other 4 permutations.} \tag{47}$$

$$\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; e^{m_\mu(t_{\rm snk}-t_{\rm src})}\sum_{x',y',z'} G_{\rho,\rho'}(x,x')G_{\sigma,\sigma'}(y,y')G_{\kappa,\kappa'}(z,z') \tag{48}$$
$$\times\sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} S_\mu(x_{\rm snk},x')i\gamma_{\rho'}S_\mu(x',y')i\gamma_{\sigma'}S_\mu(y',z')i\gamma_{\kappa'}S_\mu(z',x_{\rm src})$$

How to evaluate $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$? arXiv:1705.01067.

First, we need to regularize the infrard divergence in $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$.

$$\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \frac{1+\gamma_0}{2}[(a_{\rho,\sigma,\kappa}(x,y,z))_k\Sigma_k + i\,b_{\rho,\sigma,\kappa}(x,y,z)]\frac{1+\gamma_0}{2} \tag{49}$$

where $a_{\rho,\sigma,\kappa}(x,y,z)$ and $b_{\rho,\sigma,\kappa}(x,y,z)$ are real functions.

$$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \frac{1}{2}[\mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x)]^\dagger \tag{50}$$

It turned out that $\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z)$ is infrard finite.

$$\begin{aligned}\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) \;=\; & \frac{\gamma_0+1}{2}\,i\gamma_\sigma(\not\partial_\zeta+\gamma_0+1)i\,\gamma_\kappa(\not\partial_\xi+\gamma_0+1)i\,\gamma_\rho\,\frac{\gamma_0+1}{2} \\ & \times\int\frac{d^4\eta}{4\,\pi^2}\,\frac{f(\eta-y+\zeta)f(x-\eta+\xi)-f(y-\eta+\zeta)f(\eta-x+\xi)}{2(\eta-z)^2}\Bigg|_{\xi=\zeta=0}\end{aligned} \tag{51}$$

$$f(x)=f(|x|,x_t/|x|) \;=\; \frac{1}{8\pi^2}\int_0^1 d\,y\,e^{-yx_t}K_0(y\,|x|) \tag{52}$$

The 4 dimensional integral is calculated numerically with the CUBA library cubature rules.

Eventually, we need to compute

$$\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)\,\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \tag{53}$$

$\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ satisfies current conservation condition, which implies:

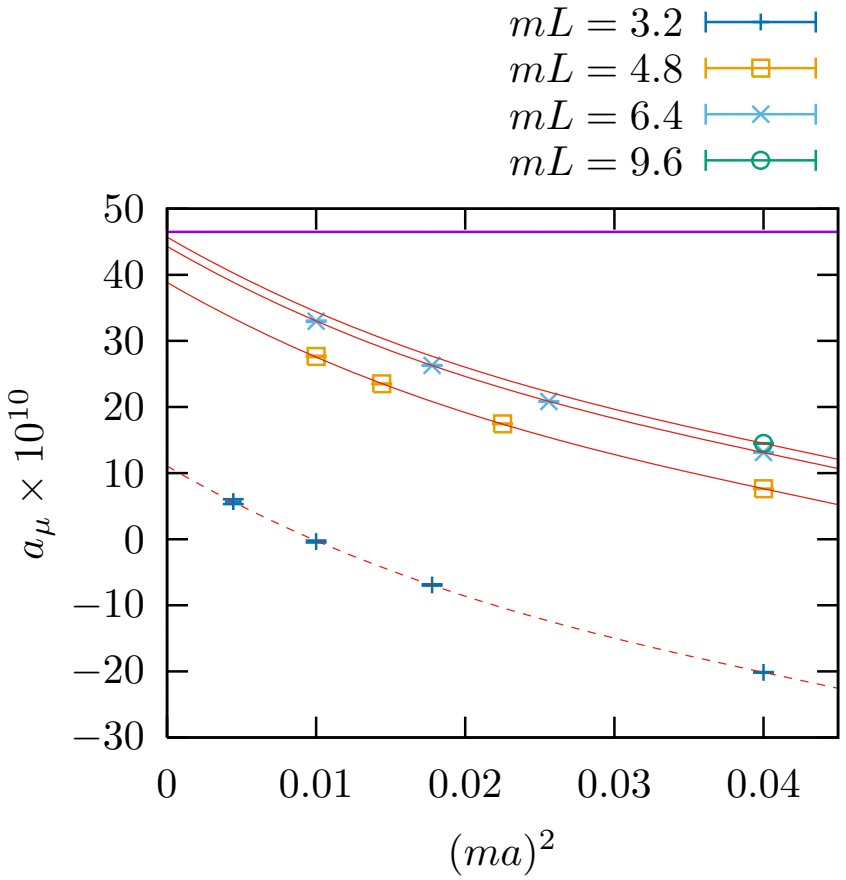
$$\sum_{x} \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \;=\; 0 \tag{54}$$

$$\sum_{z} \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \;=\; 0 \tag{55}$$

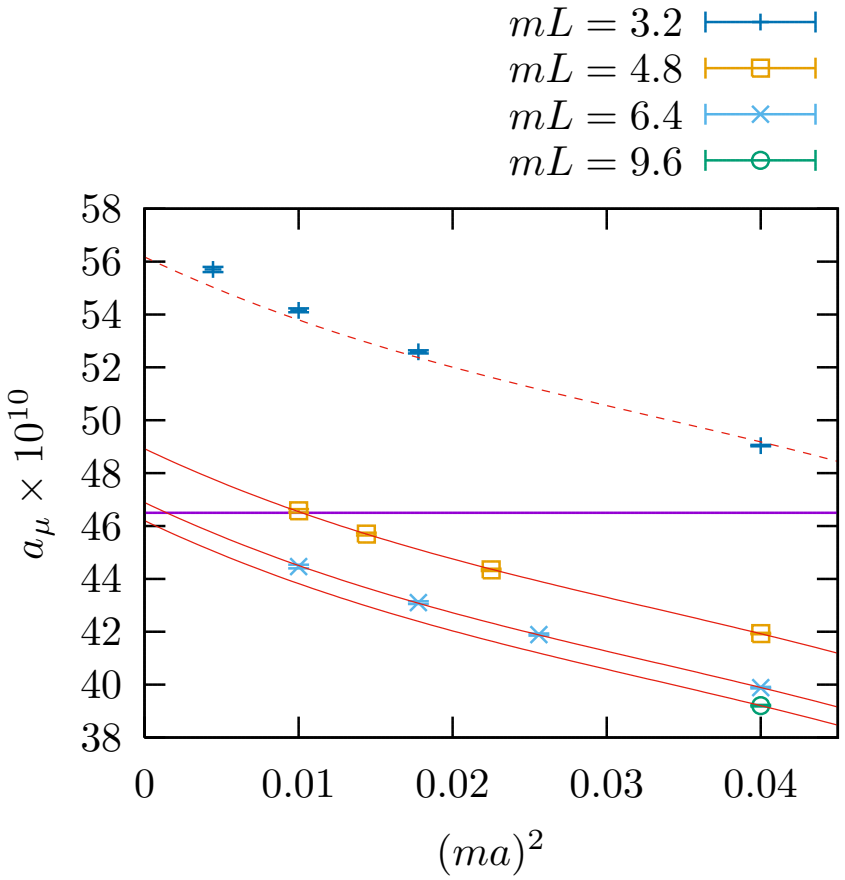
So, we have some freedom in changing $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$. One choice we find particularly helpful is:

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

- Compare the two $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$ in **pure QED computation**.



Original QED kernel



Subtracted QED kernel

- Notice the vertical scales in the two plots are different.

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- Connected diagrams:
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  The moment method
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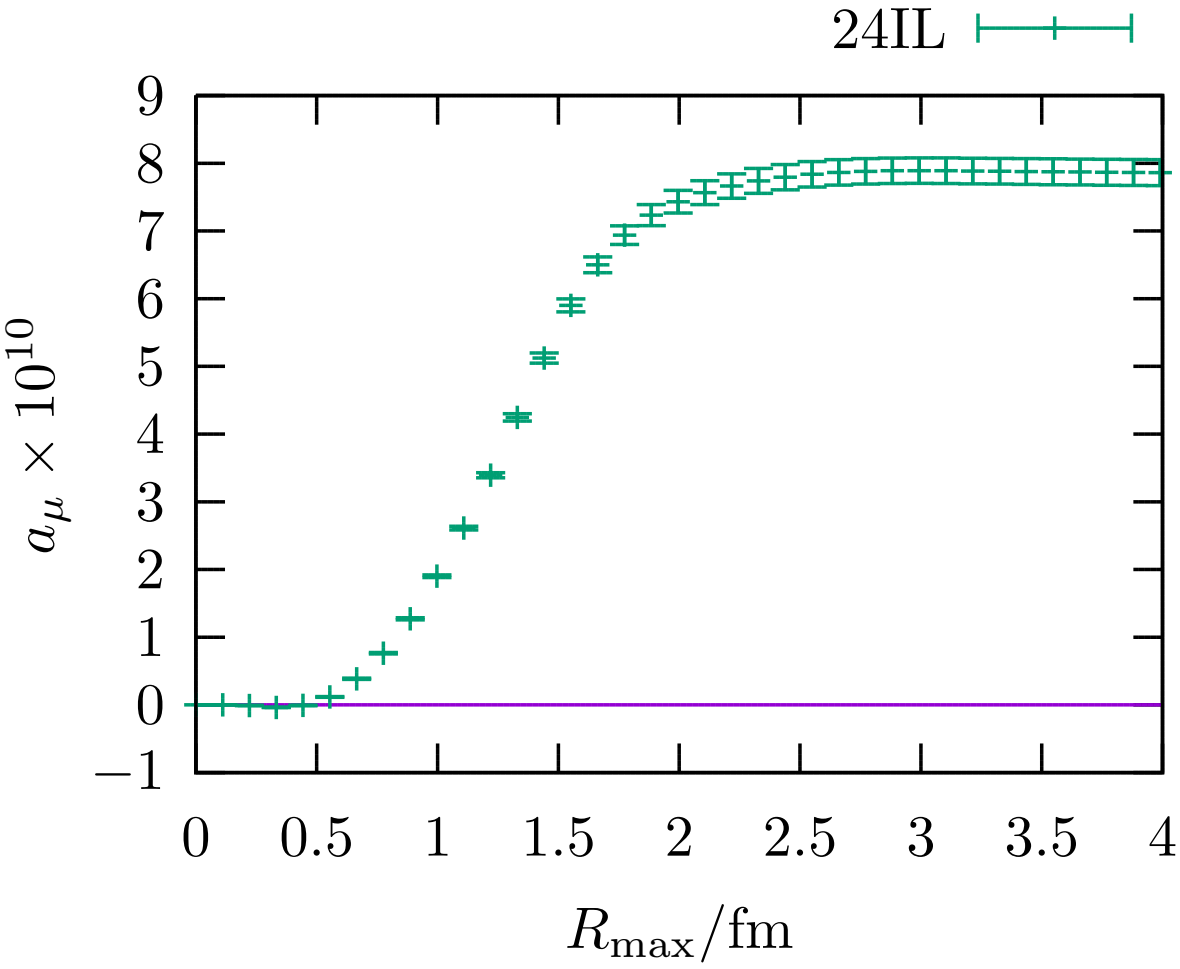
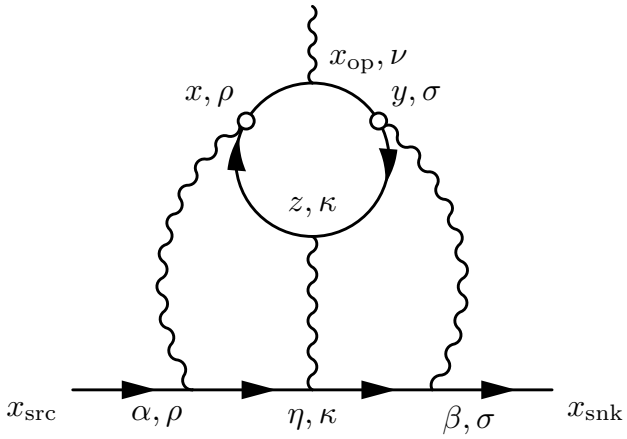
Connected @ $m_\pi = 340$ MeV (prelim)





- 24IL: $24^3 \times 64$

  $L = 2.66$ fm

- $a^{-1} = 1.78$ GeV

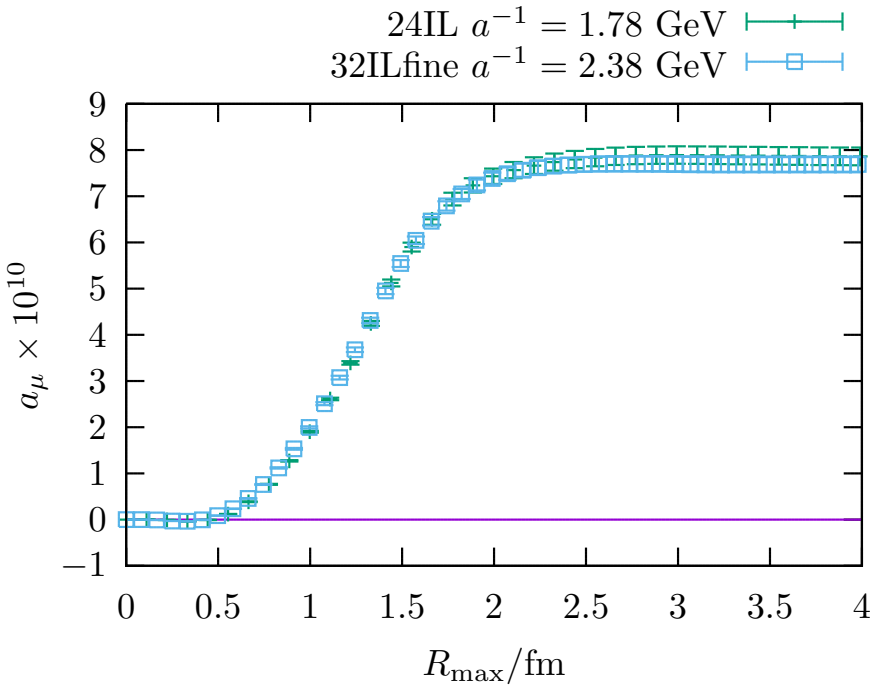
  $M_\pi = 340$ MeV

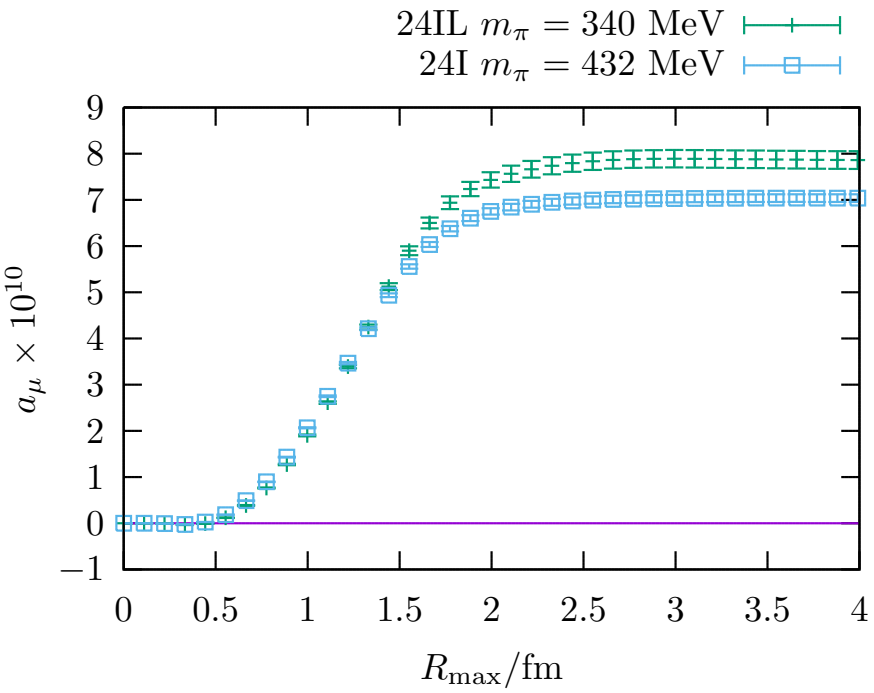
  $M_K = 594$ MeV

- Partial sum upto $R_{\max}$

  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$

Continuum @ $m_\pi = 340$ MeV (prelim)

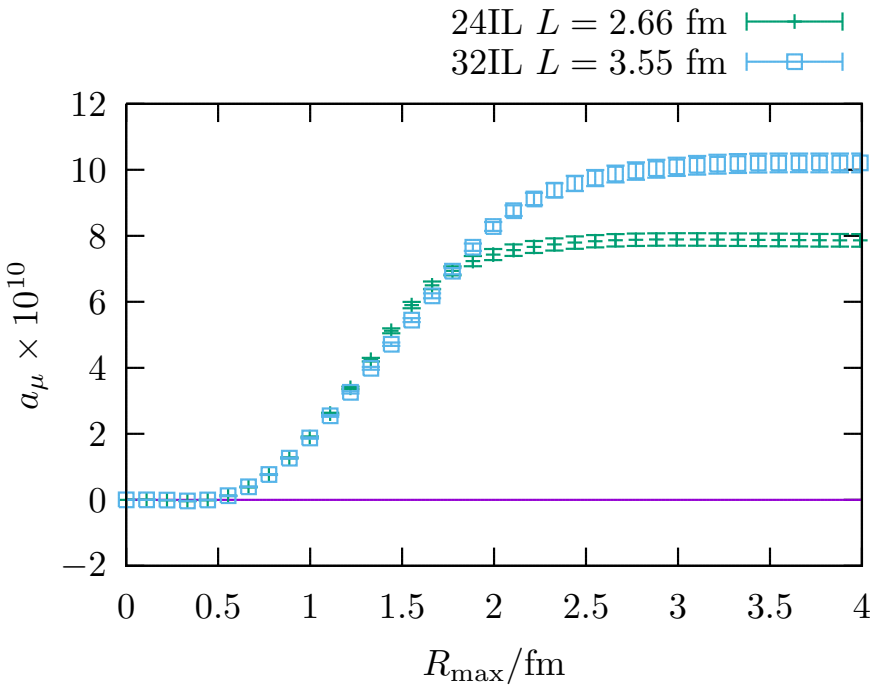


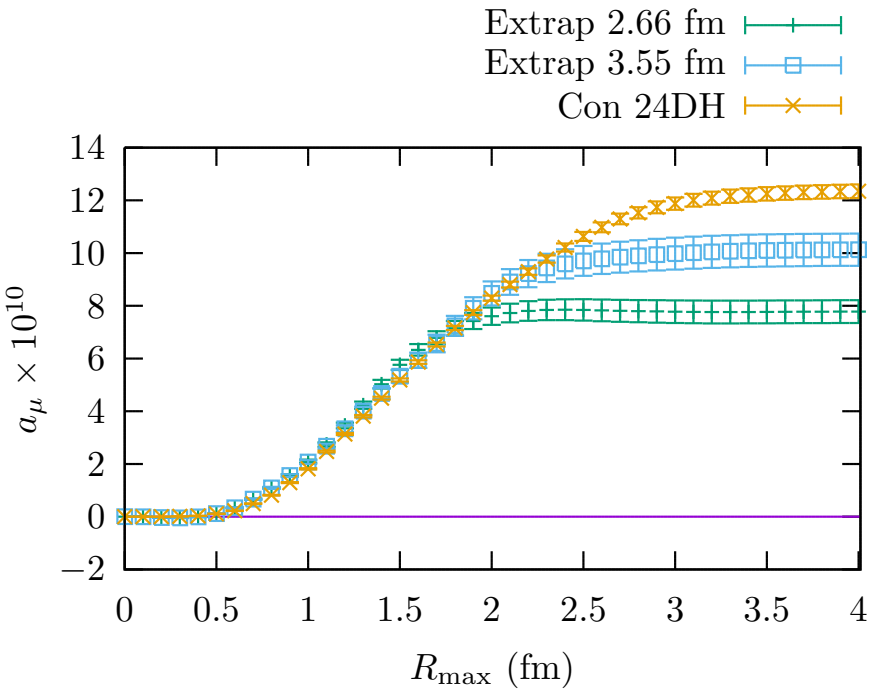


- 24IL: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 32ILfine: $32^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 2.38$ GeV
  $M_\pi = 357$ MeV
  $M_K = 590$ MeV
- 24I: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 432$ MeV
  $M_K = 626$ MeV

$$a_\mu(m_\pi, a, L) = a_\mu(m_\pi^{\text{target}}, 0, L) + c_1 a^2 + c_2(m_\pi^2 - (m_\pi^{\text{target}})^2) \quad (1)$$

Inf volume @ $m_\pi = 340$ MeV (prelim)

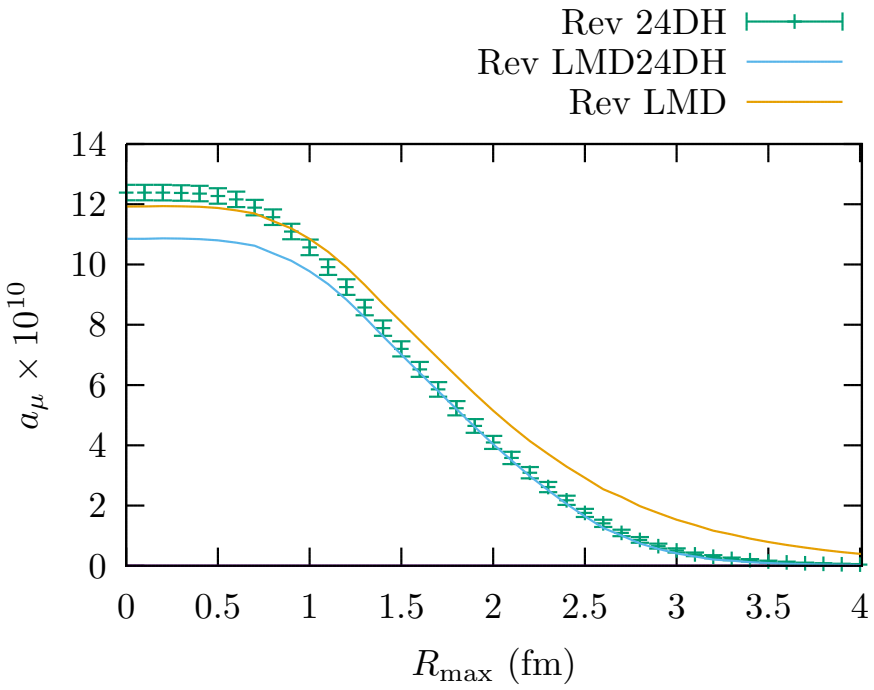


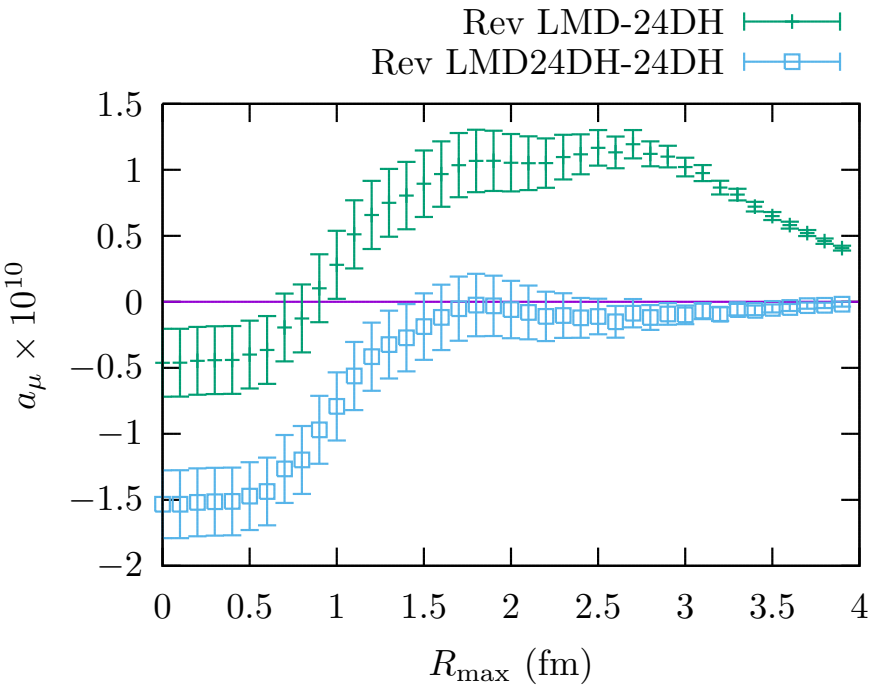


- 24IL: $24^3 \times 64$
  $L = 2.66$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 32IL: $32^3 \times 64$
  $L = 3.55$ fm
  $a^{-1} = 1.78$ GeV
  $M_\pi = 340$ MeV
  $M_K = 593$ MeV
- 24DH: $24^3 \times 64$
  $L = 4.67$ fm
  $a^{-1} = 1.015$ GeV
  $M_\pi = 340$ MeV
  $M_K \approx 593$ MeV

$$a_\mu(m_\pi, a, L) = a_\mu(m_\pi^{\text{target}}, 0, L) + c_1 a^2 + c_2(m_\pi^2 - (m_\pi^{\text{target}})^2) \quad (2)$$

Inf volume @ $m_\pi = 340$ MeV (prelim)





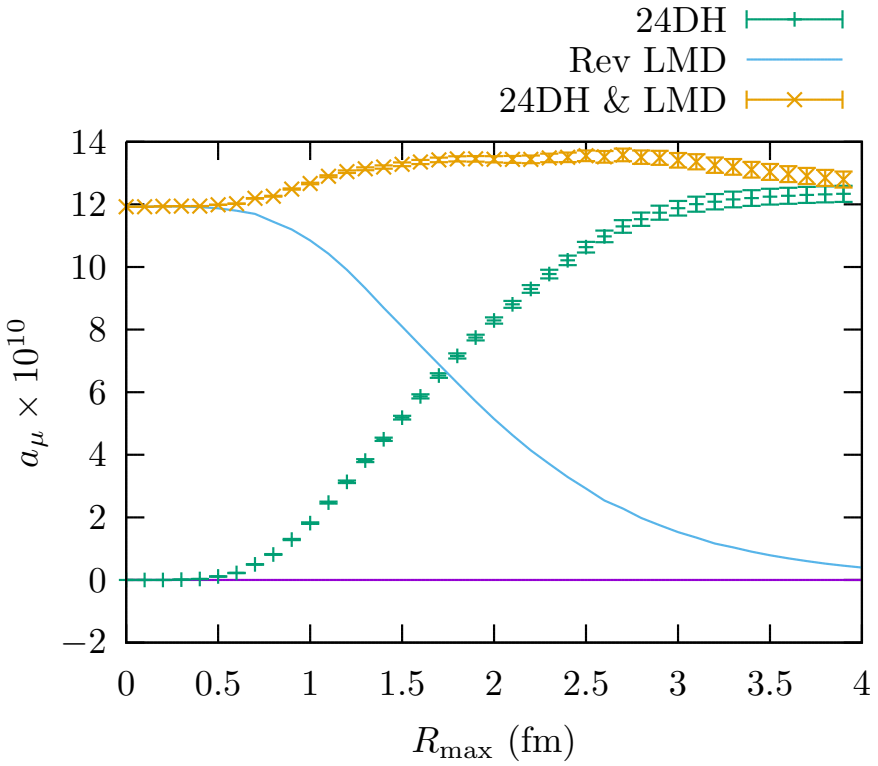
- Reverse partial sum down to $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$
- LMD: Lowest Meson Dominance Model.

  Pion-pole contribution calculated in position space.

  At $m_\pi = 340$ MeV: $f_\pi = 149$ MeV, $M_V = 830$ MeV

  The pion pole contribution should be multiplied by $-34/9$ to match with connected diagram.

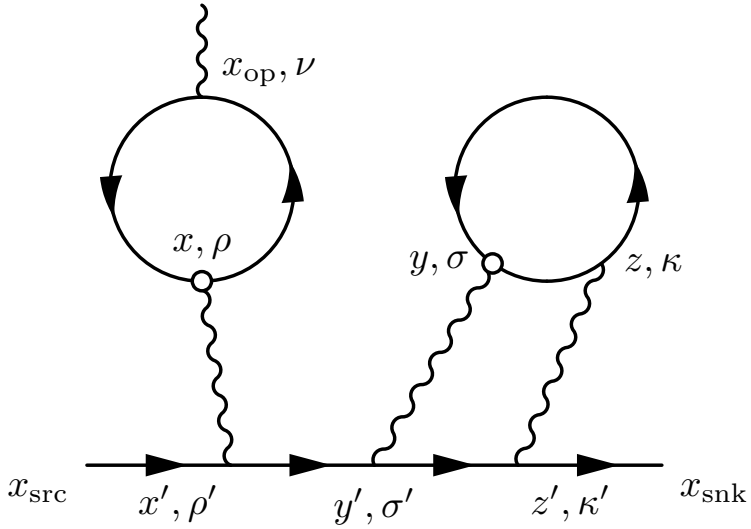
Inf volume @ $m_\pi = 340$ MeV (prelim)



- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto $R_{\max}$.
- Rev LMD:
  reverse partial sum down to $R_{\max}$.
- 24DH & LMD:
  the sum of the above two curves.

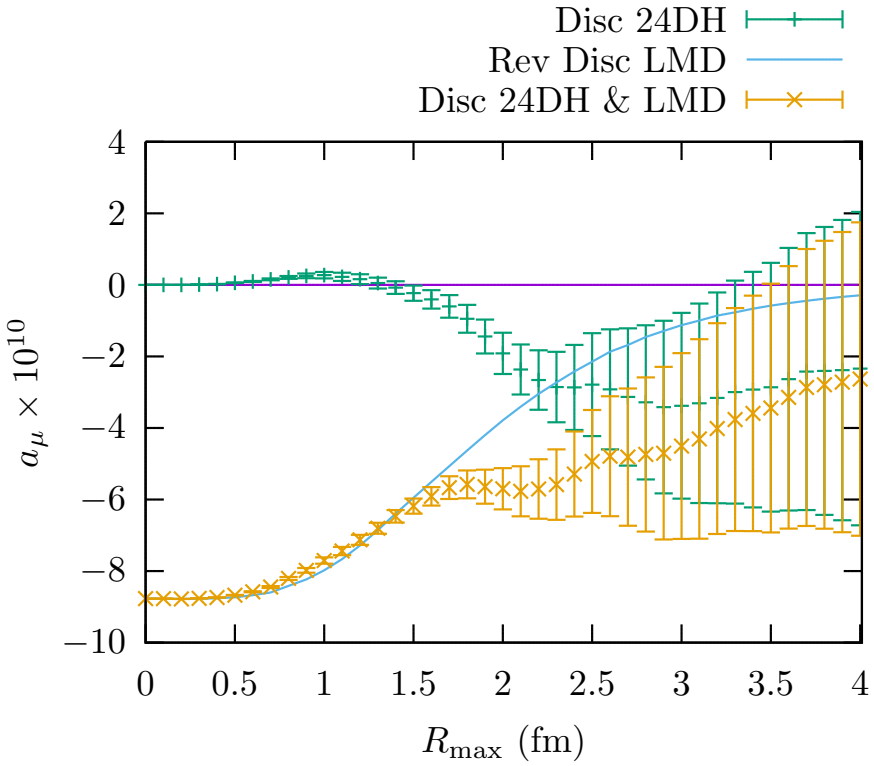
- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times 34/9$.
- At 2.0 fm, the combination gives: $a_\mu^{\text{con}} = 13.44(10)_{\text{stat}} \times 10^{-10}$.

QED_∞: disconnected diagram and M^2 trick



- For QED_L, we can compute the QED function for all x given the y location fixed and z summed over. Allow us to compute all combination of x, y with little cost.
- For QED_∞, although we can compute all the function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing y) for all x and z, simply because of its cost is proportion to Volume^2.
- However, we with QED_∞ and interpolation, we can freely choose which coordinates we compute. For example, we may compute all z for $|z - y| \leqslant 5$, and sample z for $|z - y| > 5$.

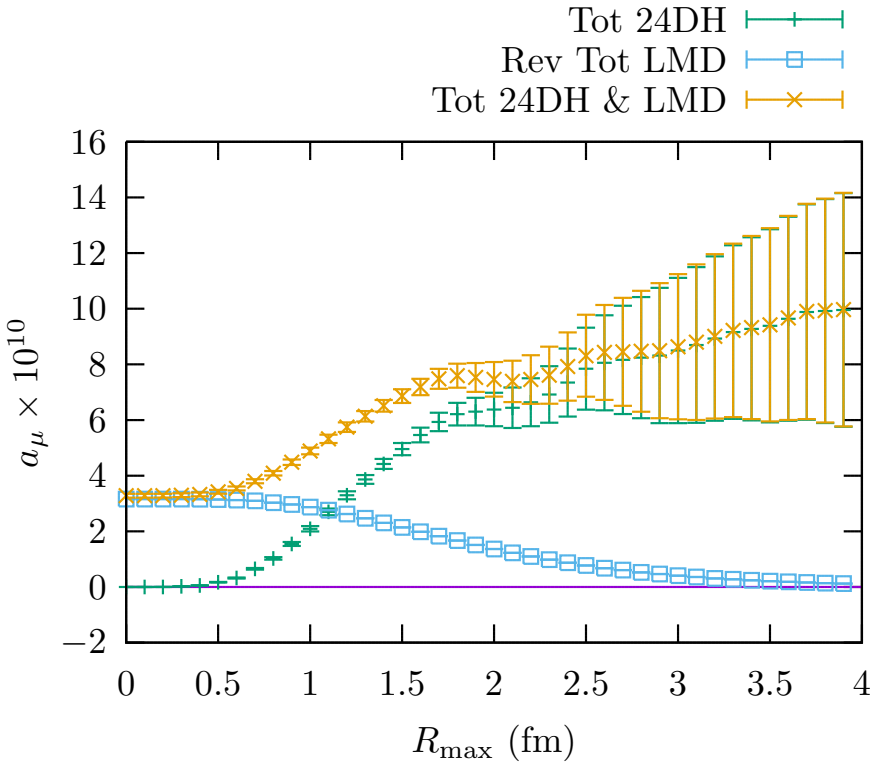
Disconnected @ $m_\pi = 340$ MeV (prelim)



- $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto $R_{\max}$.
- Rev LMD:
  reverse partial sum down to $R_{\max}$.
- 24DH & LMD:
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model $\times(-25/9)$.
- At 2.0 fm, the combination gives: $a_\mu^{\text{discon}} = -5.70(58)_{\text{stat}} \times 10^{-10}$.

Total @ $m_\pi = 340$ MeV (prelim)



- $R_{\mathsf{max}} = \max(|x-y|, |x-z|, |y-z|)$.
- 24DH: partial sum upto R_{max}.
- Rev LMD:
  reverse partial sum down to R_{max}.
- 24DH & LMD:
  the sum of the above two curves.

- Short distance part is given by lattice data.
- Long distance part is given by LMD model.
- At 2.0 fm, the combination gives: $a_\mu^{\mathsf{tot}} = 7.46(62)_{\mathsf{stat}} \times 10^{-10}$.

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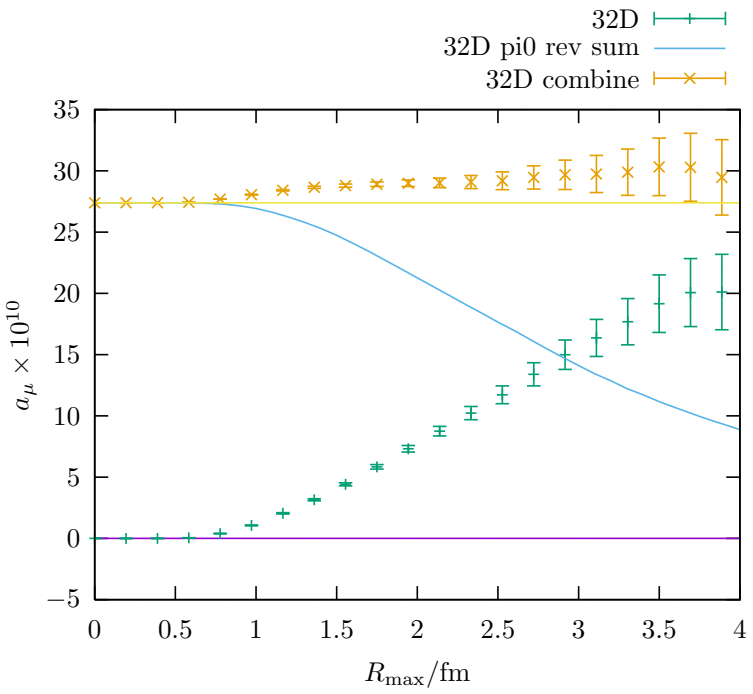
Connected @ $m_\pi = 142$ MeV (prelim)



Combine

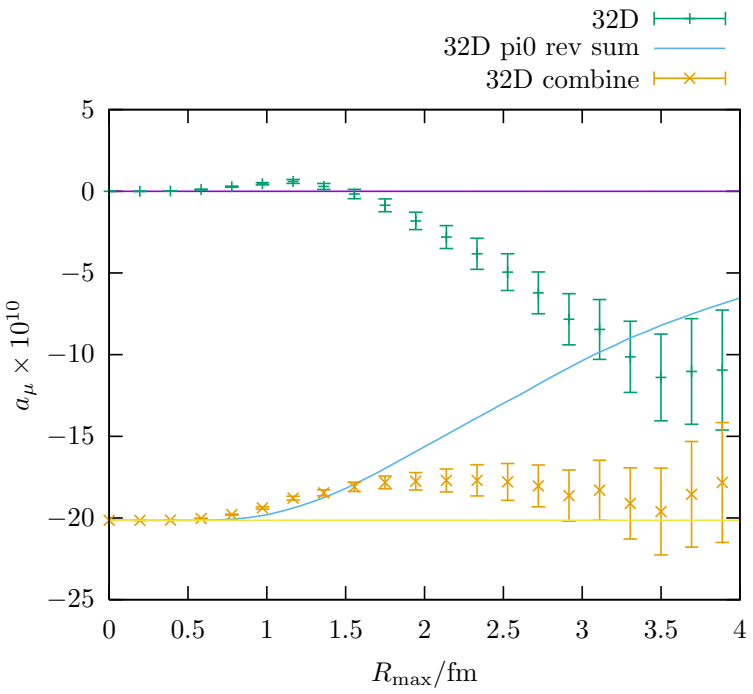
Lattice with

Pion pole

LMD model.

- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model multiplied by $34/9$ (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\mathrm{fm}$, the combined result is $a_\mu^{\mathrm{cHLbL}} = 29.19(0.73)_{\mathrm{stat}} \times 10^{-10}$.
- Previous extrapolated results with QED$_\mathrm{L}$ is $a_\mu^{\mathrm{cHLbL}} = 27.61(3.51)_{\mathrm{stat}}(0.32)_{\mathrm{sys},a^2} \times 10^{-10}$.

Disconnected @ $m_\pi = 142$ MeV (prelim) 27 / 35

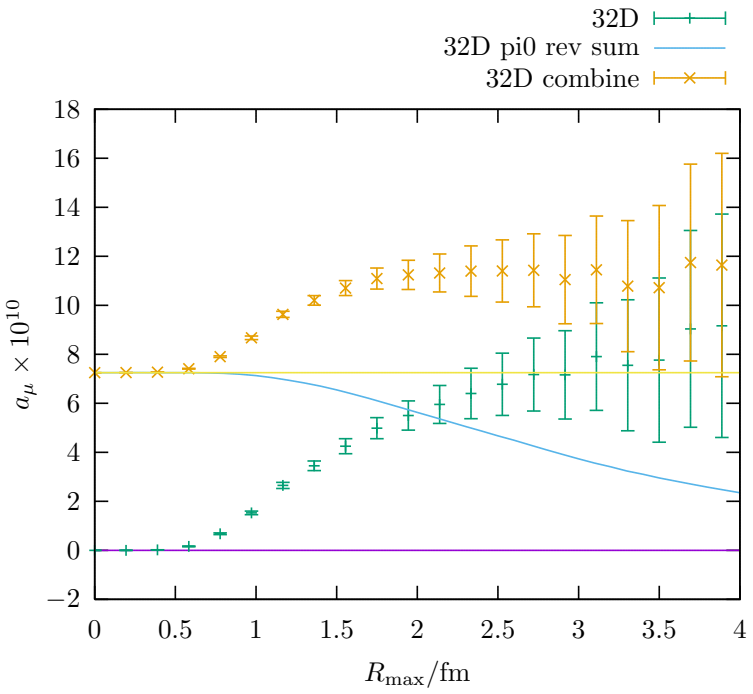


Combine Lattice with Pion pole LMD model.

- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model multiplied by $-25/9$. (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\mathrm{fm}$, the combined result is $a_\mu^{\mathrm{discon}} = -17.79(1.13)_{\mathrm{stat}} \times 10^{-10}$.

  Previous extrapolated results with QED$_\mathrm{L}$ is $a_\mu^{\mathrm{discon}} = -20.20(5.65)_{\mathrm{stat}} \times 10^{-10}$.

Total @ $m_\pi = 142$ MeV (prelim)



Combine

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- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model. (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\mathrm{fm}$ the combined results is $a_\mu^{\mathrm{total}} = 11.40(1.27)_{\mathrm{stat}} \times 10^{-10}$

  the part from lattice is $6.78(1.27)_{\mathrm{stat}} \times 10^{-10}$.

  Previous extrapolated results with QED$_\mathrm{L}$ is $a_\mu^{\mathrm{total}} = 7.41(6.32)_{\mathrm{stat}}(0.32)_{\mathrm{sys},a^2} \times 10^{-10}$.

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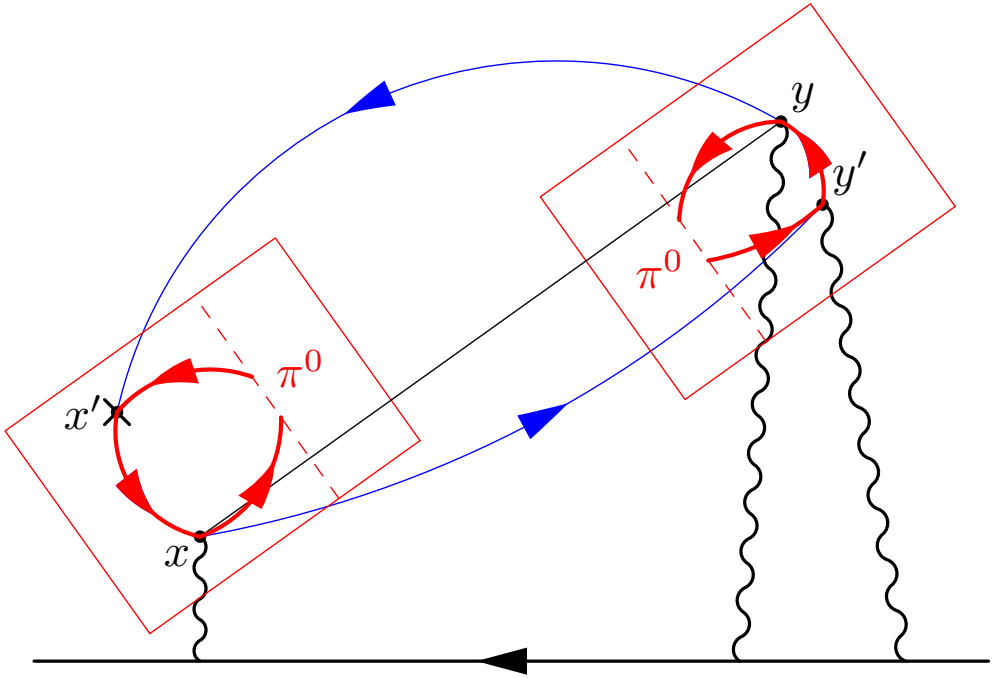
Long-distance π^0 pole: further tweaks



- Norman's initial idea for calculating the long-distance part of HLbL from π^0 pole. (Norman's talk at HLbL workshop in UCONN, 2018/03/13)
- Use two modest size lattices with time direction aligned with the pion propagating direction.

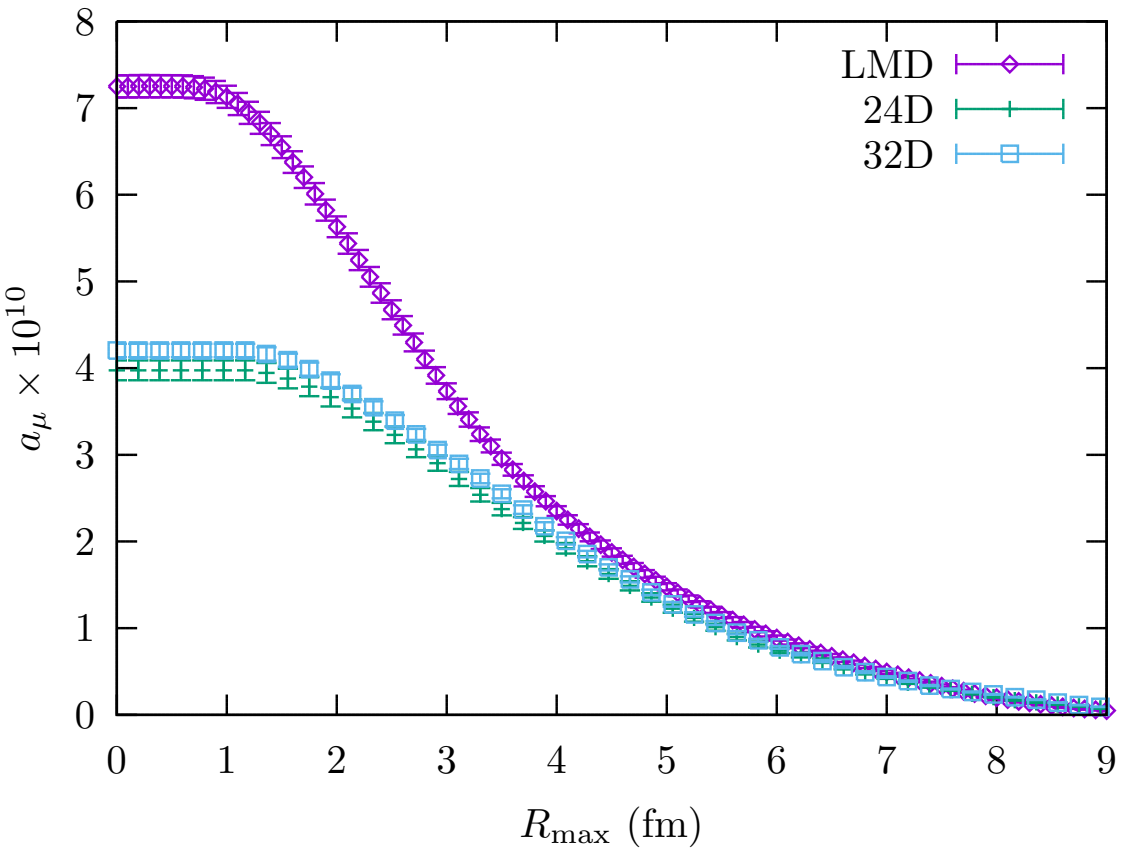
- We only need the following three-point function for $\vec{p} = 0$ pion.

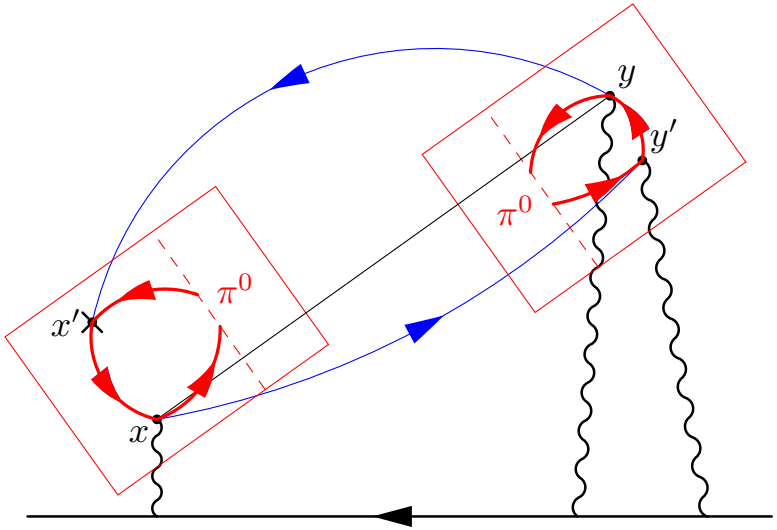
$$H_{\mu,\nu}(x-y) = \langle 0|J_\mu(x)J_\nu(y)|\pi^0\rangle$$

- The function can be saved to disk and used for other purposes.

Long-distance π^0 pole: results (prelim)

- UCONN graduate student Cheng Tu did the calculation.



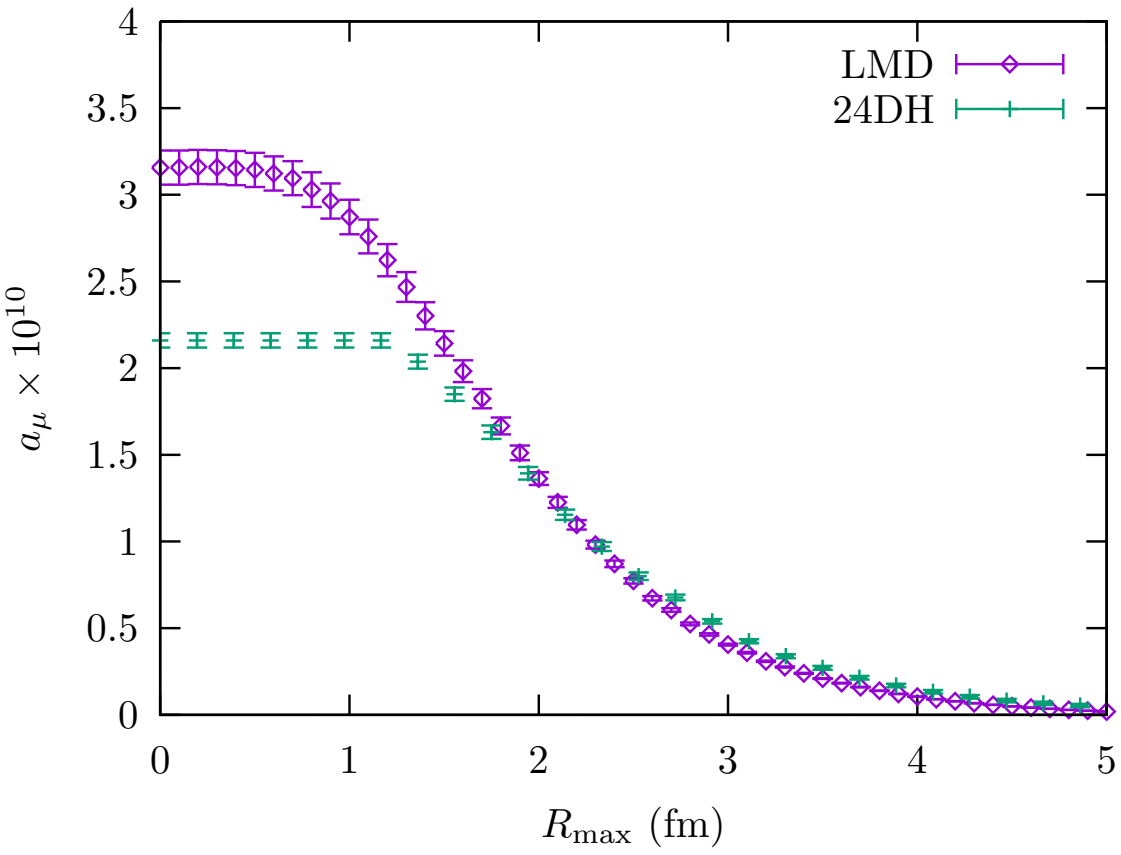


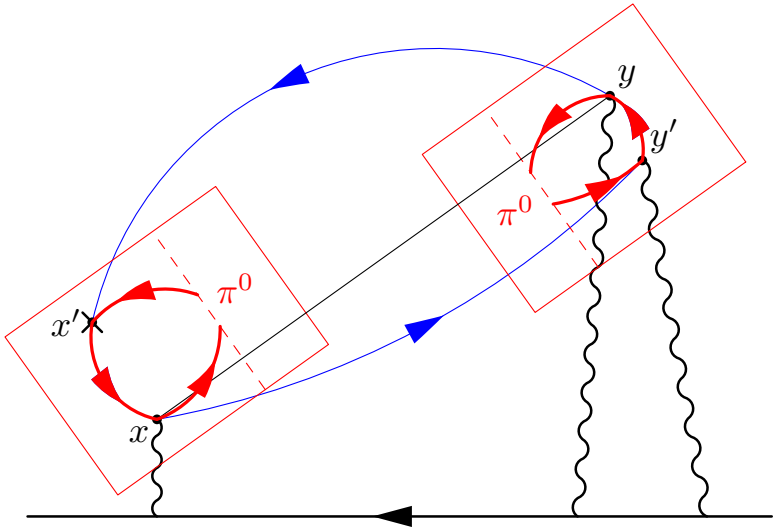
- $R_{\max} = \max(|x-y|, |x-y'|, |y-y'|)$.
  Reverse partial sum plotted.

- 24D: $24^3 \times 64$
- 32D: $32^3 \times 64$
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV

Long-distance π^0 pole: results (prelim)

- UCONN graduate student Cheng Tu did the calculation.





- $R_{\max} = \max(|x-y|, |x-y'|, |y-y'|)$.
  Reverse partial sum plotted.

- 24DH: $24^3 \times 64$
- $a^{-1} = 1.015$ GeV
  $M_\pi = 340$ MeV

Outline

- Introduction

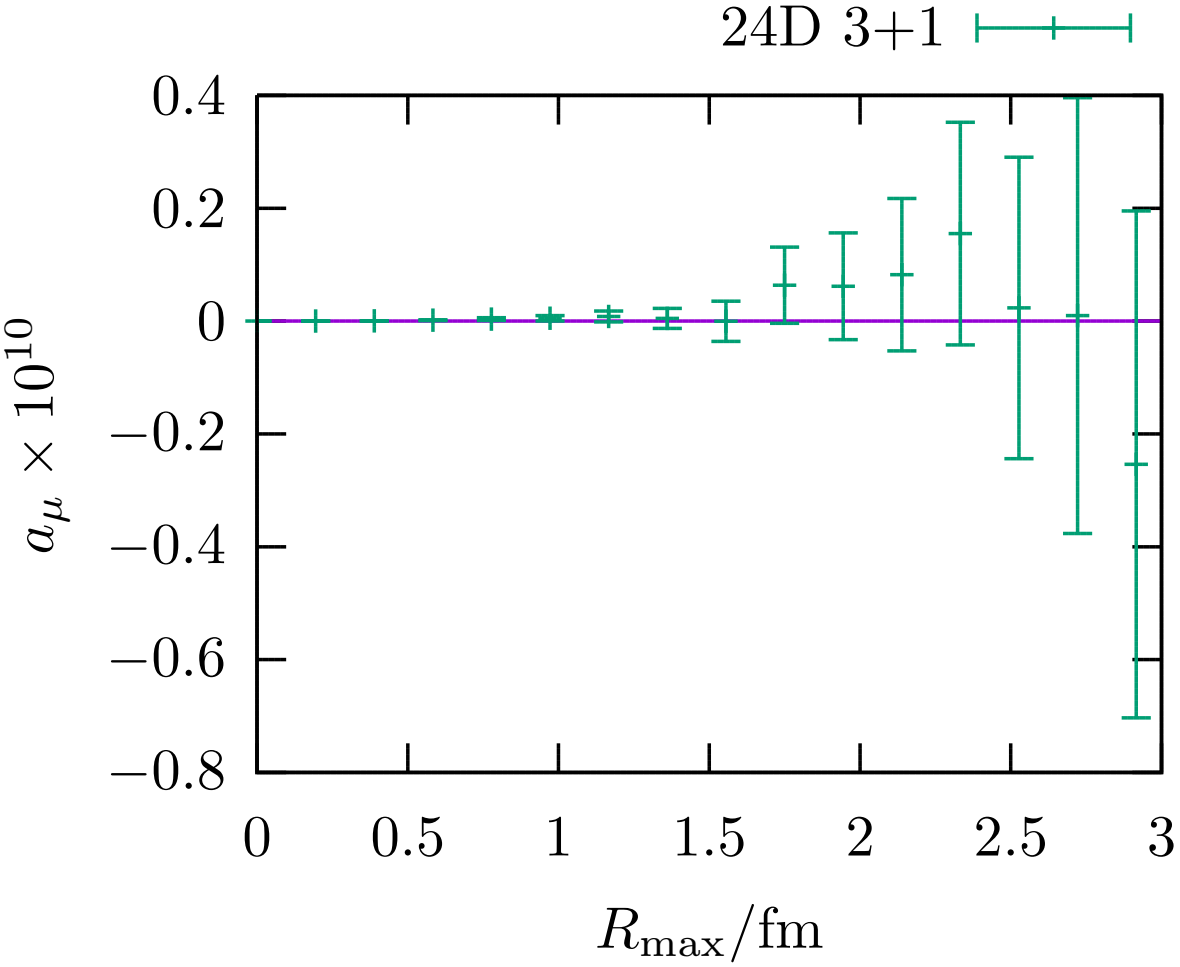
HLbL: QED$_L$ approach

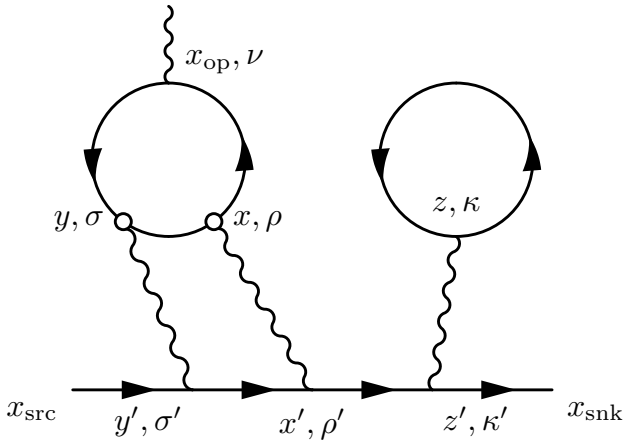
- Connected diagrams:
  Exact photon propagator
  The moment method
- Disconnected diagrams
- Results

HLbL: QED$_\infty$ approach

- QED kernel and subtraction
- QCD @ $m_\pi = 340$ MeV
  (cross check with Mainz)
- QCD @ $m_\pi = 142$ MeV
- Long distance π^0-pole
- **Subleading contributions**

- Summary and Outlook

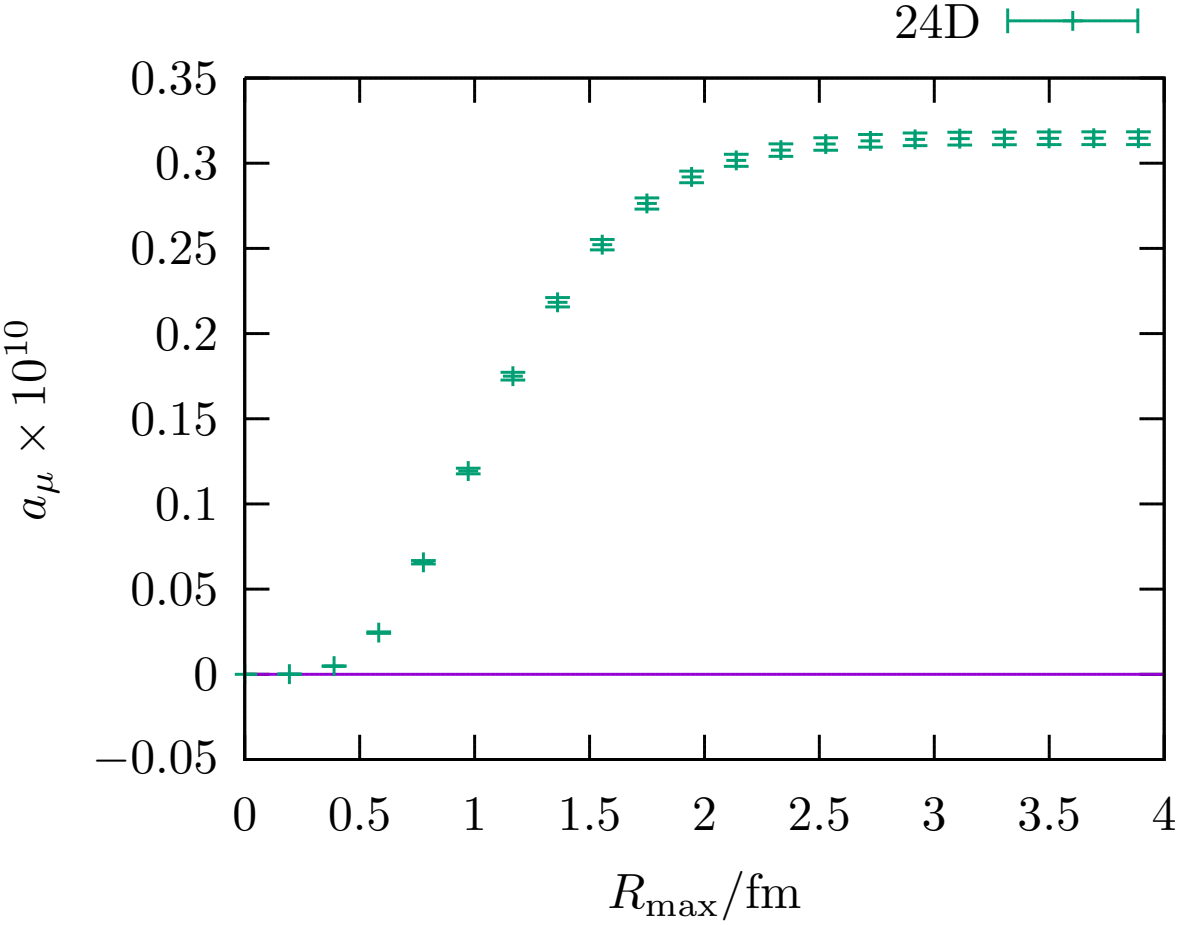


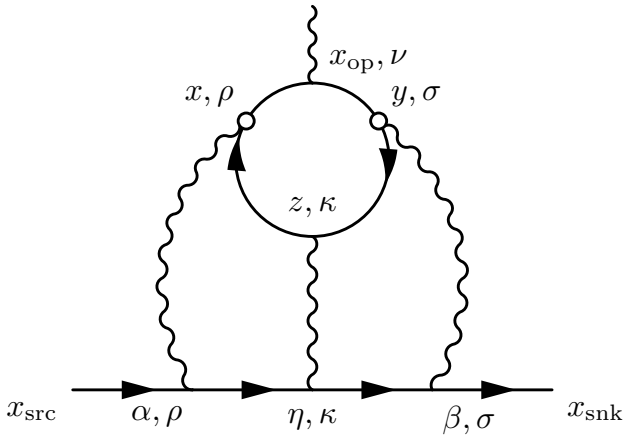


- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum upto R_{max}
  $R_{\mathrm{max}} = \max(|x-y|, |x-z|, |y-z|)$
- The tadpole part comes from Christoph's evaluation of disc HVP.

Strange connected at physical point prelim





- 24D: $24^3 \times 64$
  $L = 4.8$ fm
- $a^{-1} = 1.015$ GeV
  $M_\pi = 142$ MeV
  $M_K = 512$ MeV

- Partial sum upto $R_{\max}$
  $R_{\max} = \max(|x-y|, |x-z|, |y-z|)$
- Small. Not yet included.

Outline

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HLbL: QED$_\infty$ approach

- QED kernel and subtraction
- QCD @ $m_\pi = 340$ MeV
  (cross check with Mainz)
- QCD @ $m_\pi = 142$ MeV
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- **Summary and Outlook**

$a_\mu^{\text{con}} \times 10^{10}$	$a_\mu^{\text{discon}} \times 10^{10}$	$a_\mu^{\text{total}} \times 10^{10}$

QED_L

48I, $L = 5.5$ fm, $a^{-1} = 1.73$ GeV, $m_\pi = 139$ MeV

$11.60(96)_{\text{stat}}$	$-6.25(80)_{\text{stat}}$	$5.35(1.35)_{\text{stat}}$

Infinite volume, continuum limit, $m_\pi = 135$MeV

$27.61(3.51)_{\text{stat}}(32)_{\text{sys},a^2}$	$-20.20(5.65)_{\text{stat}}$	$7.41(6.32)_{\text{stat}}(32)_{\text{sys},a^2}$

QED_∞

$m_\pi = 340$ MeV

$R_{\text{max}} \leq 2.0$ fm 24DH: $L = 4.8$ fm, $a^{-1} = 1.015$ GeV
$R_{\text{max}} > 2.0$ fm LMD: $m_\rho = 830$ MeV, $f_\pi = 149$ MeV

$13.44(10)_{\text{stat}}$	$-5.70(58)_{\text{stat}}$	$7.46(62)_{\text{stat}}$

$m_\pi = 142$ MeV

$R_{\text{max}} \leq 2.5$ fm 32D: $L = 6.4$ fm, $a^{-1} = 1.015$ GeV
$R_{\text{max}} > 2.5$ fm LMD: $m_\rho = 770$ MeV, $f_\pi = 130$ MeV

$29.19(73)_{\text{stat}}$	$-17.79(1.13)_{\text{stat}}$	$11.40(1.27)_{\text{stat}}$

Outlook

- QED_L
  - Accumulate more statistics.
  - Study systematic error from the extrapolation formula.
- QED_∞
  - Study discretization effects of disconnected diagram.
  - Use lattice to compute long distance π^0-pole contribution. (almost there)
  - Study the long distance charged pion loop contribution.
  - Calculate remaining subleading diagrams.

Thank You!

Consequence of current conservation

Consider a vector field $J_\rho(x)$. It satisfies two conditions:

- $\partial_\rho J_\rho(x) = 0$.
- $J_\rho(x) = 0$ if $|x|$ is large.

We can conclude (the result is a little bit unexpected, but actually correct):

$$\int d^4x\, J_\rho(x) \;=\; \int d^4x\, \partial_\sigma(x_\rho\, J_\sigma(x)) = 0 \tag{56}$$

In three dimension, this result have a consequence which is well-known.

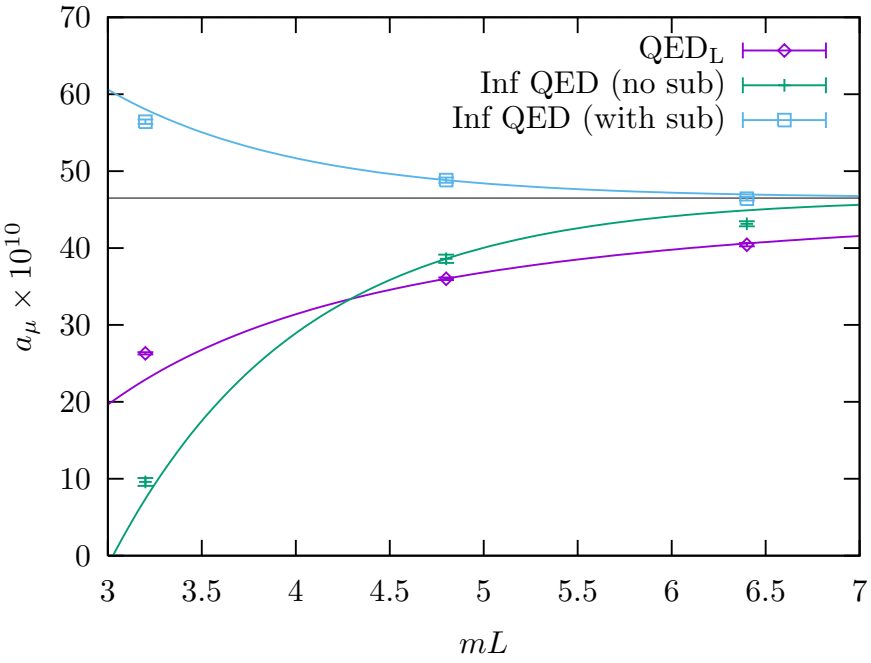
Consider a finite size system with stationary current. We then have

- $\vec{\nabla} \cdot \vec{j}(\vec{x}) = 0$, because of current conservation.
- $\vec{j}(\vec{x}) = 0$ if $|\vec{x}|$ large, because the system if of finite size.

Within a constant external magnetic field $\vec{B}$, the total magnetic force should be

$$\int \left[\vec{j}(\vec{x}) \times \vec{B}\right] d^3x \;=\; \left[\int \vec{j}(\vec{x})\, d^3x\right] \times \vec{B} = 0 \tag{57}$$

- Compare the finite volume effects in different approaches in **pure QED computation**,



- QED$_L$: $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop. Phys.Rev. D93 (2016) 1, 014503.
- Inf QED (no sub): $\mathcal{O}(e^{-mL})$ finite volume effect. Everything except the four-point-correlation function is evaluated in infinite volume. arXiv:1705.01067.
- Inf QED (with sub): smaller $\mathcal{O}(e^{-mL})$ finite volume effect. arXiv:1705.01067.