HVP contribution to $(g-2)_\mu$: status of the Mainz calculation

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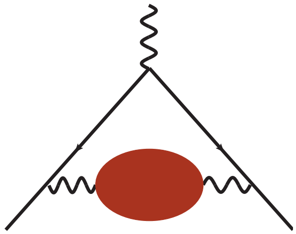
- Time-momentum representation [Bernecker, Meyer '11]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t \; K(t) \; G(t)$$

$$G(t) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$



Electromagnetic current : $V_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) - \frac{1}{3}\overline{s}(x)\gamma_\mu s(x) + \cdots$

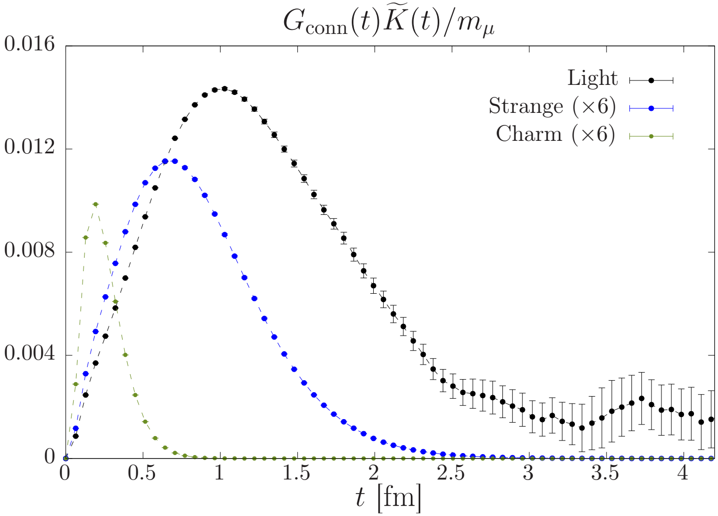
- $N_f = 2+1$ with (improved) Wilson fermions (one ensemble at the physical pion mass)

- Discretization errors : 4 lattice spacings in the range 0.050 - 0.085 fm $\ll 0.1$ fm !

$$J_\mu^{(l),a}(x) = \overline{\psi}(x)\gamma_\mu \frac{\lambda^a}{2} \psi_j(x) \,,$$

$$J_\mu^{(c),a}(x) = \frac{1}{2} \left(\overline{\psi}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)\frac{\lambda^a}{2}\psi(x) - \overline{\psi}(x)(1-\gamma_\mu)U_\mu(x)\frac{\lambda^a}{2}\psi(x+a\hat{\mu}) \right)$$

$\rightarrow$ two discretizations : combined continuum extrapolation



- Physical pion mass with $a = 0.065$ fm
- Flavor decomposition :

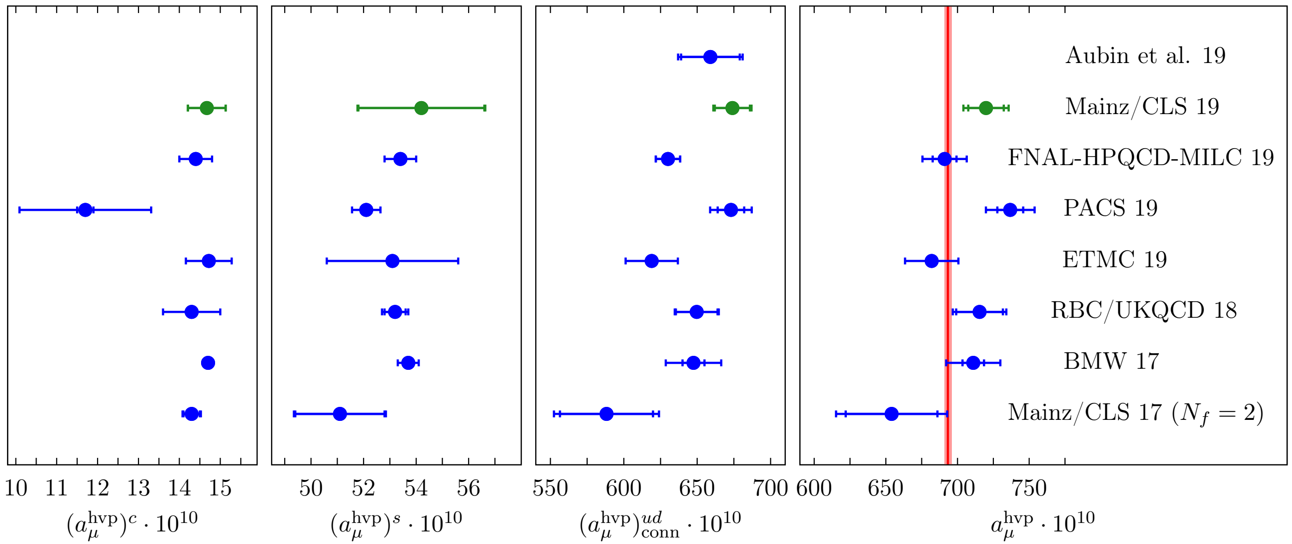
$$G(t) = G_l(t) + G_s(t) + G_c(t) + G_{\rm disc}(t)$$

$$a_\mu^{{\rm HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \sum_{t=0}^{\infty} {\rm d}t \; K(t) \; G_f(t)$$

Challenges for a high-precision calculation :

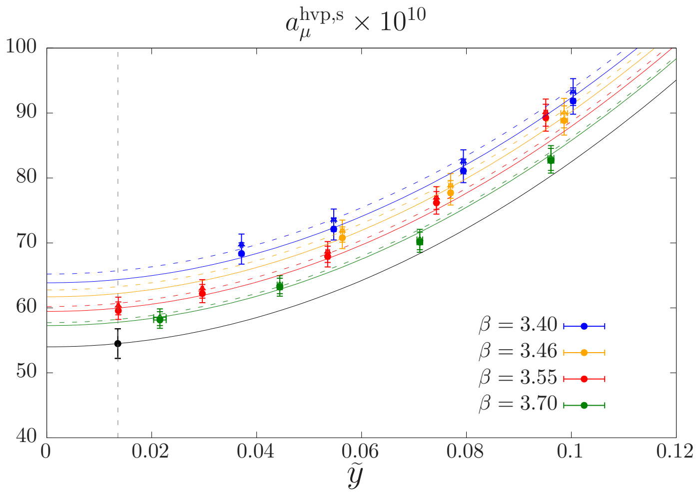
- Light contribution dominates $\rightarrow$ noise/signal increases exponentially with t
  $\rightarrow$ finite-size effects O(3%) at physical point ($m_\pi L = 4$)
- Charm quark : large discretization effects
- Disconnected diagrams of the order of O(3%)
- Continuum and chiral extrapolation (interpolation)
- QED + strong isospin corrections : O(1%) (work in progress in Mainz)

$$a_\mu^{\rm hvp} = (720.0 \pm 12.4_{\rm stat} \pm 9.9_{\rm syst}) \cdot 10^{-10}$$ [Phys.Rev. D100 (2019)]



$\rightarrow$ Precision of 2.2% (dominated by statistics)

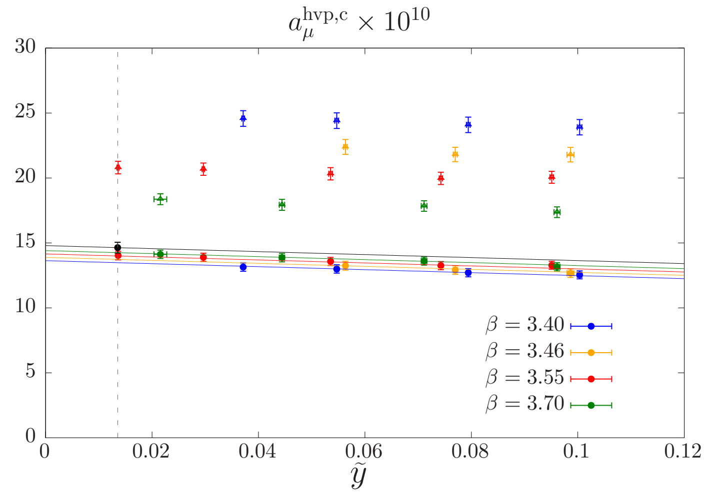
Strange quark



$$a_\mu^{\rm hvp,s} = (53.9 \pm 2.4 \pm 0.1) \times 10^{-10}$$

$$\to 0.33\%$$

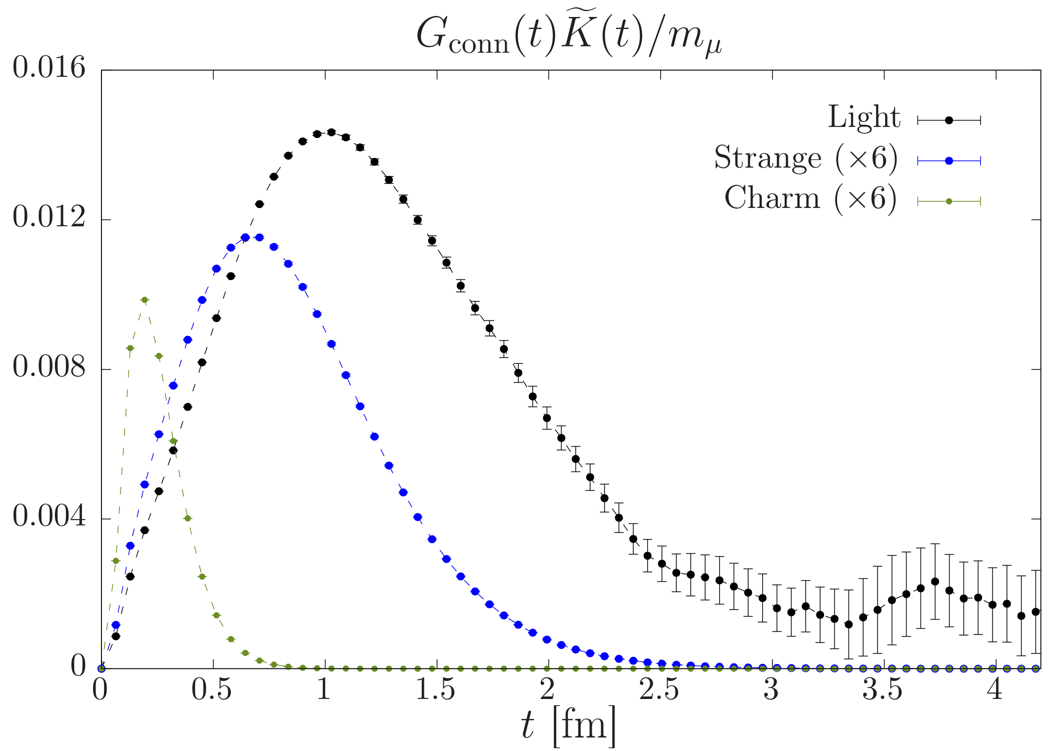
Charm quark



$$a_\mu^{\rm hvp,c} = (14.67 \pm 0.46 \pm 0.05) \times 10^{-10}$$

$$\to 0.07\%$$

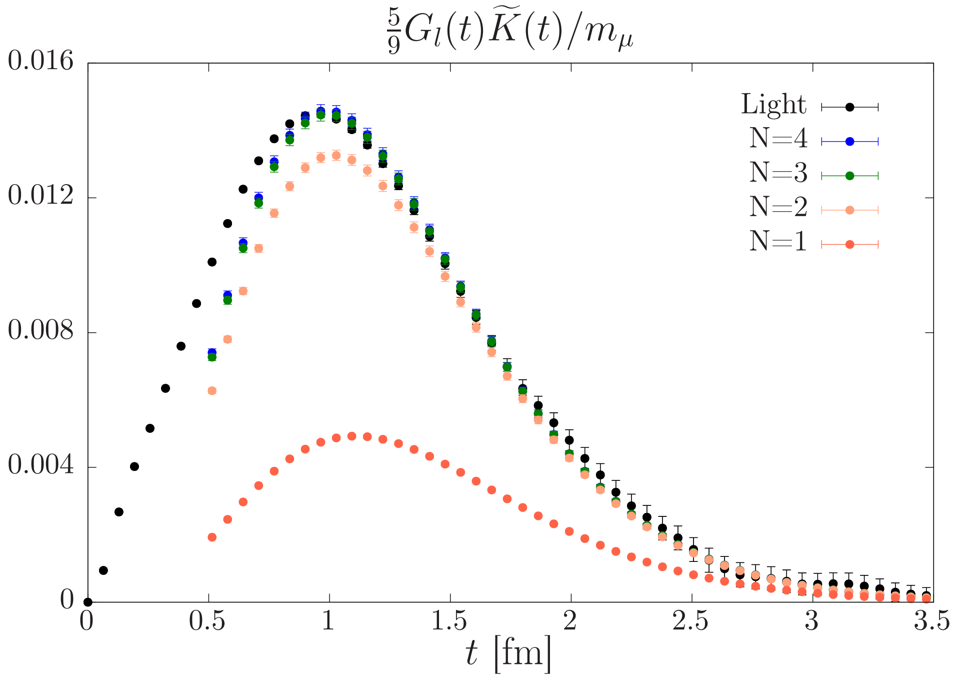
- $\tilde{y} \sim m_\pi^2$
- Error dominated by the scale setting $\to$ will be reduced in the future
- Charm quark : large discretization effects for the local discretization of the vector current
- One new ensemble at the finest lattice spacing $(m_\pi = 175$ MeV$)$


$G_{\mathrm{conn}}(t)\widetilde{K}(t)/m_\mu$
0.016
0.012
0.008
0.004
0
Light
Strange (×6)
Charm (×6)
0
0.5
1
1.5
2
2.5
3
3.5
4
t [fm]

- The vector correlator has a spectral decomposition :

$$G_l(t) = \sum_n |A_n|^2 \, e^{-E_n t}, \quad E_n = 2\sqrt{m_\pi^2 + k^2}$$

- E_n and A_n have been computed on the lattice using a dedicated spectroscopy analysis [Andersen et al. '19]

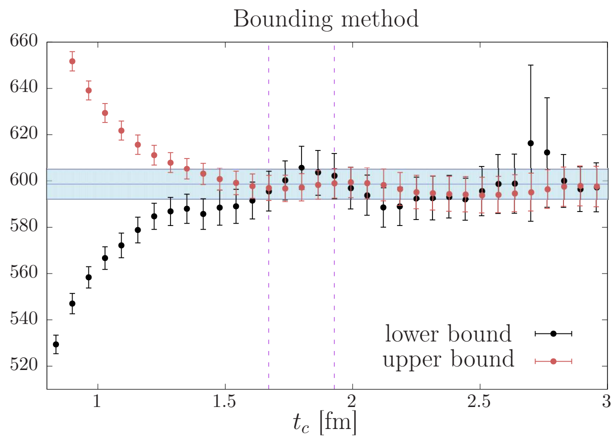


- Bounding method : [BMW '17] [RBC '18]

$$0 \le G_l(t) \le G_l(t_c)\, e^{-E_0(t-t_c)}\,, \quad t \ge t_c$$

$\rightarrow E_0$: ground state

$$a_\mu^{\mathrm{hvp},l} = 604.2(7.4) \times 10^{-10}$$

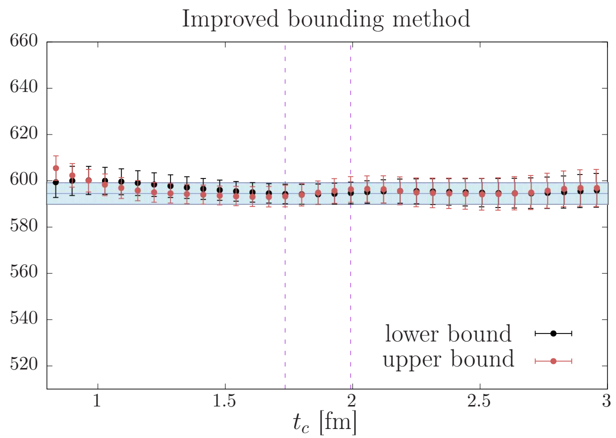


- Improved bounding method :

$$\widetilde{G}_l(t) = G_l(t) - \sum_{n=0}^{N} \frac{|A_n|^2}{2E_n} e^{-E_n t}$$

$$0 \le \widetilde{G}_l(t) \le \widetilde{G}(t_c)\, e^{-E_{N+1}(t-t_c)}\,, \quad t \ge t_c$$

$$a_\mu^{\mathrm{hvp},l} = 597.7(5.0) \times 10^{-10}$$



- The iso-vector correlator in **infinite volume**

$$G^{I=1}(t,\infty) = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, \omega^2\, \rho(\omega^2)\, e^{-\omega t}\,, \quad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1-\frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2 \qquad (1)$$

- **In finite volume** the correlator is given by the spectral decomposition

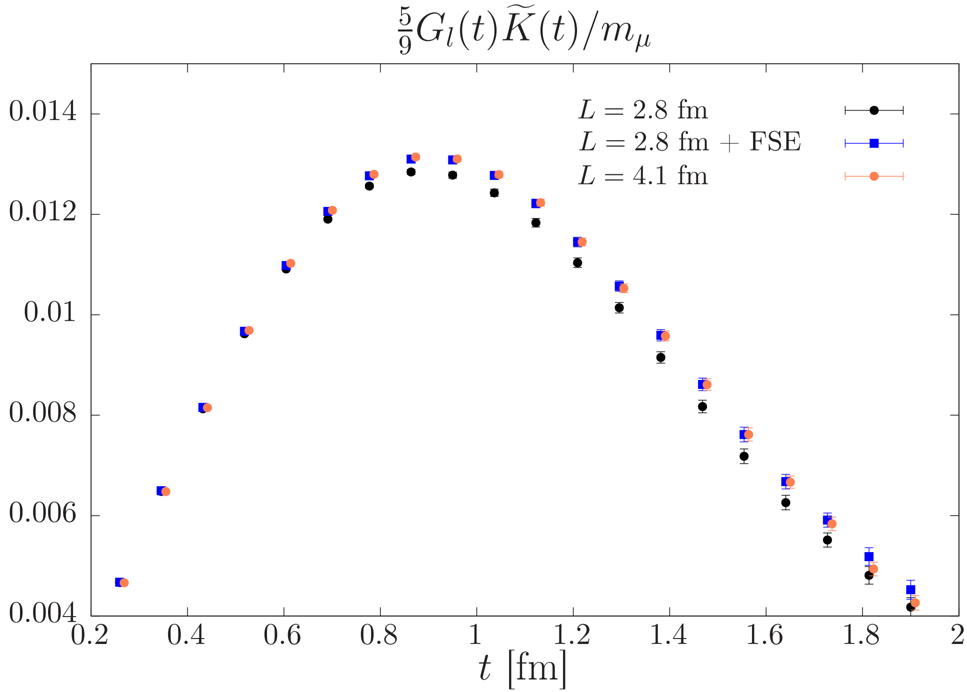
$$G^{I=1}(t,L) = \sum_i |A_i|^2\, e^{-E_i t}\,, \quad E_i = 2\sqrt{m_\pi^2 + k_i^2} \qquad (2)$$

$\rightarrow$ Discrete energy levels $\leftrightarrow$ $\pi\pi$ phase shift [Luescher '91]

$\rightarrow$ Overlaps A_i in finite volume $\leftrightarrow$ timelike pion form factor [Meyer '11]

- **Key ingredient** : timelike pion form factor computed on the same ensembles [Andersen et al. '18]

Finite-size correction = (1)−(2)



- Our procedure works remarkably well at $m_\pi = 280$ MeV
- For a subpercent precision, finite size effects are important at the physical pion mass !

$$m_\pi L = 4 \ \ (L = 6.2 \text{ fm}) \quad \Rightarrow \quad \Delta a_\mu = 22.6 \times 10^{-10} \ \ (3\%)$$

- Standard kernel : scale-setting uncertainty $\frac{\delta a_\mu^{\rm hvp}}{a_\mu^{\rm hvp}} \approx 2\,\frac{\delta a}{a}$ [M. Della Morte et al. '17]

  → solution : work with $m_\mu / f_\pi = \text{const.}$

  → idea first proposed by ETMC to get a milder chiral extrapolation [with $m_\mu / m_\rho = \text{const.}$]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty {\rm d}\left(\frac{t}{a}\right) \left(\frac{f_\pi^{\rm phys}}{m_\mu^{\rm phys}}\right)^2 \frac{1}{(af_\pi)^2}\,\widetilde{K}\left(\frac{t}{a}\cdot\frac{m_\mu^{\rm phys}}{f_\pi^{\rm phys}}\cdot af_\pi\right)\, a^3\, G\left(\frac{t}{a}\right)$$

- Massless pions : $G(t) = 1/(24\pi^2 |t|^3)$ and the QED kernel reduces to $K(t) \sim 2\pi^2 t^2$

$$a_\mu^{\rm hvp} \sim \frac{\alpha^2}{24\pi^2} \log\frac{m_\mu^2}{4m_\pi^2}, \qquad m_\pi \to 0,\ m_\mu \text{ fixed}$$

[Golterman '17]

  → not so relevant at our pion masses ($m_\pi \in$ [130 - 420] MeV)

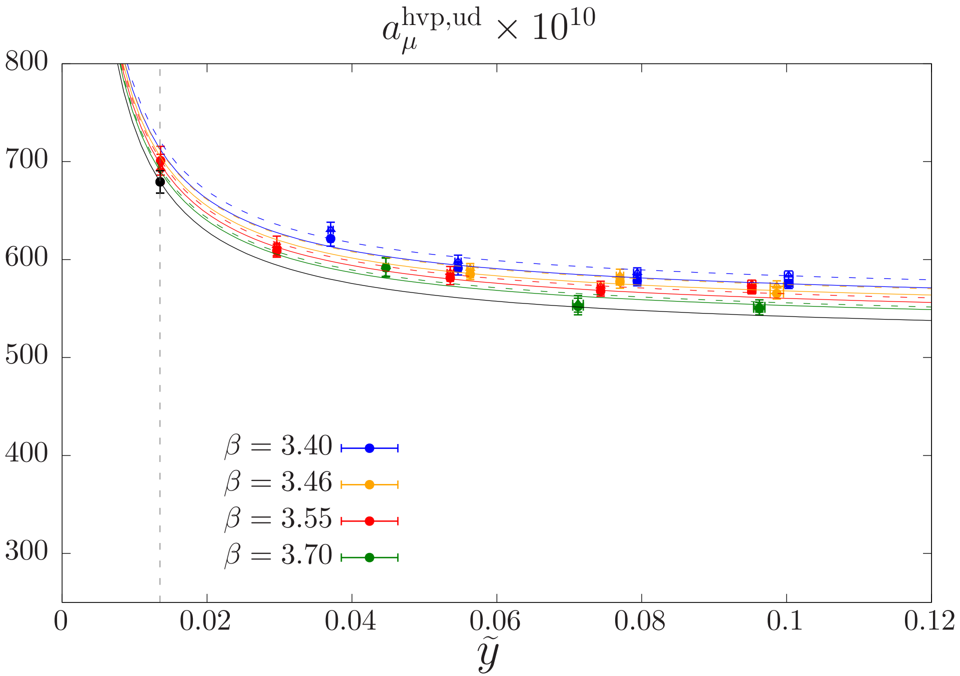
- $m_\mu \ll m_\pi$ and both are small compared to the ρ meson mass

$$a_\mu^{\rm hvp} \sim \frac{\alpha^2}{90\pi^2}\frac{m_\mu^2}{4m_\pi^2}, \qquad m_\mu \ll m_\pi \ll m_\rho.$$

  → the overall magnitude is enhanced by the (squared) timelike pion form factor

$$\text{Fit 1} : \quad a_\mu^{\text{hvp},l}(a,\widetilde{y},d) = a_\mu^{\text{hvp},l} + \delta_d\, a^2 + \gamma_1\, (\widetilde{y} - \widetilde{y}_{\text{exp}}) + \gamma_2\, (\log \widetilde{y} - \log \widetilde{y}_{\text{exp}})$$

$$\text{Fit 2} : \quad a_\mu^{\text{hvp},l}(a,\widetilde{y},d) = a_\mu^{\text{hvp},l} + \delta_d\, a^2 + \gamma_5\, (\widetilde{y} - \widetilde{y}_{\text{exp}}) + \gamma_6\, (1/\widetilde{y} - 1/\widetilde{y}_{\text{exp}})$$

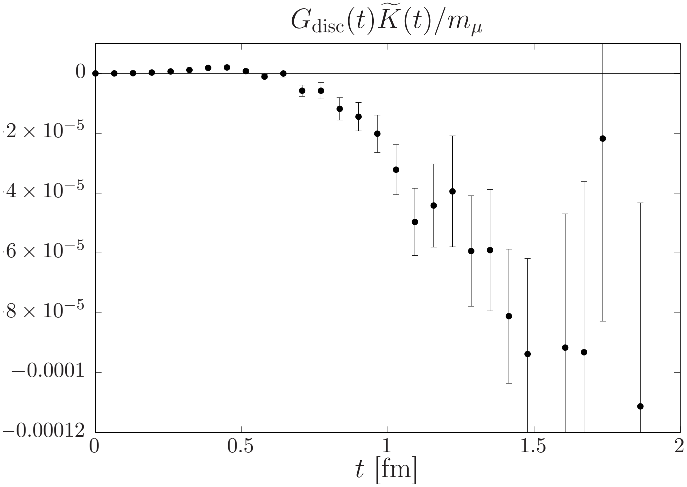


$$\widetilde{y} = m_\pi^2/(16\pi^2 f_\pi^2)$$

$$a_\mu^{\text{hvp},l} = (674 \pm 12 \pm 5) \times 10^{-10}$$

→ We are accumulating more statistics at the physical point

→ We will add a new ensemble at $m_\pi = 175$ MeV (finest lattice spacing)



- Expensive calculation
- Signal is lost at early times ($m_\pi = 280$ MeV)
- No spectral decomposition

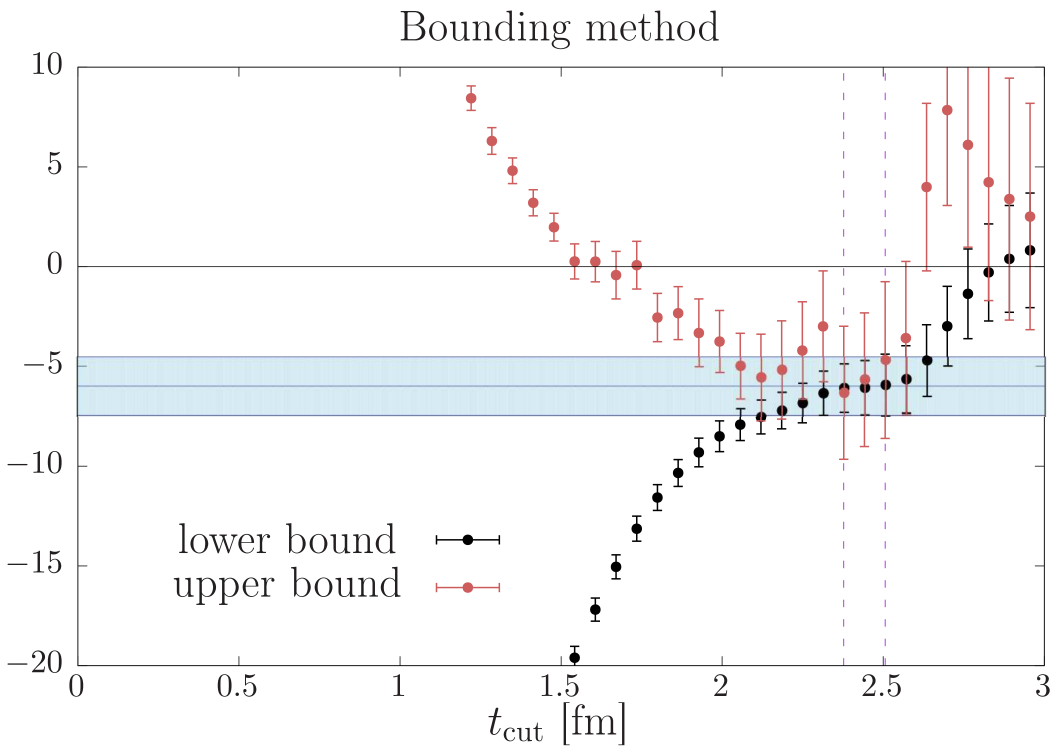
▶ Solution : isospin decomposition instead of flavor decomposition

$$G(t) = G^{I=1}(t) + G^{I=0}(t) \quad \text{with} \quad \begin{aligned} G^{I=1}(t) &= \frac{1}{2} G_l(t) \\ G^{I=0}(t) &= \frac{1}{18} G_l(t) + \frac{1}{9} G_s(t) + \frac{4}{9} G_c(t) + G_{\mathrm{disc}}(t) \end{aligned}$$

▶ Apply the bounding method to the isoscalar contribution (admits a spectral decomposition)

$$0 \leq G^{I=0,\not{d}}(t) \leq G^{I=0,\not{d}}(t_c) e^{-m_\rho (t - t_c)}, \qquad t \geq t_c.$$

▶ Subtract the (precisely known) connected contribution

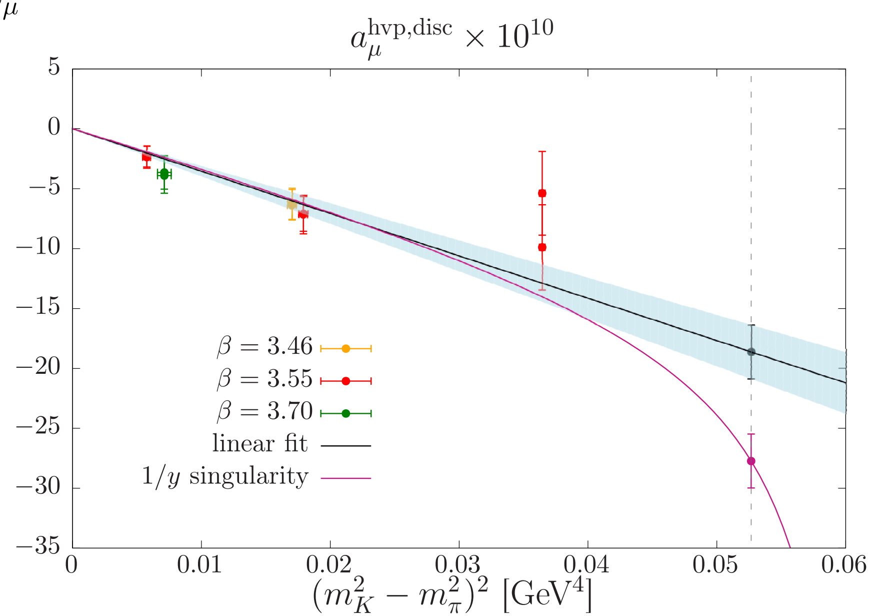

Bounding method
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
t_{cut} [fm]
lower bound
upper bound

$$a_\mu^{\mathrm{hvp,\,disc}}(\Delta_2) = \gamma_8 \Delta_2^2 - \frac{\alpha^2 m_\mu^2}{3240\pi^2} \cdot \frac{3}{2}\left[\frac{1}{\hat{M}^2 - \Delta_2} - \frac{\Delta_2}{\hat{M}^4} - \frac{1}{\hat{M}^2}\right]$$

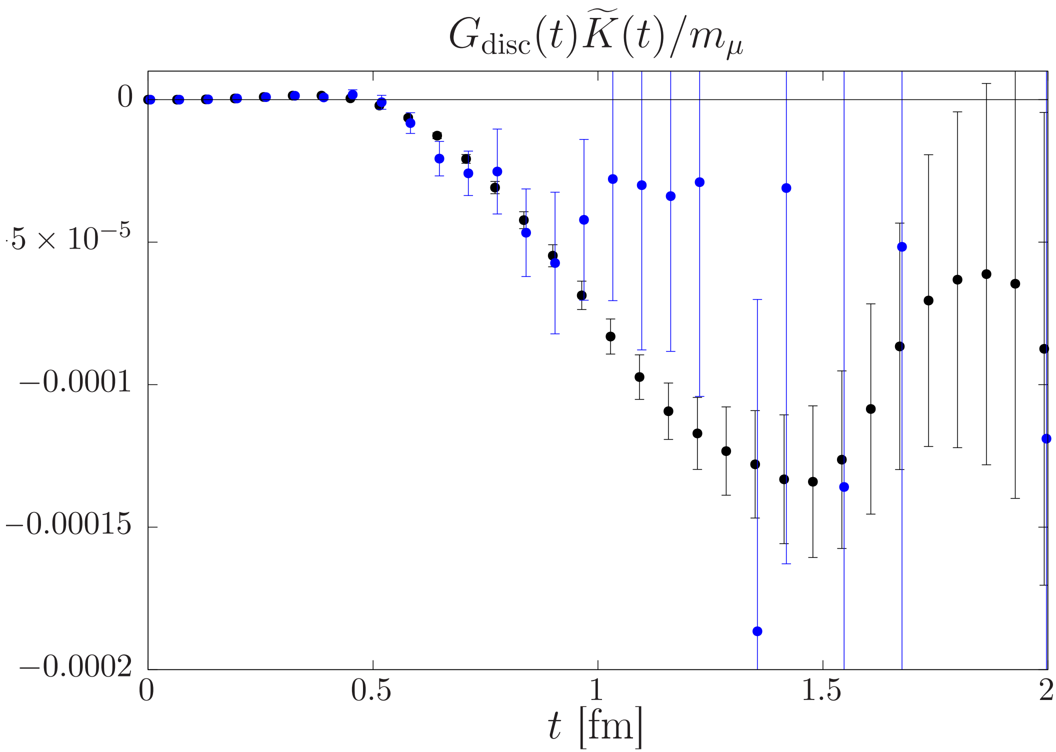
- $\Delta_2 \equiv m_K^2 - m_\pi^2$: vanishes if $m_s = m_s$
- the singular behavior of the light connected contribution in the isoscalar part must be compensated by the disconnected contribution
- Finite-size effects : $-\frac{1}{10} \times \Delta a_\mu^{\mathrm{FSE,light}}$

$$a_\mu^{\mathrm{hvp,\,disc}} \times 10^{10} =$$

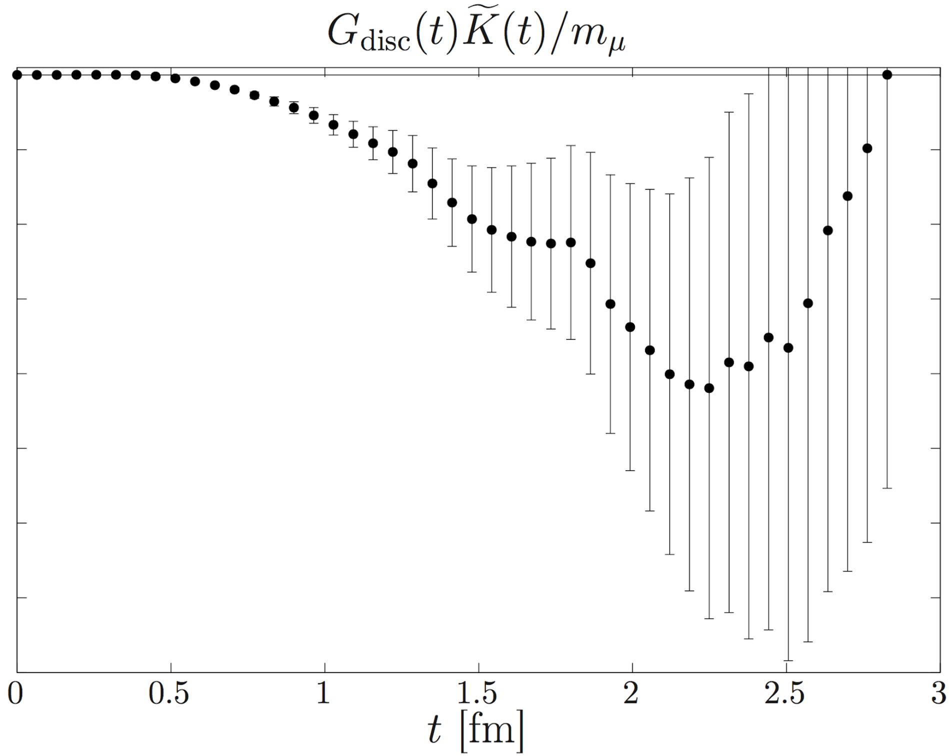
$$(-23.2 \pm 2.2 \pm 4.5)$$

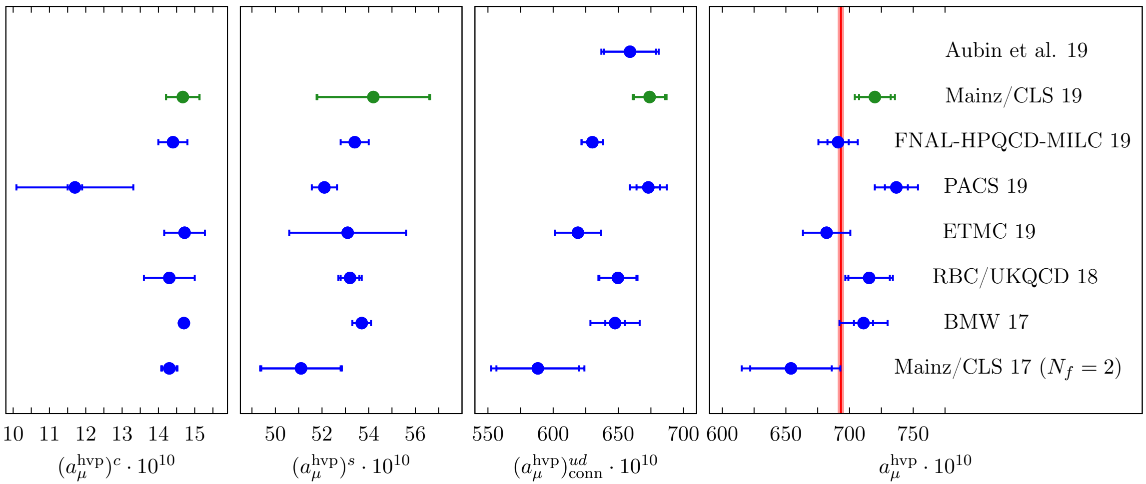


- Code faster + theoretical improvements [Talk by T. Harris @ Lattice '19] → K. Ottnad
- Much better signal : $m_\pi = 200$ MeV with ≈ twice more statistics



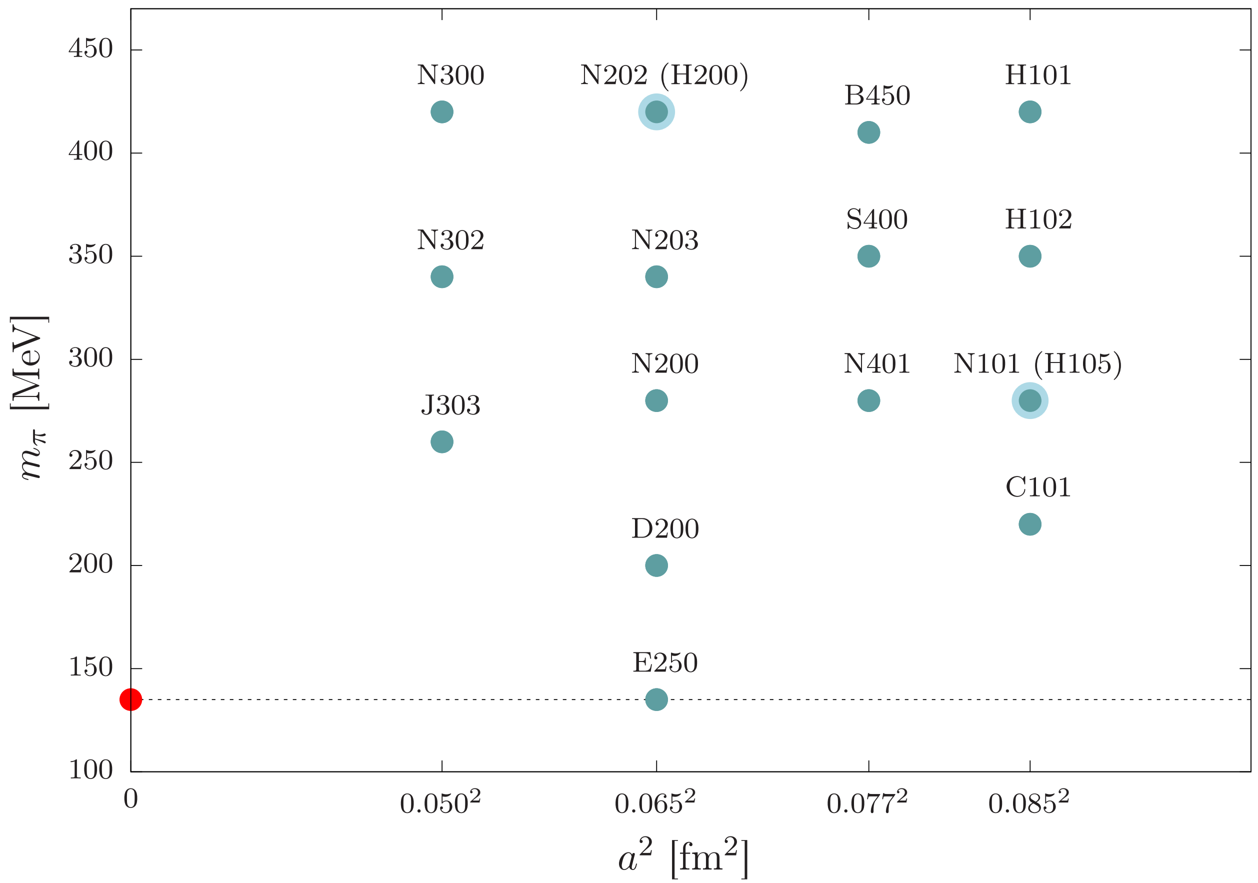
- Code faster + improvement [Talk by T. Harris @ Lattice '19] $\rightarrow$ K. Ottnad
- First results at the physical point (plan to increase stat by $\times$ 8)





$$a_\mu^{\rm hvp} = (720.0 \pm 12.4_{\rm stat} \pm 9.9_{\rm syst}) \cdot 10^{-10} \qquad (\pm\ 2.2\%)$$

- Increase the statistical precision
  - → Brute force is probably not the good option : spectroscopy can help a lot
  - → We use a dedicated spectroscopy analysis to reconstruct the vector correlator
- Sophisticated formalism available to correct for finite-size effects
- New results for the disconnected contribution
  - → Isosalar/isovector decomposition maybe more suitable
- Isospin-breaking corrections : in progress [A. Risch @ Lattice '18]

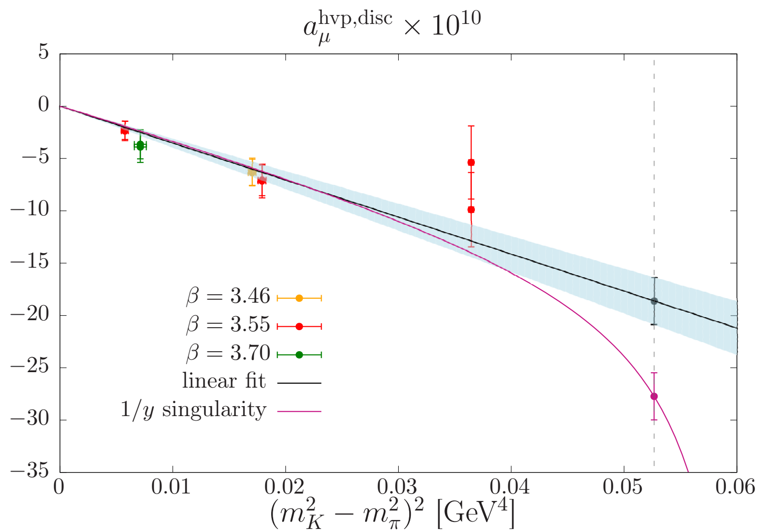

N300
N202 (H200)
B450
H101
N302
N203
S400
H102
N200
N401
N101 (H105)
J303
C101
D200
E250
450
400
350
300
250
200
150
100
0
0.050^2
0.065^2
0.077^2
0.085^2
m_π [MeV]
a^2 [fm^2]

- Vanishes exactly when $m_l = m_s$ → extrapolation with $\Delta_2 \equiv m_K^2 - m_\pi^2$
- Remember the isovector contribution is singular when $m_\pi \to 0$

→ singular behavior of the light connected contribution in the isoscalar part must be compensated by the disconnected contribution

→ Enforce this behavior in our fit *Ansatz* (we have $\hat{M}^2 \equiv \frac{1}{2}m_\pi^2 + m_K^2 = \text{const.}$)

$$a_\mu^{\text{hvp, disc}}(\Delta_2) = \gamma_8 \Delta_2^2 - \frac{\alpha^2 m_\mu^2}{3240\pi^2} \cdot \frac{3}{2}\left[\frac{1}{\hat{M}^2 - \Delta_2} - \frac{\Delta_2}{\hat{M}^4} - \frac{1}{\hat{M}^2}\right]$$



- Finite-size effects : $-\frac{1}{10} \times \Delta a_\mu^{\text{FSE,light}}$

$$a_\mu^{\text{hvp, disc}} = (-23.2 \pm 2.2 \pm 4.5) \times 10^{-10}$$

→ systematic error on this contribution must be addressed properly

→ more statistics needed