Finite size effects studies for the hadronic light-by-light scattering contribution to muon $g - 2$

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11th September 2019

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European Research Council
Established by the European Commission

Outline

1. Introduction
2. Theory computation
3. Comparison with lattice results
4. Summary and outlook

Introduction

- **Motivation** : long-ranged QED + poor signal at long distances in lattice simulations

- Master equation for computing a_μ^{HLbL} on the lattice [J. Green *et al.*, Lattice 2015]

$$a_\mu^{\mathrm{HLbL}} = -\frac{me^6}{3}\int_{|y|}\int_x 2\pi^2|y|^3\mathcal{L}_{[\rho,\sigma]\mu\nu\lambda}(x,y)\underbrace{\int_z z_\rho\langle V_\mu(x)V_\nu(y)V_\sigma(z)V_\lambda(0)\rangle}_{-i\hat{\Pi}_{[\rho,\sigma]\mu\nu\lambda}}$$

- Lattice practitioner tricks :
  - Modify the kernel to reduce the systematic errors (allowed by current conservation), we use

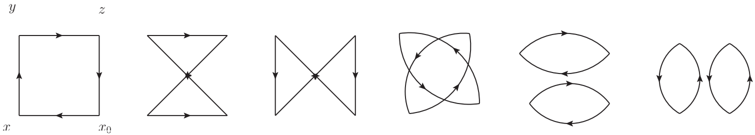
$$\mathcal{L}^{(2;\lambda)} := \mathcal{L}(x,y) - \partial_\mu^{(x)}(x_\alpha e^{-\lambda m_\mu^2 x^2/2})\mathcal{L}_{[\rho\sigma]\alpha\nu\lambda}(0,y) - \partial_\nu^{(y)}(y_\beta e^{-\lambda m_\mu^2 y^2/2})\mathcal{L}_{[\rho\sigma]\mu\beta\lambda}(x,0)$$

  - Exploit translational invariance to compute fewer Wick-contractions in the LQCD computation

- **Approach** : compute analytically $i\hat{\Pi}$ using some model on the torus and do the 4-d x-integration numerically to compare the $|y|$-integrand

Theory computation

- $SU(3)_f$ as starting point :
  - Computations on the lattice are cheaper for us
  - Non-suppressed Wick contractions in $SU(3)_f$: fully-connected and (2+2)-disconnected



- Two ways to compute the QCD 4-pt function (*cf.* R.J. Hudspith's talk) :
  - **Method 1** : compute all the Wick contractions, need sequential propagators
  - **Method 2** : compute only the "easy" Wick contractions and do change of variables in the kernel (computationally cheaper)

$$a_\mu^{\text{conn}} \propto \int_{xyz} \left\{ \Big(\mathcal{L}(x,y) + \mathcal{L}_{\mu\leftrightarrow\nu}(y,x) - \mathcal{L}_{\mu\leftrightarrow\lambda}(x,x-y) \Big) z_\rho + \mathcal{L}_{\mu\leftrightarrow\lambda}(x,x-y) x_\rho \right\} \times$$ 

$$a_\mu^{\text{disc}} \propto \int_{xyz} \Big\{ \Big(\mathcal{L}(x,y) + \mathcal{L}_{\mu\leftrightarrow\nu} \Big) \times$$  $$+ \mathcal{L}(x,y) \times$$  $$\Big\}$$

Theory computation

- Theory predictions : **pion pole** and **charged-pion loop** are expected to give major contributions to the FSE
- **Question** : how to match different contractions to different Feynman diagram in a given model ?
- Partially-Quenched ChPT (PQChPT) can be used to match the ChPT computation to different Wick contraction in Lattice QCD (this idea has been used for the HVP case)

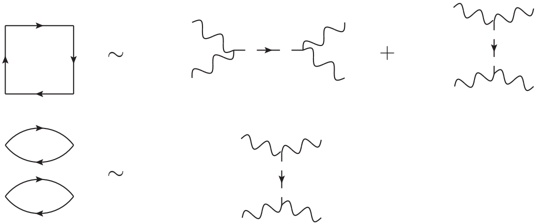
  [M. Della Morte and A. Jüttner, JHEP(2010)]
- With the Coordinated Lattice Simulations (CLS) $m_{\mathrm{light}} = m_{\mathrm{strange}}$ ensembles, one additional quark flavor is needed $\Rightarrow$ PQChPT with graded Lie-group $SU(4|1)$ as symmetry group

Theory computation

Mapping between the diagrams : pion-pole

- Use Vector-Meson-Dominance (VMD) model for the transition form factor

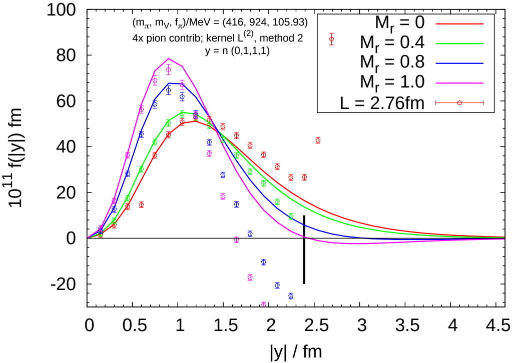
  see *eg.* [M.Knecht and A. Nyffeler, PRD 65 (2002)], parameters taken from [A. Gérardin, H. B. Meyer and A. Nyffeler, PRD 100 (2019)]
- Two ways are used to find the relevant pseudo-scalar exchange channels (with agreement) :
  - Neglect the self-contracted disconnected quark loop by large N_c argument
  - Consider Wess-Zumino-Witten term in PQChPT for $\pi^0\gamma\gamma$
    similar to [W. Detmold, B. C. Tiburzi, and A. Walker-Loud PRD 73 (2006)]
- Mappings :



- Match the charge factors to get the right weights

Theory computation

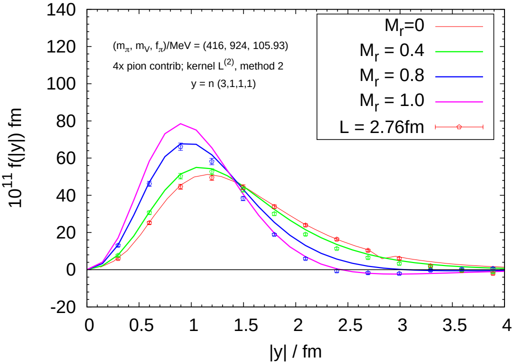
FSE from π^0-exchange : y-direction dependence



- $y = (0, n, n, n)$
- Computed at $m_\pi = 416$ MeV and $L = 2.76$ fm, with kernel $\mathcal{L}^{(2;\lambda)}$ with $\lambda = M_r^2$ with Method 2
- Severe FSE when approaching the boundary (note : the QED-kernel is not periodic)

Theory computation

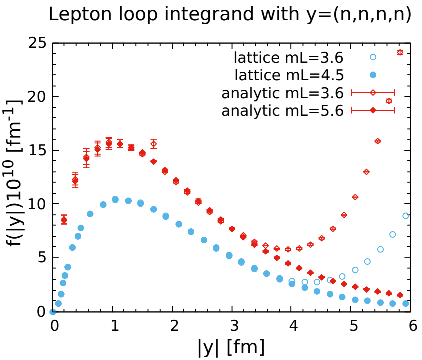
FSE from π^0-exchange : y-direction dependence



- $y = (3n, n, n, n)$
- Computed at $m_\pi = 416$ MeV and $L = 2.76$ fm, with kernel $\mathcal{L}^{(2;\lambda)}$ with $\lambda = M_r^2$ with Method 2
- One can go further in $|y|$ with mild finite size effects in the tail with much lighter FSE

Theory computation

Check for analytic method : lepton loop in free theory with method 2

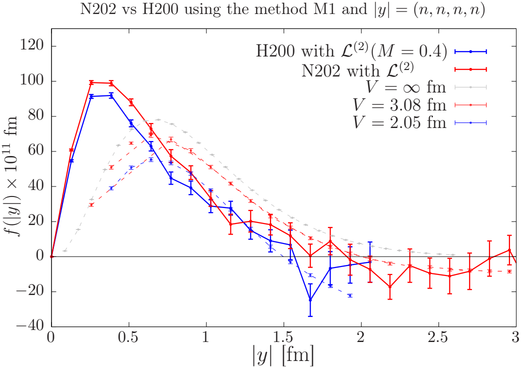


- Lepton loop in free theory with $m_{\text{lepton}} = m_\mu$: analytic approach vs. lattice (unit gauge)
- $y = (n, n, n, n)$, kernel $\mathcal{L}^{(2;0)}$
- L^4 boxes with $a = 0.1 fm \Rightarrow$ discretization effects are not totally negligeable
- Qualitative agreement of the analytical computation and lattice data for the free theory

Comparison with lattice results

Comparison with lattice results

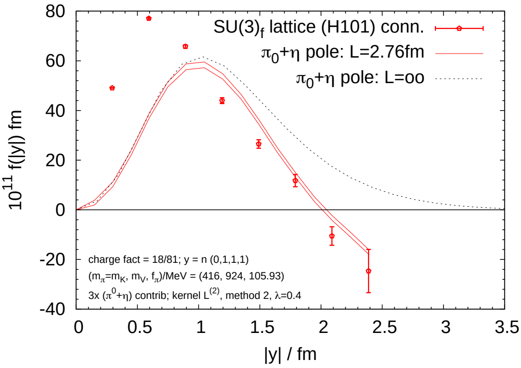
N202 and H200 from method 1 : connected contribution vs (π^0, η) exchange



- Lattice parameters (m_π, L) (MeV, fm) : H200 $(420, 2.05)$; N202 $(410, 3.08)$
- Direct check of the volume effects on the lattice
- Agreement with (π^0, η) exchange within sizeable uncertainties

Comparison with lattice results

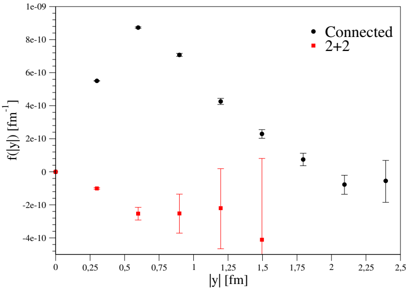
H101 from method 2 : connected contribution vs (π^0, η) exchange

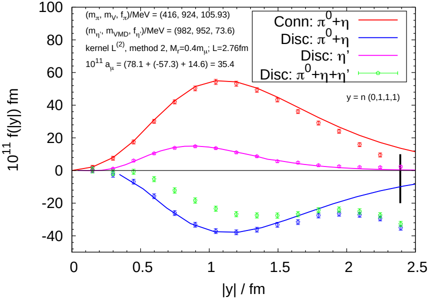


- $m_\pi = 416$ MeV $L = 2.76$ fm
- Kernel $\mathcal{L}^{(2;\lambda)}$ with $\lambda = 0.4$ is used ; y in the $(0, n, n, n)$ direction
- (π^0, η) exchange gives plausible description of the lattice data
- However, important negative contributions are missing in the tail : charged-pion loop (computed as scalar QED) appears to be tiny in the tail, what else could be responsible ?

Comparison with lattice results

H101 from method 2 : $(2+2)$ disconnected contribution vs (π^0, η) exchange





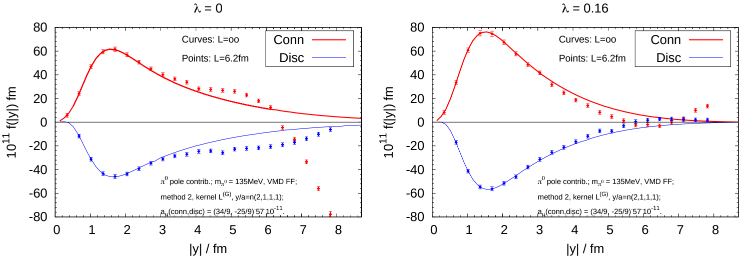
- Inclusion of η' for better prediction for the $(2+2)$-disconnected

  [A. Gérardin *et al.*, PRD 98 (2018)]

- Lattice data : 4000 measurements, kernel $\mathcal{L}^{(2;\lambda)}$ with $\lambda = 0.8$; y in the $y = (0, n, n, n)$ direction

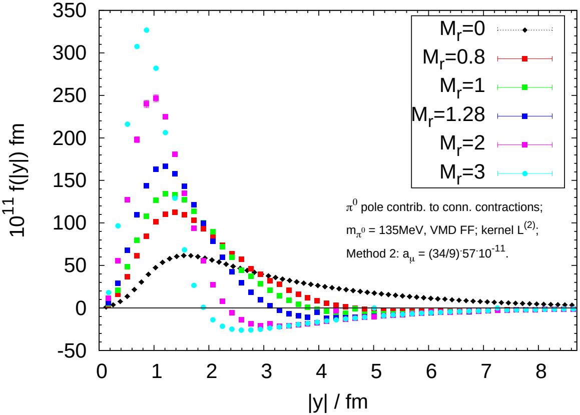
Summary and outlook

- Attempt to understand the behavior at long distance of the a_μ integrand using models: motivated by poor signals in lattice simulations
- Understand the mapping mechanism between Feynman diagrams in different models and the Lattice QCD Wick-contraction with the help of PQChPT
- Qualitative prediction for the FSE due to the choice of kernels and integration variable y
- Description still needs to be improved, especially on the missing negative contribution in the tail of the integrand (iso-vector scalar meson ?)
- Prediction for the physical pion ensemble (E250) : shows that the FSE might be under control



Back-up slides

Kernel $\mathcal{L}^{(G)}$ with different parameters



- π^0-exchange computed at physical pion mass and infinite volume
- Subtraction with different Gaussian masses helps to make the integrand short-ranged (with $\lambda = M_r^2$)
- Optimal at $M_r < 1$

Analytic computation of $i\hat{\Pi}$

- Handling a non-periodic function $f(z_\rho)$ on the lattice

$$\int_{-\infty}^{\infty} f(z)\Pi_3(y,z) \to \sum_{z=0}^{L-1} f([z])\Pi_3(y,z) \quad \text{with} \quad [z] = \begin{cases} z & \text{if} \quad z < \frac{L}{2} - 1 \\ z - L & \text{if} \quad z \geq \frac{L}{2} \end{cases} \tag{1}$$

- Starting point :

$$\int_0^L [z]f(z) = -i \sum_{q \neq 0} \frac{\hat{f}(q)}{q} \cos(\frac{qL}{2}) \tag{2}$$

where $\hat{f}$ is the discrete Fourier transform of f

$$\hat{f}(q) = \int \mathrm{d}z e^{-iqz} f(z) \tag{3}$$

we have

$$\int_z [z_\rho] e^{i(p-q)z} = -iV_3 \delta_{(p-q)_\perp,\vec{0}} \frac{L_\rho}{(p-q)_\rho} \cos(\frac{(p-q)_\rho L_\rho}{2}) \tag{4}$$

- Periodic and anti-periodic quantities computable using Poisson summation formula and Residue Theorem
- General form of the result for a four-point function
  - A term independent of the jump introduced to handle z_ρ
  - A term due to the jump
  - Sum over three 4-d and one 1-d "winding numbers"

Mapping different Wick contraction using PQ-theories

- Idea : introduce a quenched quark r and its ghost $\tilde{r}$ of the same mass as (u, d, s) to realize a specific Wick-contraction
- The partition function remains the same $\Rightarrow$ theory not modified
- Symmetry $\Rightarrow$ same propagator for the quenched quark as for the other quarks
- Example : fully connected

$$\begin{aligned}&\Big(\langle(\bar{u}\gamma_\mu d)(x)(\bar{d}\gamma_\nu s)(y)(\bar{s}\gamma_\sigma r)(z)(\bar{r}\gamma_\lambda u)(0)\rangle + h.c.\Big)\\&= 16\frac{\delta^4 \mathcal{Z}_{PQQCD}}{\delta A_\mu^{(ud,1)}(x)\delta A_\nu^{(ds,1)}(y)\delta A_\sigma^{(sr,1)}(z)\delta A_\lambda^{(ru,1)}(0)}\end{aligned} \tag{5}$$

- PQChPT as EFT $\Rightarrow$ same partition function as $\mathcal{Z}_{\mathrm{PQQCD}}$

Wess-Zumino-Witten term in PQChPT

- Effective action in presence of an external source [S. Scherer, Adv.Nucl.Phys. 27 (2003)]

$$S^{\mathrm{ext}}_{\mathrm{WZW}} = -\frac{i}{48}\int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(Z_{\mu\nu\rho\sigma}) \tag{6}$$

$$\begin{aligned} Z_{\mu\nu\rho\sigma} \supset & \mathcal{U}^L_\mu U^\dagger \partial_\nu r_\rho U l_\sigma - \mathcal{U}^R_\mu U \partial_\nu l_\rho U^\dagger r_\sigma \\ & - \mathcal{U}^L_\mu \mathcal{U}^L_\nu U^\dagger r_\rho U l_\sigma + \mathcal{U}^R_\mu \mathcal{U}^R_\nu U l_\rho U^\dagger r_\sigma \\ & + \mathcal{U}^L_\mu l_\nu \partial_\rho l_\sigma - \mathcal{U}^R_\mu r_\nu \partial_\rho r_\sigma \\ & \mathcal{U}^L_\mu \partial_\nu l_\rho l_\sigma - \mathcal{U}^R_\mu \partial_\nu r_r hor_\sigma \end{aligned} \tag{7}$$

- In order to get the 4-pt function that we are interested in, we set

$$l_\mu = r_\mu = v^a_\mu T^a \tag{8}$$

- $\mathrm{str}(T^a) = 0 \Rightarrow$ the only relevant term for $\pi^0 \to \gamma\gamma$ is thus

$$\frac{1}{96\pi^2 F_0}\mathrm{str}(T^a T^b T^c)\int \mathrm{d}^4 x \phi^c \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma} \tag{9}$$

Mapping between the diagrams : charged-pion loop

- Consider ChPT (point-like pions) ⇒ has contact terms
- Mappings :
  - Connected :