Simulating dark matter with the Schrödinger equation

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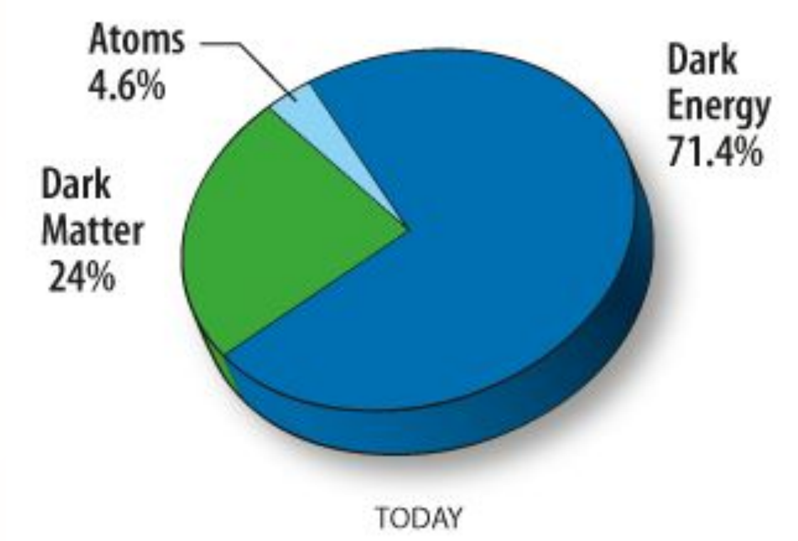
The (very) big picture

- Cosmology - the study of the origin and evolution of the Universe and the stuff in it

- Universe
  - Homogeneous at large scales
  - Expanded from a hot, radiation dominated state after inflation
- Stuff
  - ~70% dark energy
  - ~23% dark matter
  - ~ 7% everything else



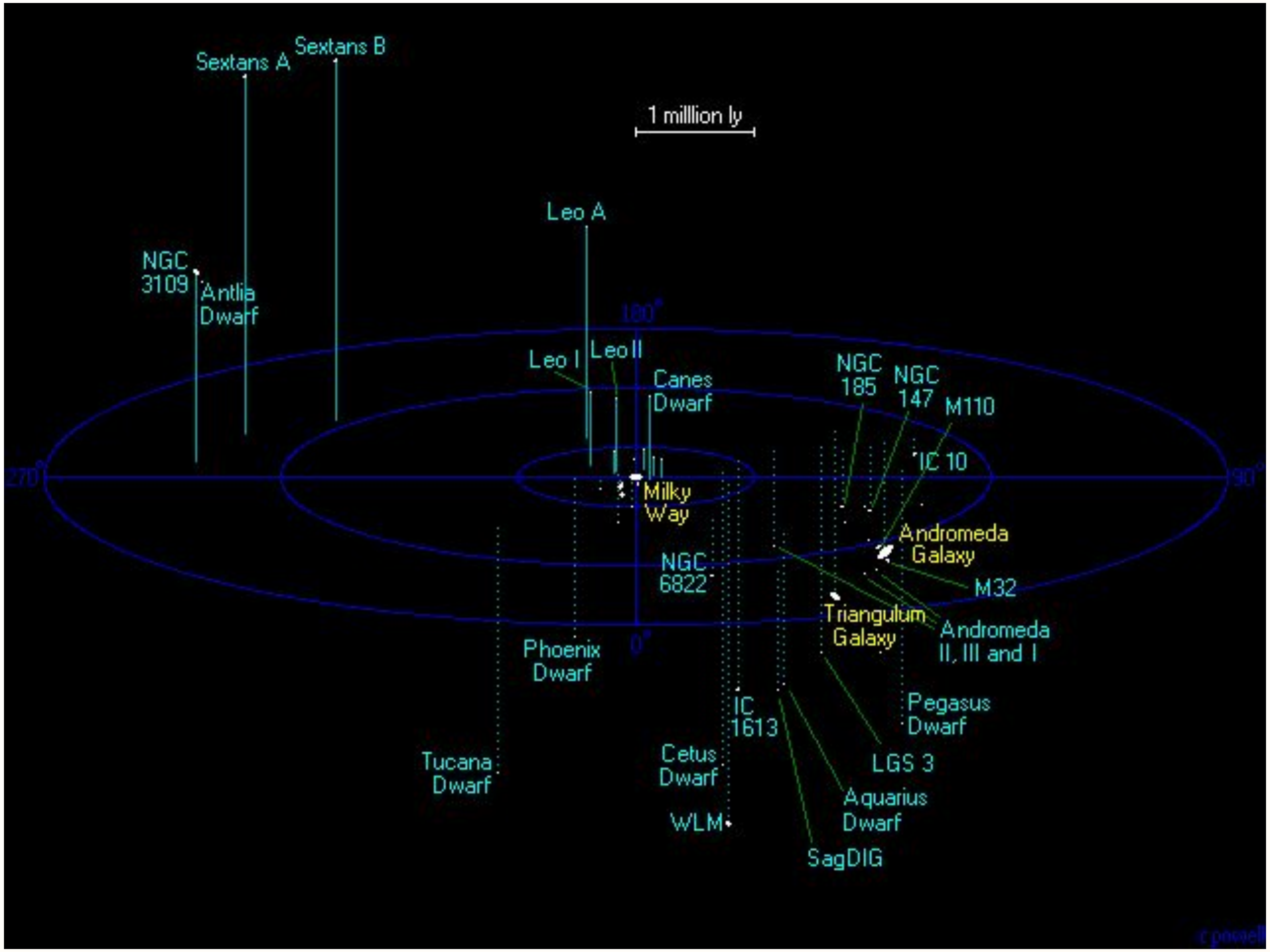
$$\left\{\Omega_b h^2, \Omega_c h^2, t_0, n_s, \Delta_R^2, \tau\right\}$$

Images: NASA

Large scales

- Universe is homogeneous at large scales. How large?

~ 1.0 Mpc →



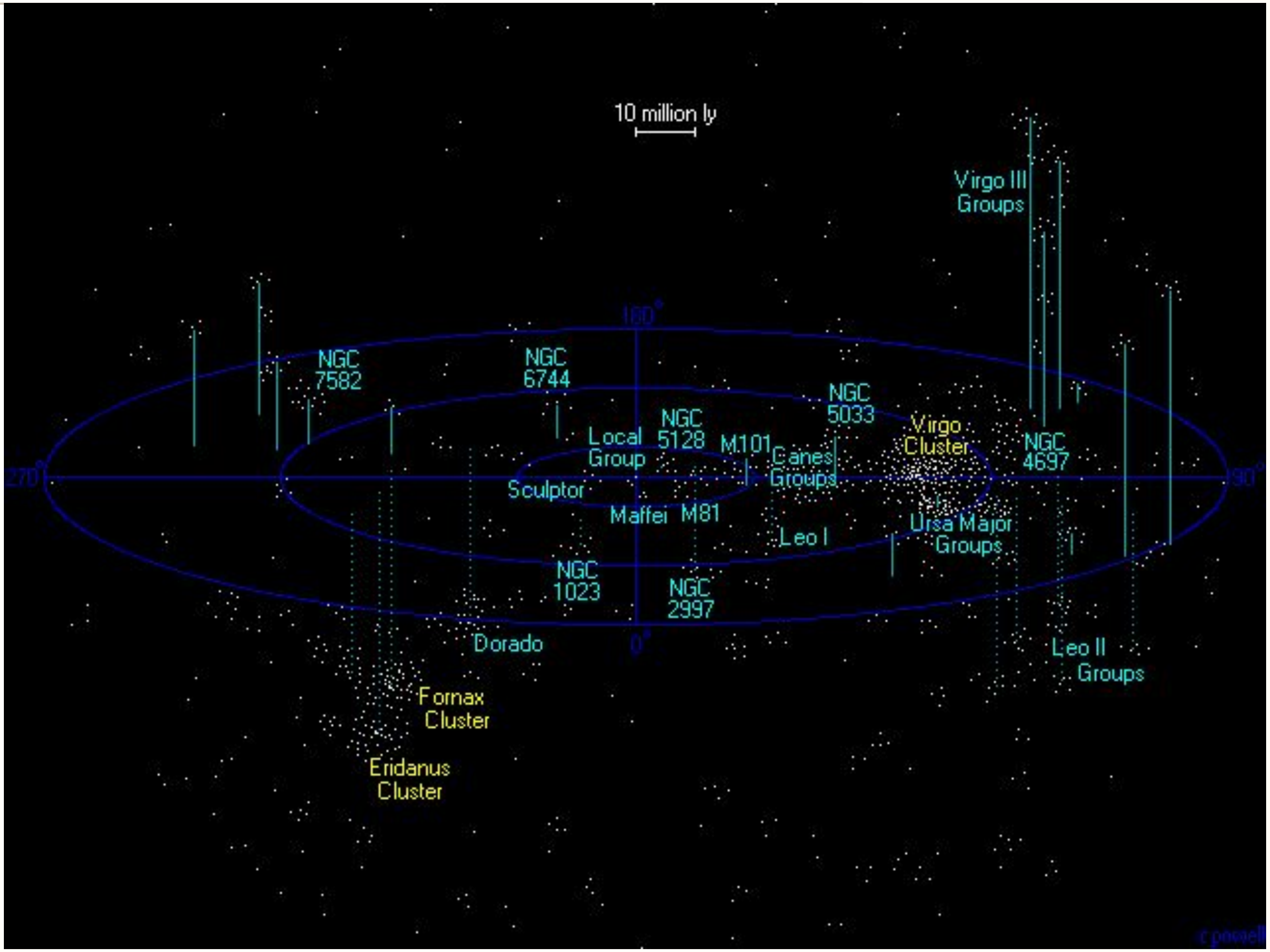
Images: atlasoftheuniverse

Large scales

- Universe is homogeneous at large scales. How large?

~ 30 Mpc

$$\frac{\delta M}{M} \sim 1$$



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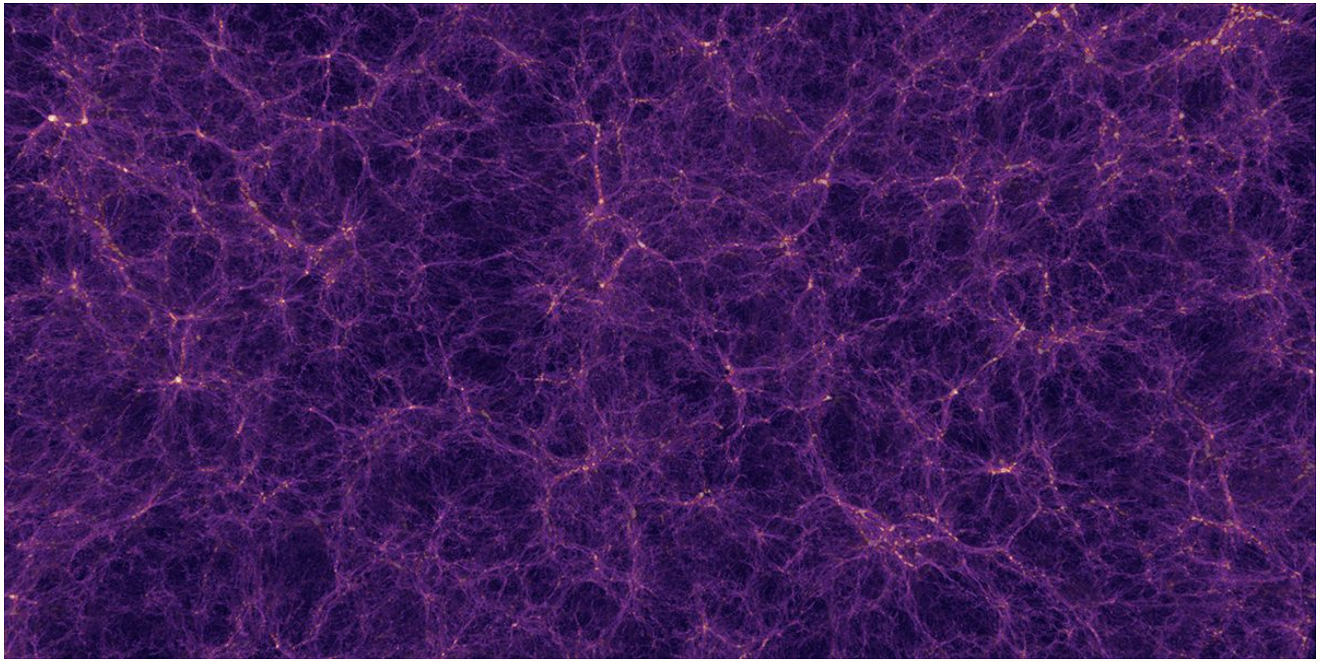


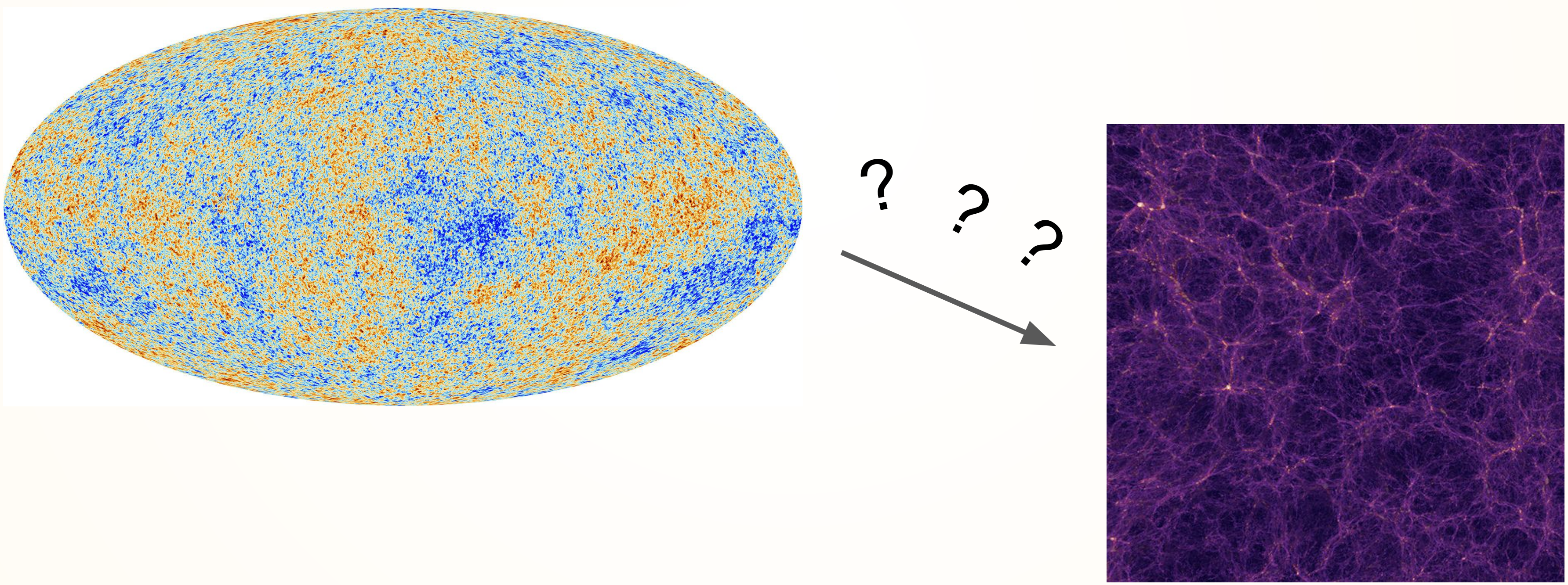
~ 4000 Mpc →

$$\frac{\delta M}{M} < 10^{-4}$$

Images: HACC team

Computational cosmology?

- Universe starts in a near uniform, hot, and dense state and evolves to have complex structures at large scales. How?



Images: ESA, HACC team

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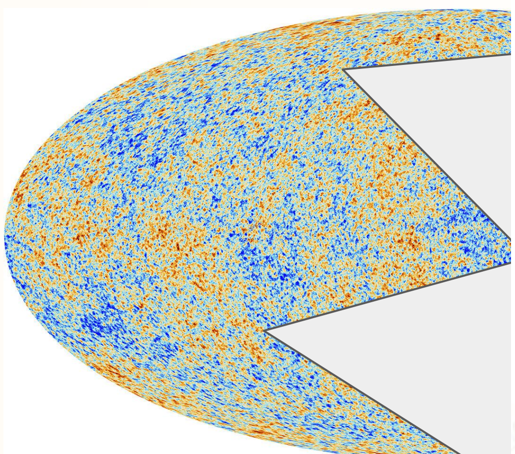

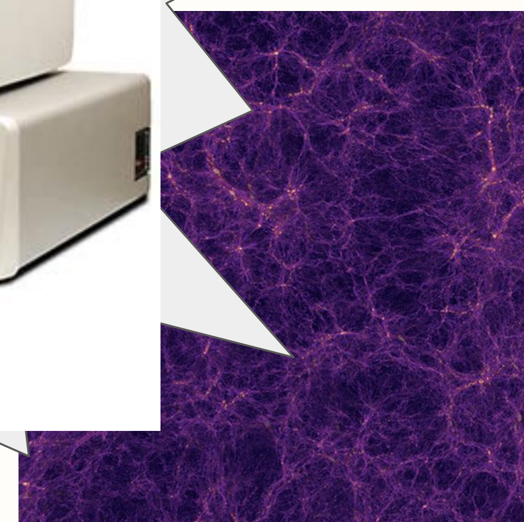

Images: ESA, HACC team

Motivation: BEC dark matter

- Consider the Hamiltonian of many interacting bosons in a condensate:

$$\hat{H} = \int dr \hat{\Psi}^\dagger(r) \left(\frac{\hbar^2}{2m} \nabla^2 + m\phi \right) \hat{\Psi}(r)$$
$$+ \frac{1}{2} \int dr dr' \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r') V_i(r - r') \hat{\Psi}(r') \hat{\Psi}(r)$$

- Under reasonable assumptions, the condensate wave-function evolves according to:

$$\frac{i\hbar}{a^{3/2}} \partial_t \left(a^{3/2} \psi(x,t) \right) = \frac{-\hbar^2}{2ma^2} \nabla^2 \psi(x,t) + m\phi\psi(x,t) + g|\psi(x,t)|^2$$

Moti… tion: BEC dark m… atter

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$$\hat{H} = \int \ldots$$

$$+ \ldots \hat{\;}(x)$$

All DM dynamics are contained in ψ

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Even more motivation:

- Write wavefunction as a complex amplitude and phase:

$$\psi = \sqrt{\rho(x,t)}e^{iS(x,t)/m}$$

- Two fluid-like equations from equating real and imaginary parts:

$$Q \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = -\nabla\phi - \frac{1}{m}\nabla Q$$

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We have a dictionary between the quantum and fluid formulations

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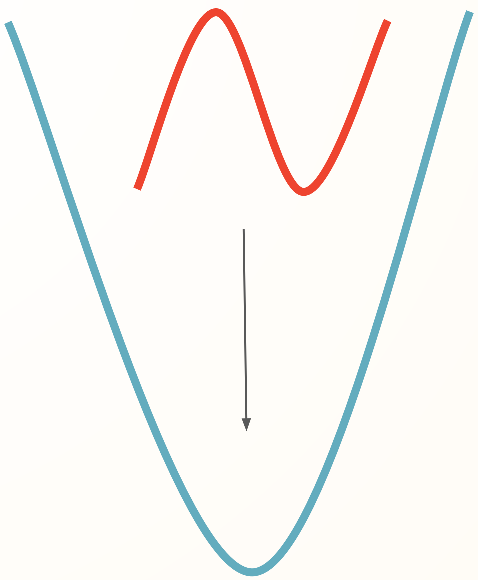
Simulating BEC dark matter: Size requirements

- Recall: $\frac{\delta M}{M} \sim 1$ averaged over scales of ~30 Mpc
  - This is the smallest box size we would want to simulate
- For a particle mass of 1e-22 eV, we expect a macroscopic de Broglie wavelength:

$$\frac{\lambda_{deb}}{2\pi} = \frac{\hbar}{mv} \sim 10 \text{ kpc}$$



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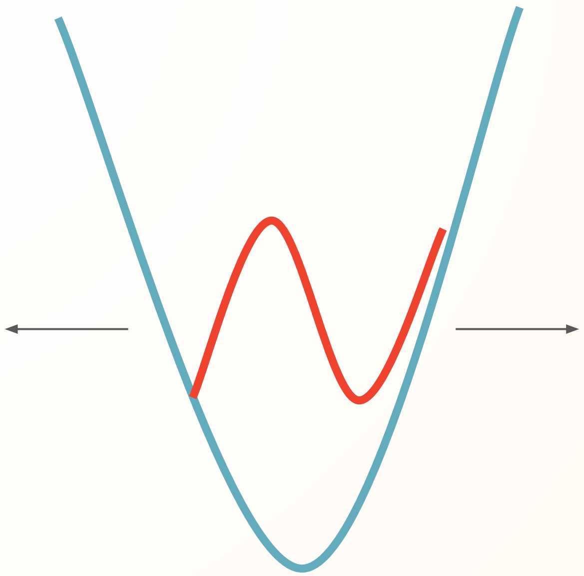
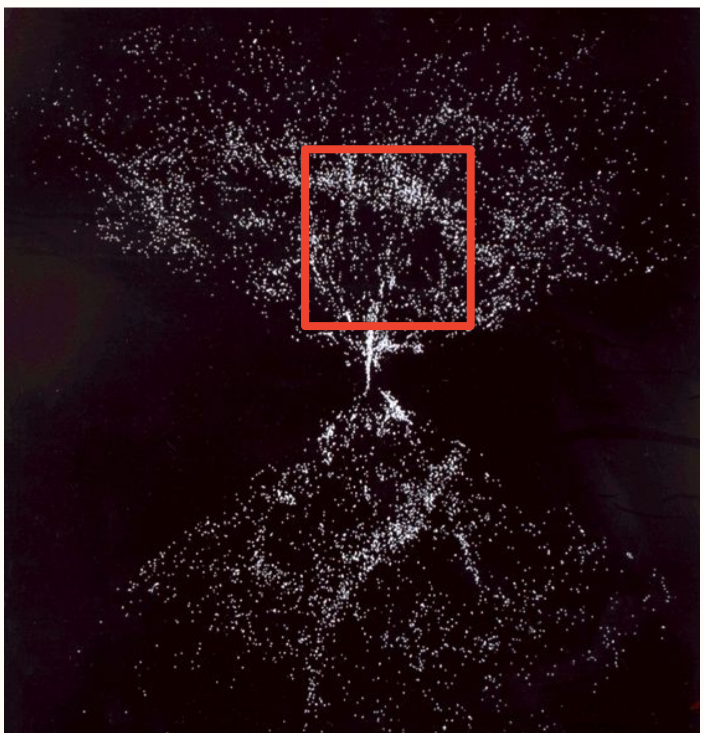
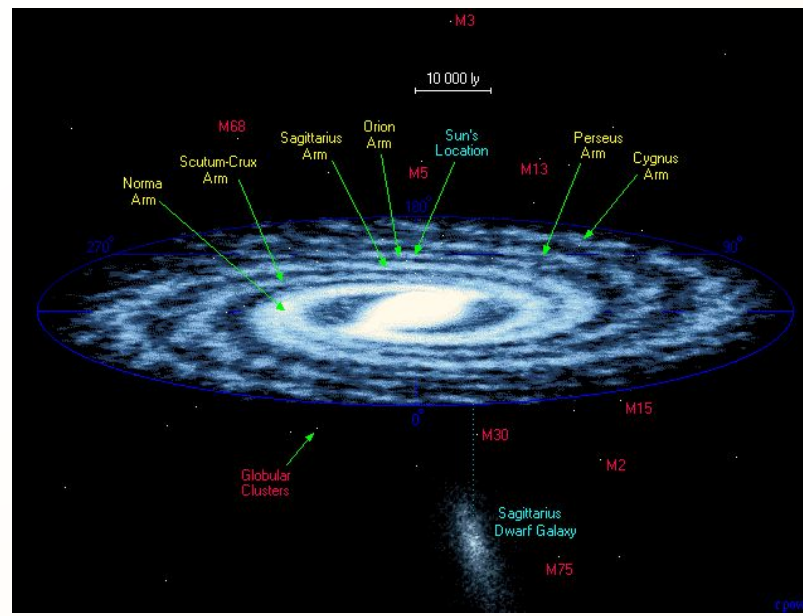


Simulating BEC dark matter: Size requirements

- We want to simulate a ~100 Mpc box with a resolution of ~10 kpc
  - This requires **~10,000^3 grid points.**





Images: CfA

The Plan:

- Avoid spatial error introduced by sampling the wavefunction with particle tracers and solve for the wavefunction exactly using a spectral method
    - Think of this as a “no-body” simulation

- Use SWFFT for FFTs
    - O(10) seconds for a $10k^3$ FFT - fast enough to make this approach feasible in the first place

https://xgitlab.cels.anl.gov/hacc/SWFFT

Method: Split operator spectral method

- Solving the Schrödinger equation gives:

$$\psi(a_0 + \Delta a) = \exp\left(-\frac{i}{\hbar}\int_{a_0}^{a_0+\Delta a} H(a')da'\right)\psi(a_0)$$

Where $H(a') = \hat{K}(a') + \hat{V}(a') = \dfrac{-i\hbar}{2ma^2\dot{a}}\nabla^2 + i\dfrac{m}{\dot{a}\hbar}\phi.$

- We can split, accruing a Δt^3 error:

$$\psi(a_0 + \Delta a) = \exp\left(i\frac{m}{\hbar}\phi\frac{\Delta t}{2}\right)\exp\left(i\frac{\hbar}{m}t\nabla^2\right) \times$$

$$\times \quad \exp\left(i\frac{m}{\hbar}\phi\frac{\Delta t}{2}\right)\psi(a_0) + \mathcal{O}(\Delta t^3)$$

Test: Gaussian Wavepacket:

- Algorithm:

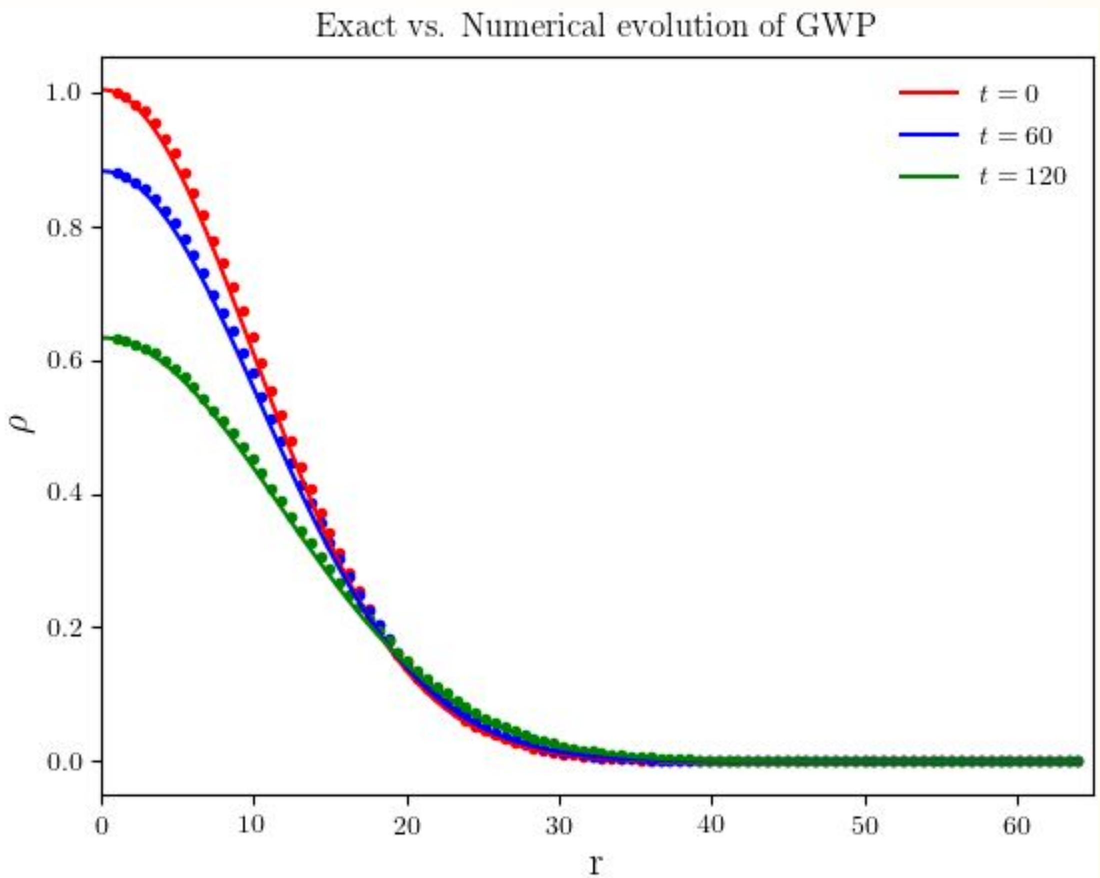
$$\mathrm{FFT}\left(\exp\left(V(x,\Delta T/2)\right)\psi(x,a_0)\right) = \hat{\psi}$$

$$\downarrow$$

$$\mathrm{FFT}^{-1}\left(\exp\left(\hat{K}(k,\Delta T)\right)\hat{\psi}(k)\right) = \psi_{int}$$

$$\downarrow$$

$$\exp\left(V(x,\Delta T/2)\right)\psi_{int} = \psi(x,t+\Delta T)$$



Initial Conditions

- The initial conditions are a Gaussian random field with a power spectrum given by:

$$P(k, z_{in}) = Bk^{n_s} T^2(k) D^2(z_{in})$$

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Captures all of the complicated physics until matter-domination

Growth function that scales the power spectrum from z_{eq} to z_{in}

Growth Function

- In ΛCDM, the growth function is given by a system of ODEs:

$$\frac{\partial D}{\partial a} = \frac{1}{\dot{a}}\dot{D}$$

$$\frac{\partial \dot{D}}{\partial a} = -2\frac{\dot{D}}{a} + \frac{3}{2}\frac{\Omega_m}{H a^4} D$$

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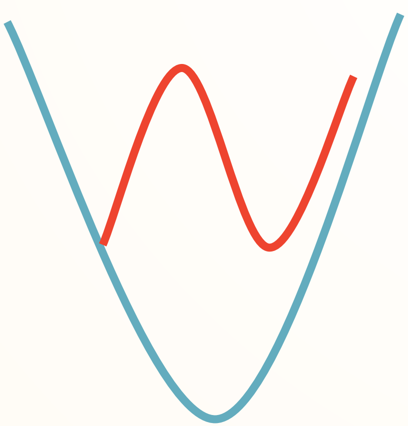
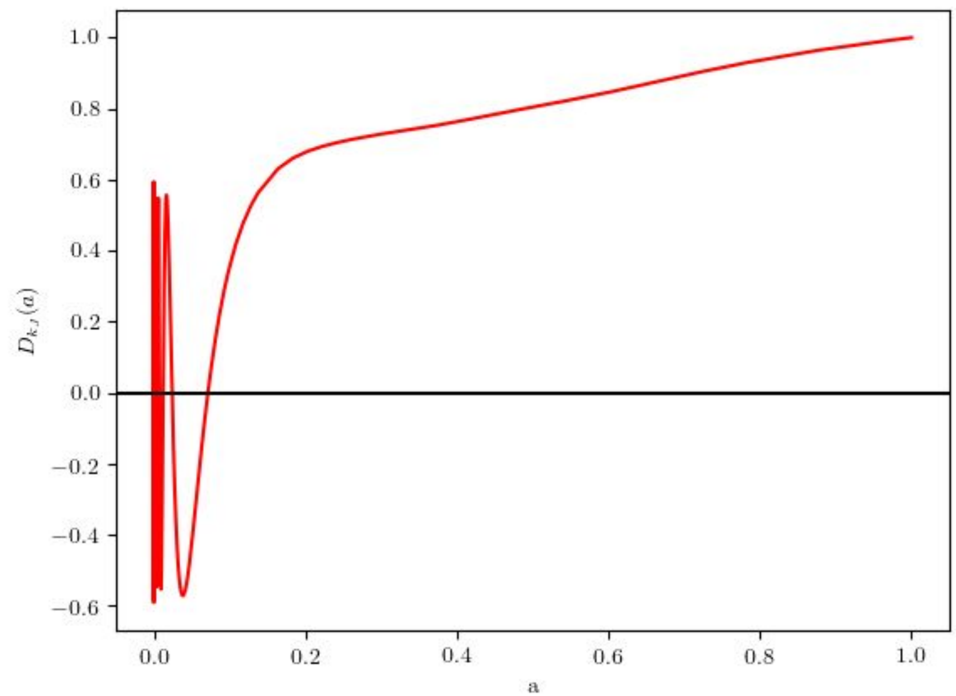
Growth Function

- Jean’s scale is the scale at which the quantum pressure balances gravity

$$k_J = (a/\beta)^{(1/4)}$$





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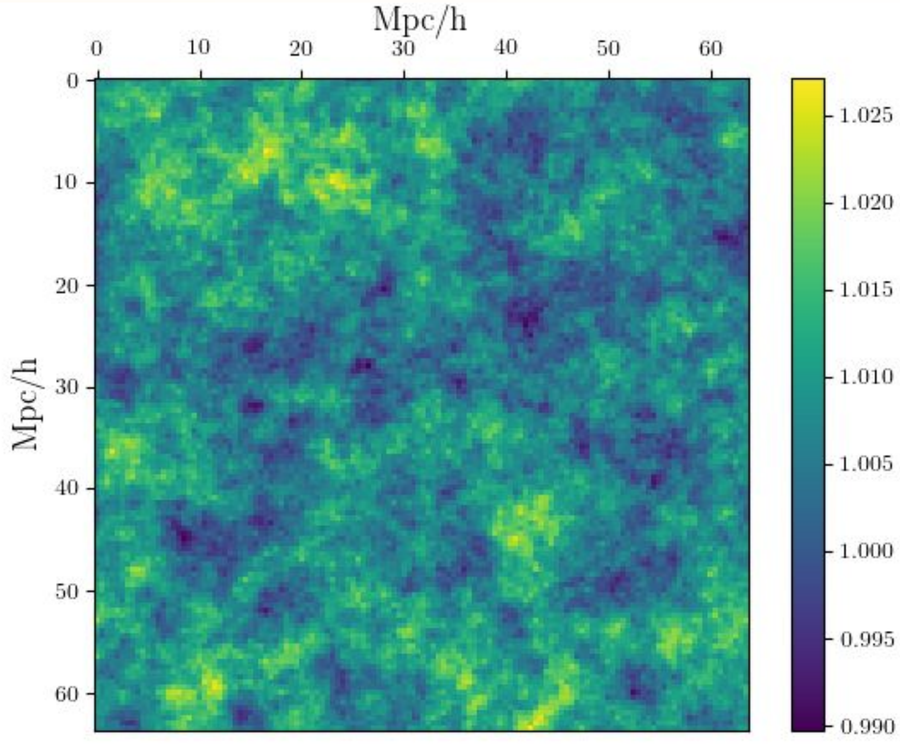

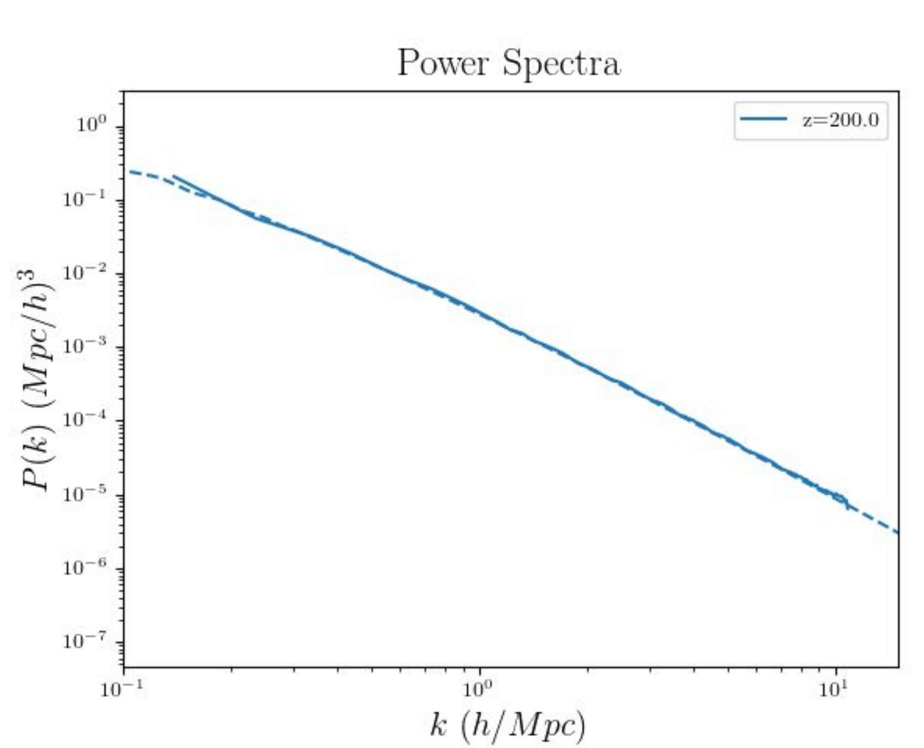

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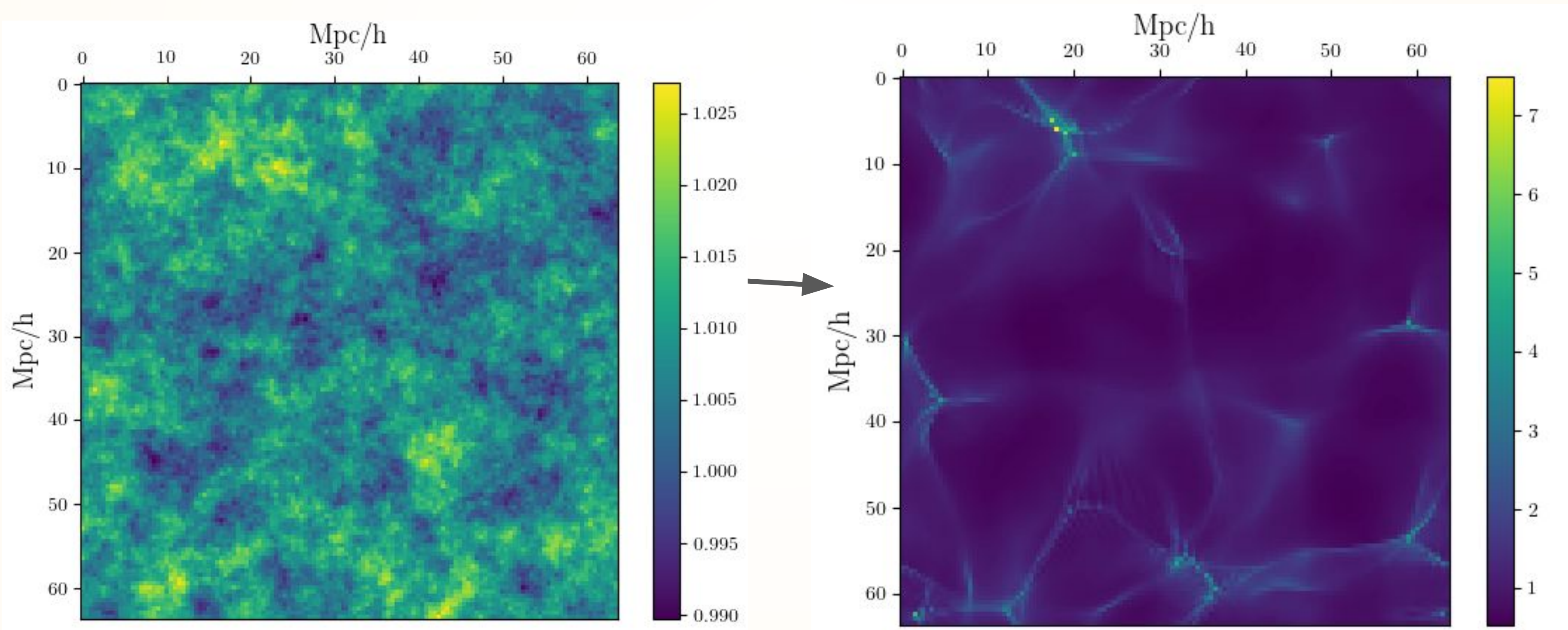

Images: ESA, HACC team

Future Work

- Incorporate background expansion (current problem area)
- Calculate halo profiles from cosmological sized boxes
    - NFW profiles?
- Small scale structure suppression

- And more...

Thank you!