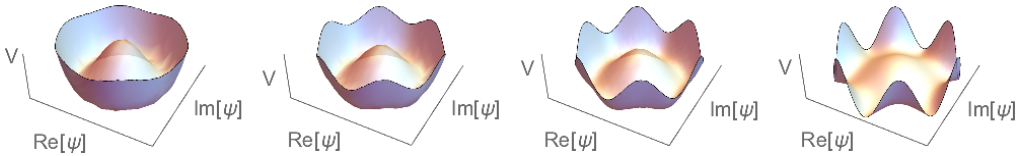
Some New Ideas in QFT and Condensed Matter Physics
or
The Generalized Landau Paradigm

John McGreevy (UCSD)

based on 2204.03045
and partly on work with
Nabil Iqbal (Durham)
2003.04349, 2106.12610.



hep-th and cond-mat

There is a long and illustrious history of cross-fertilization of ideas between high-energy physics and condensed matter physics.
(e.g. Nambu-Goldstone bosons, Anderson-Higgs mechanism, renormalization group, effective field theory, topological defects, ...)

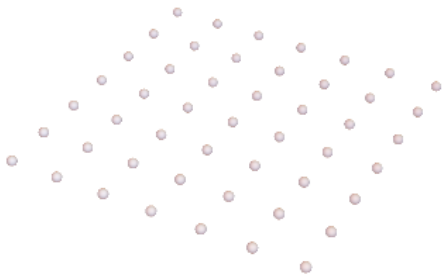
This talk:
Recent ideas from hep-th have helped us organize our understanding of quantum phases of matter.

Why?:
QFT is a universal language for systems with extensive degrees of freedom.

Landau Paradigm.

(basis of most condensed matter understanding!)

1. Phases of matter are classified by how they represent their symmetries.



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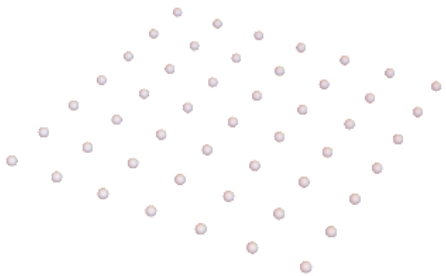

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Landau-Ginzburg theory is an implementation of this point of view for finding representative states, for understanding gross phase structure; it is a starting point for understanding phase transitions.

Landau-Ginzburg-Wilson reminder.

A basic application of Effective Field Theory: once we know
(1) the symmetries (2) the degrees of freedom (3) the cutoff

the dynamics is determined.

Order parameter for $\mathsf{U}(1)$ 0-form symmetry-breaking, $\phi(x) \mapsto e^{\mathbf{i}\alpha}\phi(x)$.

ϕ is a coarse-grained object, this is an effective long-wavelength description.
All local, symmetric terms, organized by derivative expansion

(what else could it be):

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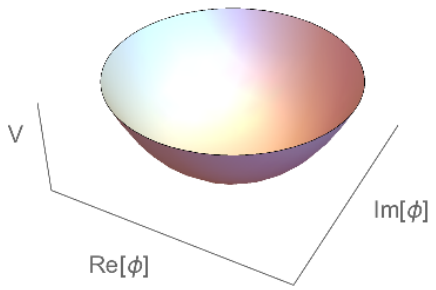
All local, symmetric terms, organized by derivative expansion (what else could it be):

$$S_{\text{Landau-Ginzburg-Wilson}}[\phi] = \int d^D x \left(r|\phi|^2 + u|\phi|^4 + \cdots + |\partial\phi|^2 + \cdots \right) \ .$$



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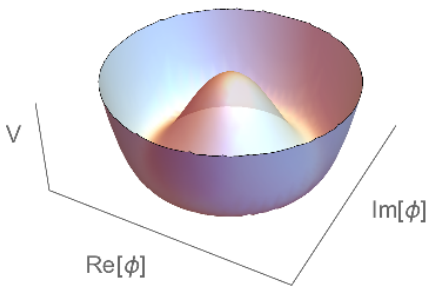

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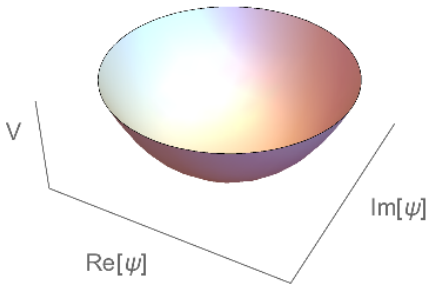

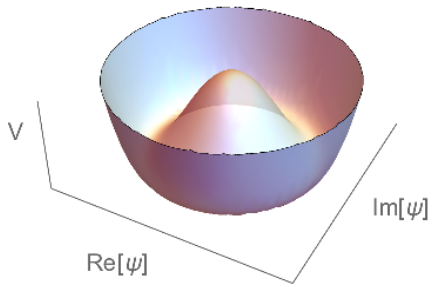

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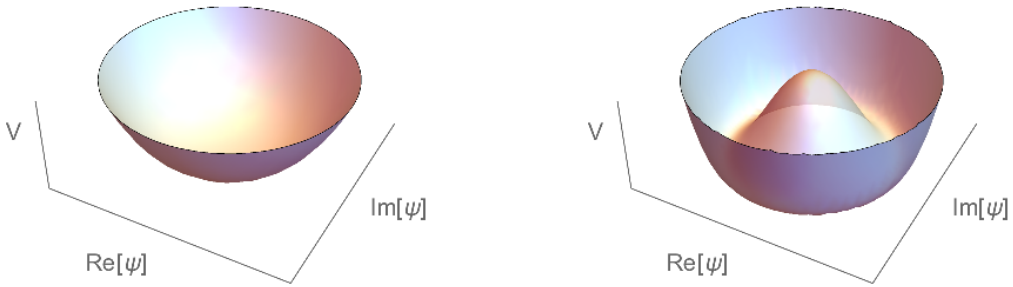

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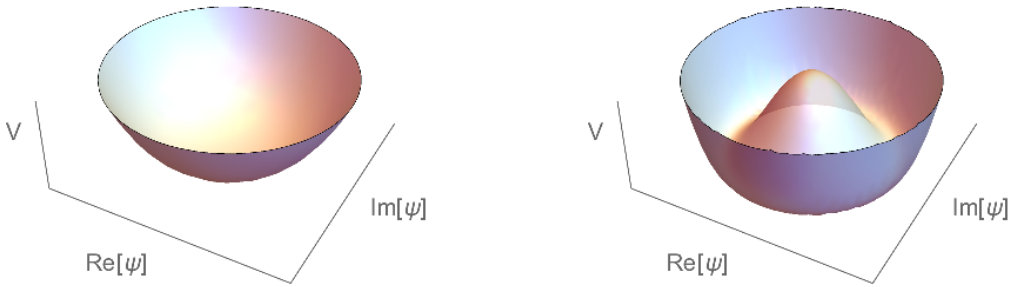

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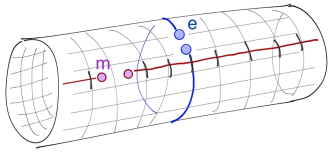
Some apparent exceptions:

- topological order [Wegner, Wen]



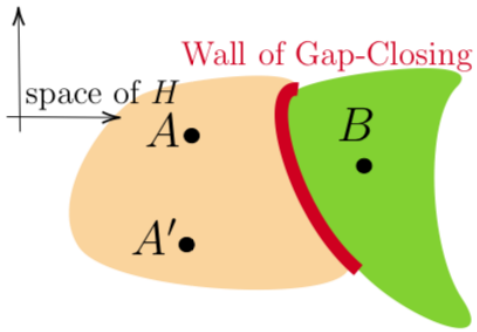
- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M, gapless spin liquids).
- fracton phases.
- topological insulators and integer quantum Hall states.
- (Landau) Fermi liquid.

Brief non-symmetry accounting of gapped phases.

Def: A **gapped phase** is an equivalence class of gapped groundstates or Hamiltonians,

$H_A \simeq H_{A'}$ if they are related by adiabatic evolution and/or inclusion of product states.

Crucial Q: How to label phases?



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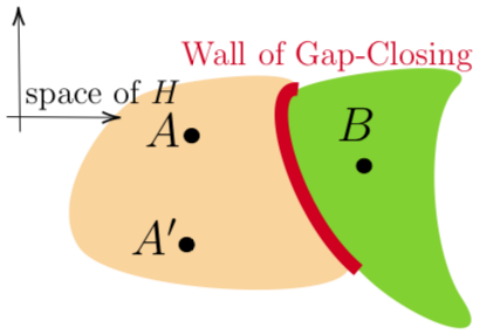
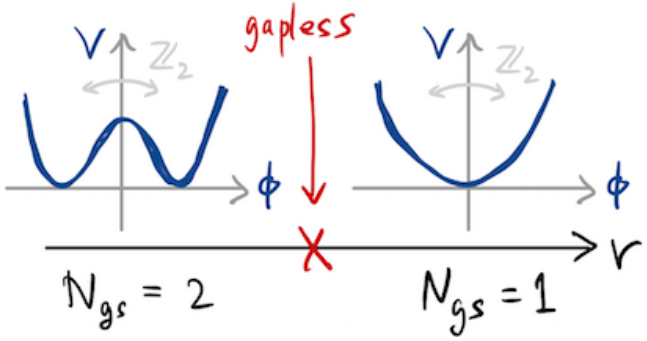


They can break a discrete symmetry. Landau.

(Possible response: even SSB phases are distinguished by topology.)



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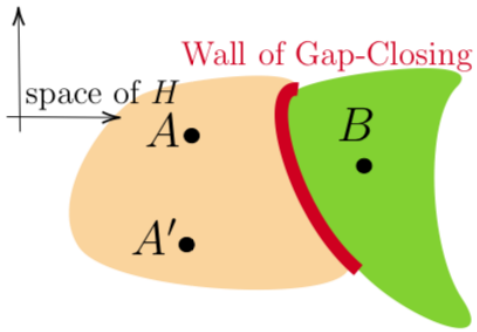

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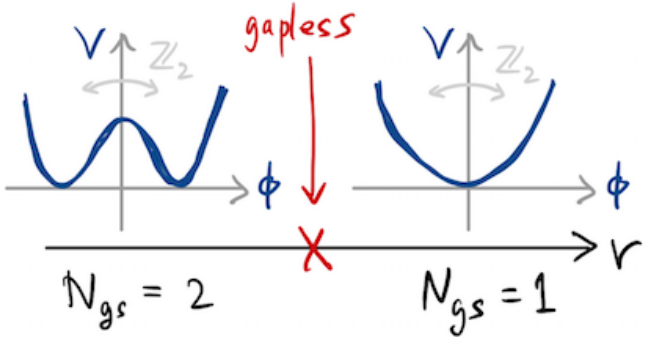

A quantity is *topological* if it doesn't change under continuous deformations.

Nontrivial phases that don't break any (ordinary) symmetries are often called **topological phases**.

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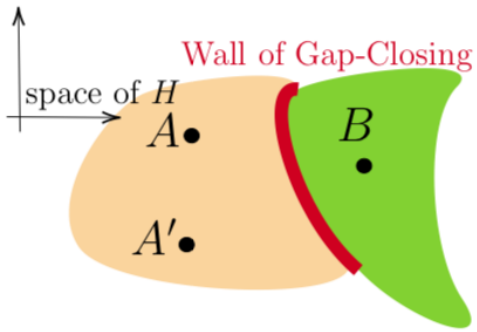

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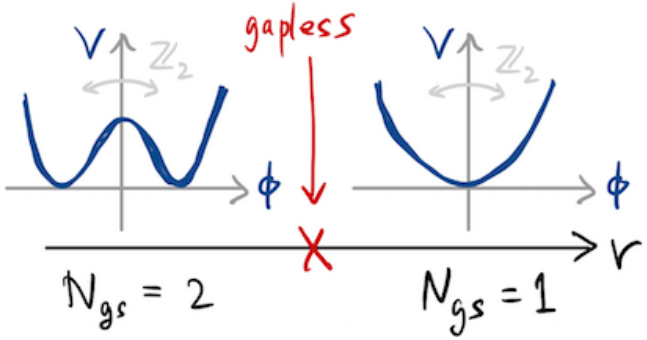

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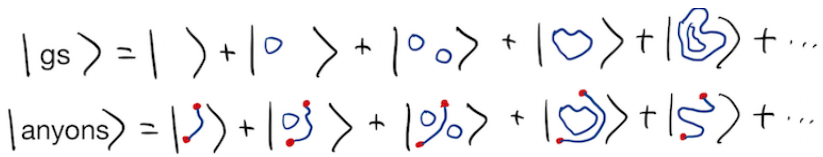
Topological phases can be divided into two classes: those with *topological order* and those without.

Brief non-symmetry accounting of gapped phases.

Topological order [Wen]: localized excitations that can't be created by any local operator (anyons).



e.g.: fractional quantum Hall (FQH) states, gapped spin liquids

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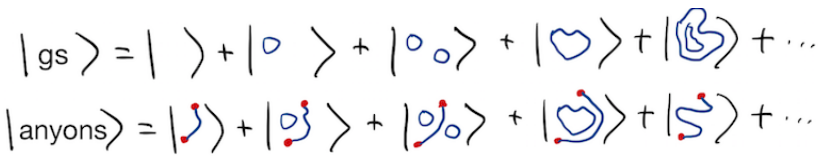

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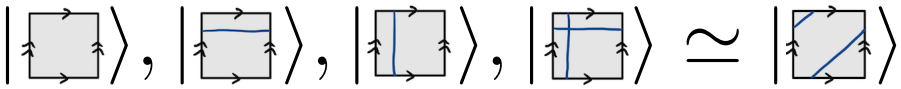
Other important symptoms:

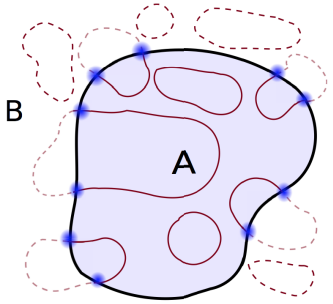
- Topology-dependent groundstate degeneracy

These groundstates are *locally indistinguishable*:

$\langle$  $|\mathcal{O}_x|$  $\rangle = 0 \; \forall$ local ops $\mathcal{O}_x$.



- Long-range entanglement



[Fig: Tarun Grover]

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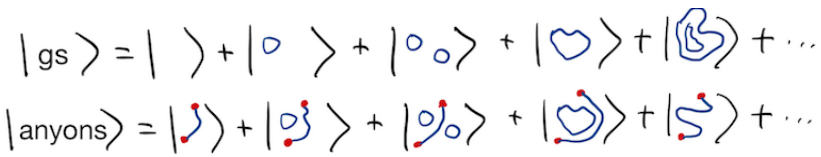

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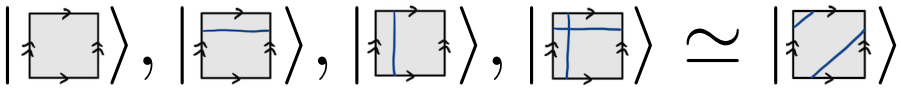

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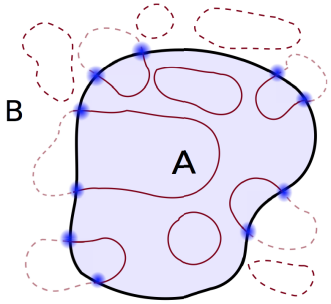


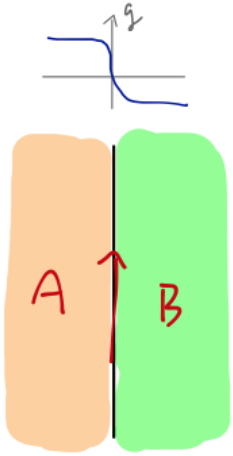
[Fig: Tarun Grover]

(Interesting new special case: **Fracton phases** have excitations that can't be *moved* by any local operator.)

Brief non-symmetry accounting of gapped phases.

Even without TO, there can still be phases distinct from the trivial phase. One way in which they can be distinguished is by what happens if we put them on a space with boundary.

very rough idea:



e.g.: integer quantum Hall (IQH) states, topological insulators, symmetry-protected topological states (SPTs) such as Haldane phase of spin-1 chain, polyacetylene

Generalized Landau paradigm.

The idea is that by suitably refining and generalizing our notions of symmetry, we can incorporate all of these "beyond-Landau" examples into a *Generalized Landau Paradigm*.

[Wen Gaiotto Seiberg Kapustin Willett Iqbal Hofman Cordova...]

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Two important steps:

1. Generalized symmetries
2. Anomalies

Persnickety but important comment: I am talking about actual symmetries. There is no such thing as "gauge symmetry".

New Ingredient 1:

Generalized Symmetries

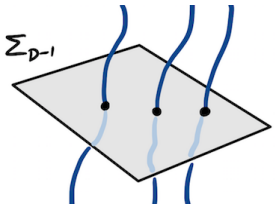
Higher-form symmetries.

[Gaiotto-Kapustin-Seiberg-Willett, Sharpe, Hofman-Iqbal, Lake...]

($D \equiv d + 1 =$ number of spacetime dimensions.)

0-form symmetry:

$\partial^\mu J_\mu = 0$ (*i.e.* $d \star J = 0$)
$\implies Q = \int_{\Sigma_{D-1}} \star J$ is independent of time-slice Σ,
i.e. **is topological.**



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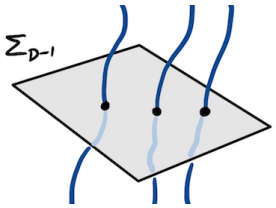

Charged particle worldlines can't end
(except on charged operators).
Charged objects are local operators
$\mathcal{O}(x) \to e^{\mathbf{i}\alpha}\mathcal{O}(x), \quad d\alpha = 0.$

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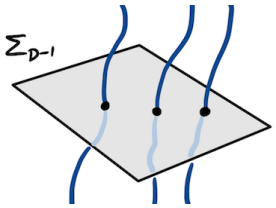

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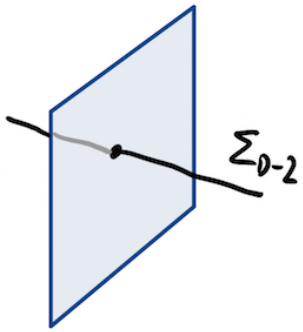
1-form symmetry:

$J_{\mu\nu} = -J_{\nu\mu}$ with $\partial^\mu J_{\mu\nu} = 0$
(*i.e.* $d \star J = 0$)
$\Longrightarrow$ $Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ.



Charged string worldsheets can't end (except on charged operators).

Charged objects are loop operators:
$W[C] \to e^{\mathbf{i}\oint_C \Gamma} W[C], \quad d\Gamma = 0.$

Physics examples of exact one-form symmetries:

- Maxwell theory with only electric charges:
  $J^{\mu\nu}_{(m)} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = (d\tilde{A})^{\mu\nu}$ is conserved: $\nabla_\mu J^{\mu\nu}_{(m)} = 0$ (no monopoles).
  The symmetry operator is $U^{(m)}_\alpha(\Sigma) = e^{\frac{\mathbf{i}\alpha}{2\pi}\int_\Sigma F}$. (Charged operator is the 't Hooft line, $W^E = e^{\mathbf{i}\oint_C \tilde{A}}$, $\tilde{A} \to \tilde{A} + \Gamma, d\Gamma = 0$.)
  Without electric charge: $J_{(e)} = F$ is also conserved.
  Symmetry op: $U^{(e)}_\alpha(\Sigma_2) = e^{\mathbf{i}\frac{2\alpha}{g^2}\int_{\Sigma_2} \star F}$.
  (The charged operator is the Wegner-Wilson loop $e^{\mathbf{i}\oint_C A}$, $A \to A + \Gamma, d\Gamma = 0$.)

- Pure $\mathsf{SU}(N)$ gauge theory
  or $\mathbb{Z}_N$ gauge theory
  or $\mathsf{U}(1)$ gauge theory with charge-N matter
  has a $\mathbb{Z}_N$ 1-form symmetry ('center symmetry').
  (Charged line operator is the Wegner-Wilson line in the minimal irrep, $W[C] = \mathrm{tr}Pe^{\mathbf{i}\oint_C A}$.)

Physics examples of one-form symmetries:

- Many condensed matter systems have emergent higher-form symmetries.
- When we spontaneously break a 0-form U(1) symmetry in $d = 2$, there is an emergent 1-form U(1) symmetry whose charge counts the winding number of the Goldstone phase φ around an arbitrary closed loop C, $Q[C] = \oint_C \star J = \oint_C d\varphi/(2\pi)$.
  (In d spatial dimensions, this is a $(d-1)$-form symmetry.)
  The charged operator creates a vortex (in $d = 2$, or a vortex line or sheet in $d > 2$).

  Not an exact symmetry: broken by vortices.

p-form symmetry for general p.

A p-form symmetry operator $U_\alpha(\Sigma_{D-p-1})$ is a topological operator supported on a closed codimension $p+1$ surface Σ_{D-p-1} in spacetime.

- The charged operators are supported on p-dimensional loci C_p.
- For $p \geq 1$ in $D \geq 2+1$: $U_\alpha(\Sigma)U_\beta(\Sigma) = U_\beta(\Sigma)U_\alpha(\Sigma) = U_{\alpha+\beta}(\Sigma)$: abelian.

Higher-form symmetries can be broken spontaneously.

[Kovner-Rosenstein, Nussinov-Ortiz, Gaiotto-Kapustin-Seiberg-Willett, Hofman-Iqbal, Lake]

0-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object (S^0 = two points) grows.

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle \sim e^{-m|x|}$

($|x| = \mathrm{Area}(S^0(x))$.)

Broken phase for 0-form sym:

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger \rangle \langle \mathcal{O} \rangle + \ldots$

independent of size of S^0.

1-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object grows.

$\langle W(C) \rangle \sim e^{-T_{p+1}\mathrm{Area}(C)}$

For E&M, area law for $\langle W^E(C) \rangle$ is the superconducting phase.

Broken phase for 1-form sym:

$\langle W(C) \rangle = e^{-T_p \mathrm{Perimeter}(C)} + \ldots$

(set to 1 by counterterms local to C: large loop has a vev)

(or Coulomb law)

Higher-form symmetries, a fruitful idea:

- Topological order as SSB [Nussinov-Ortiz 06, Gaiotto-Kapustin-Seiberg-Willett 14]
- Photon as Goldstone boson [Kovner-Rosenstein 92, Gaiotto et al, Hofman-Iqbal, Lake 18]
- A new organizing principle for magnetohydrodynamics [Grozdanov-Hofman-Iqbal 16, Vardhan-Grozdanov-Leutheusser-Liu 22]
- New anomaly constraints on IR behavior of QFT [Gaiotto-Kapustin-Komargodski-Seiberg 17, Wan, Wang, Zheng, Cordova, Ohmori, Dumitrescu, many others]

Landau was even more right than we thought.

[Nussinov-Ortiz 07, Gaiotto et al 14, Wen 18]

- Topological order = SSB of *discrete* higher-form symmetry

≡ degenerate groundstates which are *locally indistinguishable*.

implies topological order, since the algebra of loop (or surface) operators must be realized on the vacuum.

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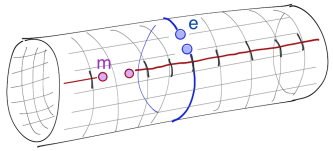
- eg 1 ($\mathbb{Z}_p$ gauge theory/toric code): in D spacetime dimensions with $\mathbb{Z}_p^{(1)}$ 1-form symmetry: $(C_1, M_{D-2}, m, n = 1..p)$

$$U^m(M)V^n(C) = e^{2\pi \mathbf{i} \frac{mn}{p} \#(C,M)} V^n(C)U^m(M). \quad (\#(C,M) \equiv \text{intersection } \#)$$

$U^m(M)$ = symmetry operator, $V^n(C)$ =charged object.

This is the algebra of electric and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory.



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[Nussinov-Ortiz 07, Gaiotto et al 14, Wen 18]

- Topological order = SSB of *discrete* higher-form symmetry

≡ degenerate groundstates which are *locally indistinguishable.*

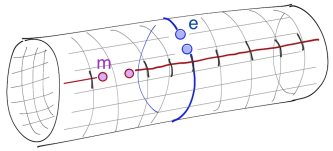
implies topological order, since the algebra of loop (or surface) operators must be realized on the vacuum.

- eg 1 ($\mathbb{Z}_p$ gauge theory/toric code): in D spacetime dimensions with $\mathbb{Z}_p^{(1)}$ 1-form symmetry: $(C_1, M_{D-2}, m, n = 1..p)$

$$U^m(M)V^n(C) = e^{2\pi \mathbf{i} \frac{mn}{p} \#(C,M)} V^n(C)U^m(M). \quad (\#(C,M) \equiv \text{intersection } \#)$$

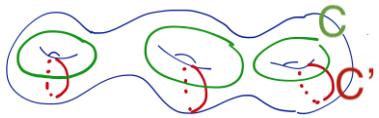
$U^m(M)$ = symmetry operator, $V^n(C)$ =charged object.

This is the algebra of electric and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory.



- eg 2 (Laughlin FQHE): in $D = 2+1$, $\mathbb{Z}_k^{(1)}$ 1-form symmetry with an 't Hooft anomaly

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn \#(C,C')}{k}} U^n(C')U^m(C).$$



(The flux carries charge.) Gives k groundstates on T^2.

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[Kovner-Rosenstein, Gaiotto et al, Hofman-Iqbal, Lake]

- The gaplessness of the photon can be understood as required by spontaneously broken $\mathsf{U}(1)$ 1-form symmetry.

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0-form symmetry:

If we couple to a bg field $\Delta L = j_\mu \mathcal{A}^\mu$,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi g}\left(\underbrace{d\varphi}_{\text{Goldstone}} + \mathcal{A}\right)^2 .$$

The goldstone transforms nonlinearly $\varphi \to \varphi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

(By $(\text{form})^2$ I mean $(\text{form}) \wedge \star(\text{form})$.)

$$\langle 0 | j_\mu(x) | \zeta, p \rangle = \mathbf{i} p_\mu f e^{\mathbf{i} p x}$$

Particle condensation.

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Particle condensation.

1-form symmetry:

If we couple to a bg field $\Delta L = J_{\mu\nu}\mathcal{B}^{\mu\nu}$,

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The goldstone transforms nonlinearly $a \to a + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

Maxwell term for a. g^{-2} = stiffness.

$\langle 0|j_{\mu\nu}(x)|p\rangle = (\zeta_\mu p_\nu - \zeta_\nu p_\mu) f e^{\mathbf{i}px}$

String condensation.

Robustness of higher-form symmetries.

We are used to the idea that consequences of emergent (aka accidental) symmetries are only approximate:

Explicitly breaking a 0-form symmetry gives a mass to the Goldstone boson.

Q: The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

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If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass? No!

Cheap explanation #1: By dimensional analysis (take $m_e \to \infty$).

$m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$.

Cheap explanation #2: By dimensional reduction.

$m_\gamma \overset{\text{[Polyakov]}}{\sim} e^{-M_{\text{monopole}} R} \overset{R\to\infty}{\to} 0.$

Cheap explanation #3: The operators that are charged under a 1-form symmetry are loop operators – they are not local. We can't add non-local operators to the action at all.

An application of this perspective

Landau-Ginzburg mean field theory is our zeroth order tool for understanding symmetry-breaking phases and their neighbors.

0-form symmetry : mean field theory

::

1-form symmetry : ?

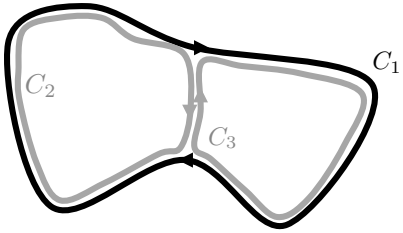
Mean String Field Theory.

[N. Iqbal, JM 2106.12610]

All terms consistent with basic principles in (area) derivative expansion:

$$S_{\rm LGW}[\psi] = \int [dC] \left(V\left(|\psi[C]|^2\right) + \tfrac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi],$$

$v(x) \equiv rx + ux^2 + \cdots,$ $\quad \frac{\delta}{\delta C^{\mu\nu}}$: area derivative [Migdal, Polyakov]



Topology-changing recombination terms:

$$S_r[\psi] = \int [dC_{1,2,3}] \delta[C_1 - (C_2 + C_3)] \left(\lambda \psi[C_1] \psi^\star[C_2] \psi^\star[C_3] + h.c.\right)$$

$+\cdots$ also respect 1-form symmetry.

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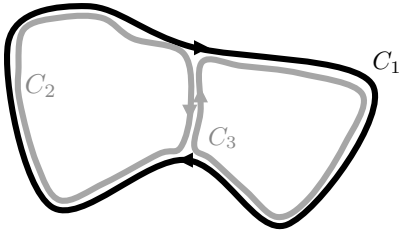

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$+ \cdots$ also respect 1-form symmetry.

- Important disclaimer: Not UV complete, no gravity.
- It gives an interesting new perspective on phase transitions of gauge theories. The presence of the cubic recombination term gives a reason that they are often first order.
- It motivates an interesting analogy between 4d U(1) gauge theory and the Kosterlitz-Thouless transition in 2d.
- What is a gauge theory?

  Contact with work of Polyakov, Migdal, Makeenko and others reformulating a particular gauge theory as a field theory in loop space.

New Ingredient 2:

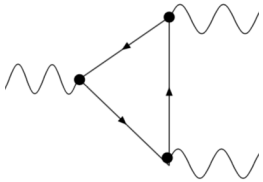
Anomalies

Perspectives on anomalies.

[Adler-Bell-Jackiw, Fujikawa, Atiyah-Singer]

hep-th perspective: A QFT is specified by a path integral

$$Z = \int [D(\text{fields})] e^{\mathbf{i}S[\text{fields}]}.$$



Anomaly = symmetry of action that's not a symmetry of the measure.

e.g.: chiral anomaly, $\partial_\mu j^\mu_A = \frac{\mathbf{N}}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \equiv \mathcal{A}$ controls $\pi \to \gamma\gamma$.

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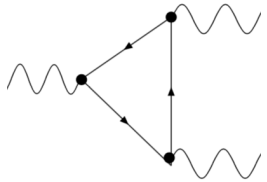

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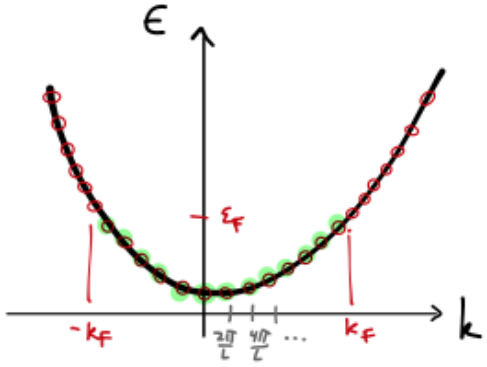
cond-mat perspective: the chiral symmetry is emergent, violated by UV physics in a definite way.

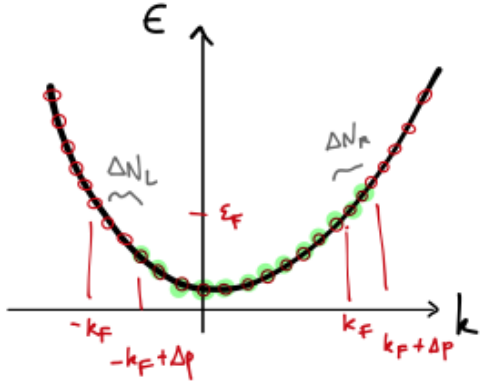
e.g.: free fermions in 1+1d. Apply E_x adiabatically.



$$\Delta Q_A = \Delta(N_L - N_R) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}e\int \mathrm{d}t E_x(t) = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}$$

On the other hand, the LHS is $\Delta Q_A = \int \partial^\mu J^A_\mu$.

$$\partial_\mu J^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$



Anomaly matching.

Reason for excitement: $Z \to e^{\mathbf{i}\int \alpha \mathcal{A}} Z \implies$ RG invariant.

Much of physics is about trying to match microscopic (UV) and long-wavelength (IR) descriptions. Anomalies are precious to us, because they are RG-invariant information ['t Hooft 1979]: any anomaly in the UV description must be realized somehow in the correct IR description.

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Comments:

• Useful perspective: an anomaly is an obstruction to gauging the symmetry.

• I've described an example of an anomaly of a continuous symmetry; discrete symmetries can also be anomalous.

• Anomaly is actually a more basic notion than phase of matter: multiple phases of matter can carry the same anomaly. The anomaly is a property of the degrees of freedom (of the Hilbert space) and how the symmetry acts on them, independent of a choice of Hamiltonian.

Anomaly inflow and SPTs.

SPT (Symmetry-Protected Topological phase) $\equiv$ nontrivial phase of matter (with some symmetry G) without TO.

Can be characterized by its edge states (interface with vacuum).

The idea is that the edge theory has to represent an anomaly.

It is really the anomaly that labels the bulk phase.

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e.g.: topological insulators, integer quantum Hall (IQH), polyacetylene, Haldane phase of spin-1 chain.

An effective field theory for IQH, regarded as an SPT for charge conservation symmetry:
Solve $\vec{\nabla} \cdot \vec{j} = 0$ by $\vec{j} = \vec{\nabla} \times \vec{a}$.
$S_{\mathrm{IQH}}[a, A] = \frac{1}{4\pi} \int_M \epsilon^{\mu\nu\rho} \left(a_\mu \partial_\nu a_\rho + 2A_\mu \partial_\nu a_\rho\right)$
Under $A \to A + d\lambda$, $\delta S_{\mathrm{IQH}} = \int_{\partial M} \frac{\epsilon^{ij}}{4\pi} f_{ij} \lambda$.
This is the contribution to the chiral anomaly from a single right-moving edge mode.

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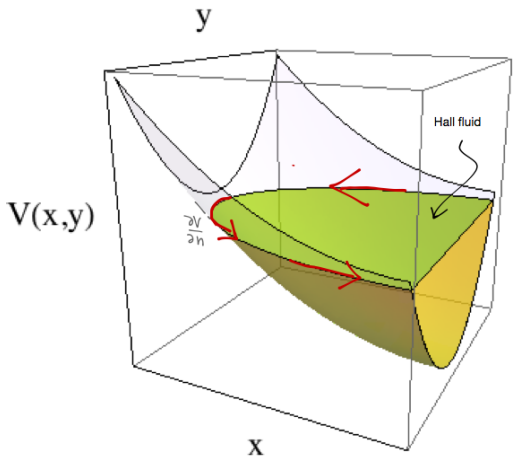

The variation of the bulk action cancels the anomaly of the edge theory.

Anomaly as a label on SPTs.

[Chen-Gu-Wen, Vishwanath-Senthil, Kapustin, ... review: 1405.4015]

The edge theory cannot be trivial: it has to be either

- gapless
- symmetry-broken
- or topologically ordered.

Such a statement is called an LSMOH theorem

(Lieb-Schultz-Mattis-Oshikawa-Hastings).

There is by now a sophisticated mathematical classification of SPTs for various G in various dimensions about which I will say nothing.

The Point: We are still using the realization of symmetries to label these phases!

Anomalies of higher-form symmetries.

Example 1: [Gaiotto et al, Hsin-Lam-Seiberg, ...] Abelian anyons in $D = 2 + 1$.
Gauging a symmetry involves summing over background fields

$\ni$ arbitrary insertions of symmetry operators.

For a 1-form symmetry in $D = 2 + 1$, this means summing over anyon worldlines = *anyon condensation.* [Bais-Slingerland Kong Burnell]

But in order to condense an anyon, it must be a self-boson.

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e.g. Laughlin state ($\mathsf{U}(1)_k$ Chern-Simons theory):
$e^{\mathbf{i}S[a]}$ with $S[a] = \frac{k}{4\pi}\int a \wedge da$
is invariant under $\mathbb{Z}_k^{(1)} : a \mapsto a + \frac{1}{k}\Gamma$: Γ a flat connection with $\oint_C \Gamma \in \mathbb{Z}$.
Wegner-Wilson line $W_n(C) = e^{n\mathbf{i}\oint_C a} \mapsto e^{\mathbf{i}\frac{2\pi n}{k}\int_C \Gamma} e^{\mathbf{i}n\oint_C a}$.
Gauging $\mathbb{Z}_k^{(1)}$, the invariant connection is $a' = ka$.
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Mutual statistics between a and b is a mixed anomaly.

Anomalies of higher-form symmetries.

Example 2: [Hofman-Iqbal, Delecretaz-Hofman-Mathys, Else-Senthil] The $(d-1)$-form ('dual') symmetry emerging in a superfluid:
$(\star J)_\mu = \partial_\mu \varphi \rightsquigarrow D_\mu \varphi = \partial_\mu \varphi - qA_\mu$ is not conserved: $d \star J = -qF$.

(Applying an electric field leads to a linearly-growing supercurrent.)

This is a mixed anomaly between a $(d-1)-$form symmetry and a 0-form symmetry.

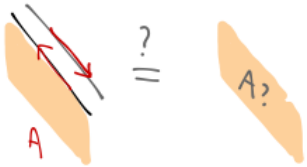
- [Delacretaz et al] Converse statement: Any system with such an anomaly has a Goldstone boson.
- [Else-Senthil] Existence of equilibrium states with non-dissipating current.

An application of this perspective

Anomaly as obstruction to symmetric regulator.

Edge theories of an SPT_G^{d+1} cannot be regularized in d dimensions, exactly preserving on-site G symmetry.

If they could be, they wouldn't characterize the bulk state.



The most famous example of such an obstruction to a symmetric regulator was articulated by Nielsen and Ninomiya:

It is not possible to regulate free fermions while preserving on-site chiral symmetry.

A reason to care about this is that the Standard Model is a chiral gauge theory, and this is an obstruction to a naive lattice definition.

Recasting the NN result as a statement about SPTs.

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

$\pm m$ label distinct SPT phases.

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One proof of this:
Couple to external gauge field
$\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$

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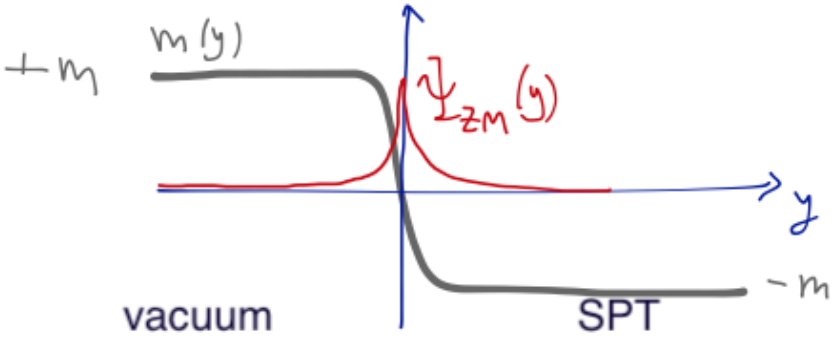

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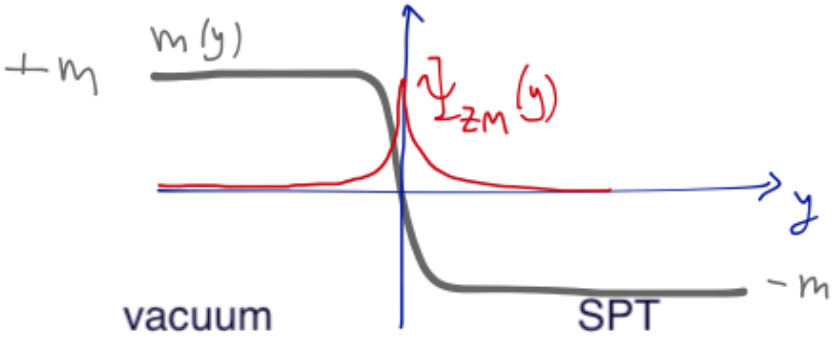


Galling fact: if we want the extra dimension to be finite, there's another domain wall with the antichiral fermions ('mirror fermions').

Anomaly as obstruction to symmetric regulator.

Converse: The SM (with RH neutrinos) has no anomalies. This suggests that there should be a symmetric regulator!

Loophole: Interactions between the fermions. [Eichten-Preskill, Wen, J Wang]

This means that the classification of *interacting* fermion SPTs differs from that of free fermions. [Fidkowski-Kitaev]

More generally, each SPT in $d+1$ dimensions represents such an obstruction in d dimensions. This is a physical application of $4+1$-dimensional models!

• A certain $4+1$-dimensional SPT implies that there is no lattice realization of Maxwell theory with exact duality symmetry. [Kravec-JM]

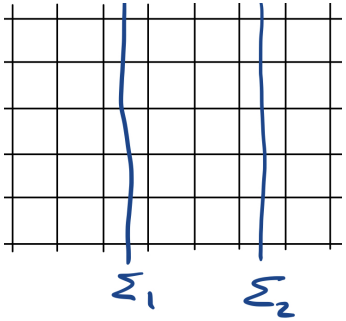
• A certain $4+1$-dimensional SPT implies that there is no lattice realization of QED where the electron and magnetic monopole are both fermions. [Kravec-JM-Swingle]

Further Generalizations of the Notion of Symmetry

Subsystem symmetries and fracton phases, briefly.

Symmetry $\not\Rightarrow$ fully-topological defect operators.

So far: symmetry operators were fully topological. But there can exist operators $U(\Sigma)$ with $[U(\Sigma), H] = 0$, but which are not topological.



For example:

$\Sigma_1 \simeq \Sigma_2$ but $\langle U(\Sigma_1)...\rangle \neq \langle U(\Sigma_2)\rangle$.

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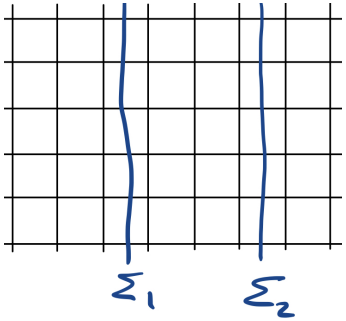
Symmetry $\not\Rightarrow$ fully-topological defect operators.

So far: symmetry operators were fully topological. But there can exist operators $U(\Sigma)$ with $[U(\Sigma), H] = 0$, but which are not topological.



For example:

$\Sigma_1 \simeq \Sigma_2$ but $\langle U(\Sigma_1)...\rangle \neq \langle U(\Sigma_2)\rangle$.

Subsystem (or 'faithful') symmetry: symmetry operators act independently on *rigid* subspaces.

• Gapped fracton phases: spontaneously break a discrete subsystem higher-form symmetry. [Qi-Hermele-Radzihovsky, Rayhaun-Williamson]

Charged objects are stuck where the symmetry acts.

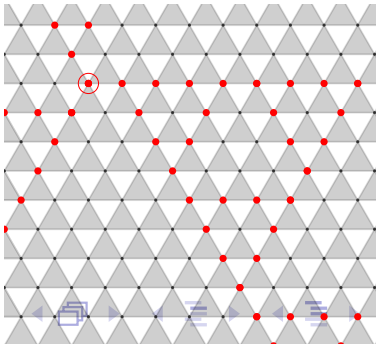
• Multipole symms: (e.g. $\dot{J}^0 + \partial_i \partial_j J^{ij} = 0$) $\xrightarrow{\text{SSB}}$ gapless fracton phases.

[Pretko Seiberg Shao Gorantla Gromov Bulmash Barkeshli ...]

• The subsystem Σ could be a fractal:

e.g. $H = \sum_{\Delta(ijk)} Z_i Z_j Z_k + g \sum_i X_i$

[Newman Moore Yoshida Williamson Zhou Zhang Pollmann You Devakul Burnell Sondhi]



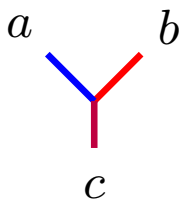
Fusion category symmetries.

[...Chang-Lin-Shao-Wang-Yin 2018, Bhardwaj-Tachikawa, Wang-Thorngren, Lin-Shao, Gaiotto-Kulp, Kikuchi, Pal-Sun...]

(also known as: categorical symms
or algebraic higher symms
or non-invertible symms)

Suppose we have topological operators (associated to each closed $(D-p-1)$-manifold Σ) satisfying a fusion algebra

$$T_a T_b = \sum_c N_{ab}^c T_c \qquad N_{ab}^c \neq 0 \implies$$



Not a group! Still $T_1 = \mathbb{1}, T_{\bar{a}} = T_a^\dagger$.

$\implies T_a T_a^\dagger = \sum_c N_{a\bar{a}}^c T_c$. If $N_{a\bar{a}}^c \neq 0$ for $c \neq 1$, then T_a is not unitary.

Fusion category symmetries.

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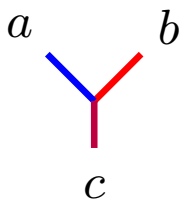

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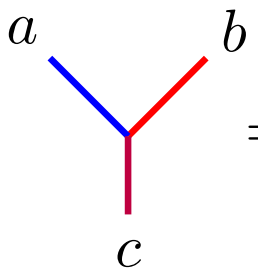
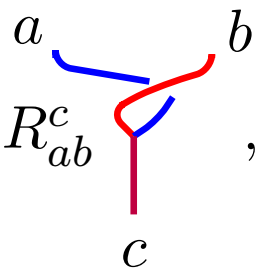
Application 1: Non-abelian topological order as SSB.



$= R_{ab}^c$



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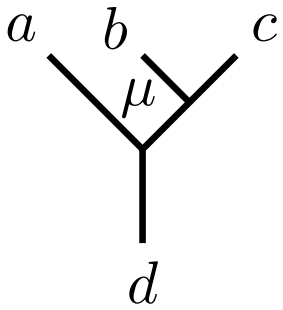
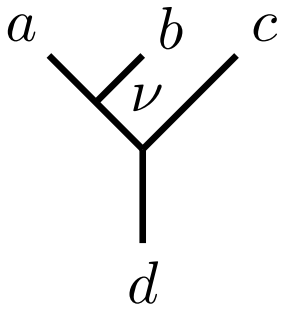


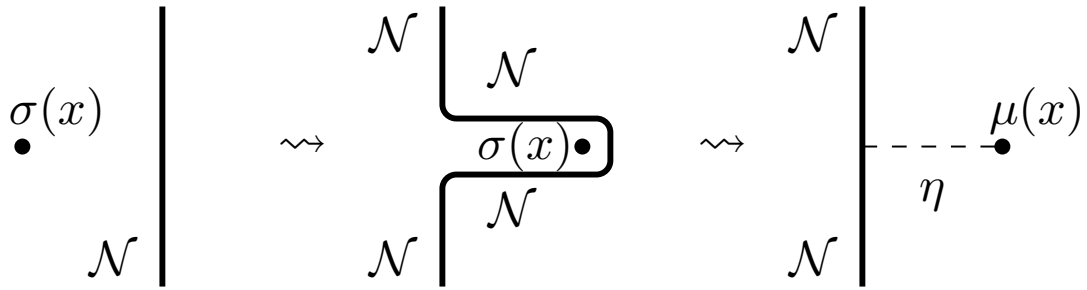
$$= \sum_\nu \left(F_d^{abc}\right)_{\mu\nu}$$



Fusion category symmetries.

Example ($D = 1+1$ Ising CFT): $\eta\eta = 1, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}, \quad \mathcal{N}\mathcal{N} = 1 + \eta$

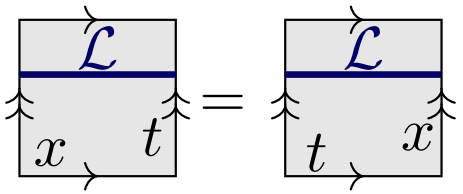
$\eta = \mathbb{Z}_2$ symmetry, $\mathcal{N}$ = duality wall.



Application 2: [Chang et al 2018] An LSMOH theorem in 1+1d CFT.

Perturb a CFT by a relevant operator that commutes with a line op. $\mathcal{L}$.

$$\mathrm{tr}\mathcal{L}e^{-\frac{2\pi}{\beta}(H-E_0)} =$$



$$= \mathrm{tr}_{\mathcal{H}_{\mathcal{L}}} e^{-\beta(H-E_0)}$$

If gapped, evaluate BHS in TQFT: $\langle\mathcal{L}\rangle = \mathrm{tr}_{\mathcal{L}} 1 \in \mathbb{Z}_{\geq 0}$. If eigenvalues of $\mathcal{L}$ are not non-negative integers, there must be a groundstate degeneracy.

e.g. $\mathcal{L}^2 = 1 + \mathcal{L} \implies$ evals are $\frac{1}{2}(1 \pm \sqrt{5})$, requires at least 2 groundstates.

Fusion category symmetries, FAQ.

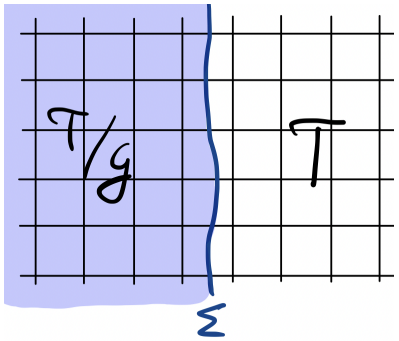
Why should we call this a symmetry?

- Commutes with H.
- Can sometimes be gauged.
- Obstruction to gauging is a useful RG-invariant.
- Their inclusion consolidates conjectures about the absence of symmetry in quantum gravity.

Where to find them in general D?

[Choi et al 2111.01139, Kaidi et al 2111.01141]

- Find a self-dual theory: $\mathcal{T} \simeq \mathcal{T}/\mathcal{G}$, gauge the symmetry in part of spacetime. The interface is a (non-invertible) duality wall.



or: • [Roumpedakis et al 2204.02407] Gauge the symmetry on some finite codimension locus.

Further generalization: the fusion coefficients N_{ab}^{c} may be a TQFT (number depending on topology of Σ).

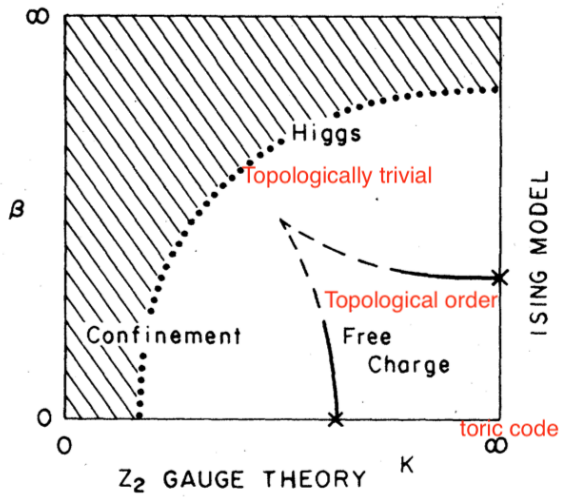
Generalized Landau paradigm, part 2

‘Beyond-Landau’ critical points?

Landau paradigm part 2:
At a critical point, the critical dofs are the fluctuations of the order parameter.

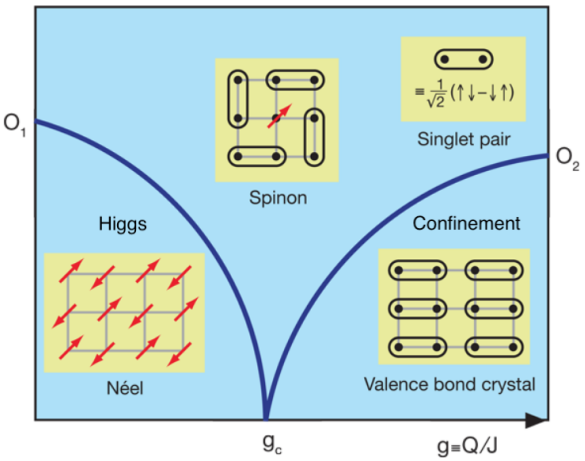
Apparent exceptions:

- Transitions out of deconfined phases, such as topologically-ordered states (no local order parameter).



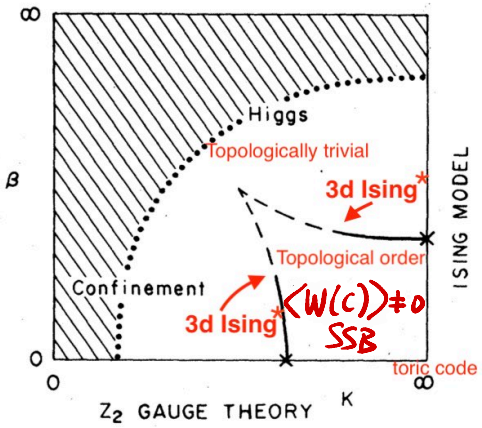
[Image: Fradkin-Shenker]

- Direct transitions between states which break *different symmetries* (deconfined quantum critical points), *e.g.* Neel to VBS in $D = 2 + 1$.



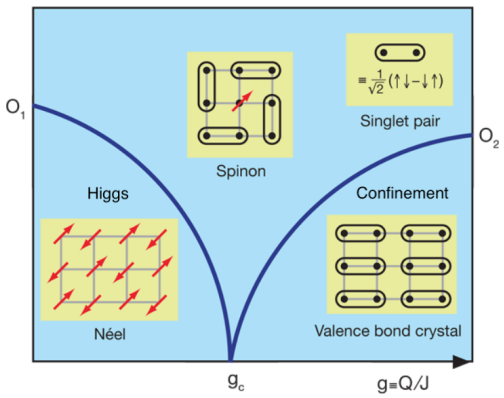
[Balents-Senthil-Vishwanath-Sachdev-Fisher] [Image: Alan Stonebraker]

‘Beyond-Landau’ critical points?

- 

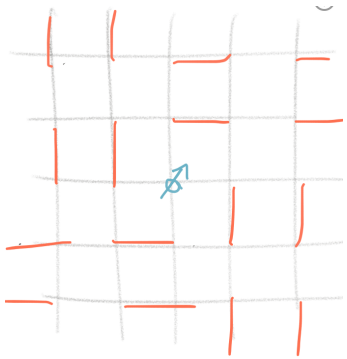
  Can we understand the critical theory in terms of fluctuations of the string order parameter $W(C)$? But by Wegner’s duality, this theory (up to global data) is in the same universality class as the 3d Ising model.

This suggests that the near-critical 3d Ising model should have a description as a string theory. [Polyakov ... Iqbal,JM]

- 

  Can be understood as a consequence of symmetries with mixed ’t Hooft anomalies [Metlitski-Thorngren 18, Wang et al 17]

  $\Longrightarrow$ defects in one order carry charge of the other!



[Levin-Senthil]

Final thought.

Q: Does the enlarged Landau paradigm

(including all generalizations of symmetries, and their anomalies)

incorporate all phases of matter

(and transitions between them)

as consequences of symmetry?

Some apparent exceptions:

- topological order. = SSB of higher-form symmetries.
- fracton phases = SSB of *subsystem* higher-form symmetries.
- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M).
- topological insulator and integer quantum Hall states.
- (Landau) Fermi liquid. See [Else-Senthil].
- CFTs with no (symmetric) relevant operators (*e.g.* Dirac spin liquid or Stiefel liquid [Zou-He-Wang 2101.07805]).

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Landau was even more right than we thought.
This seems to be a fruitful principle.

The end.

Thanks for listening.

Thanks to Nabil Iqbal, Diego Hofman, Tarun Grover, Yi-Zhuang You for helpful discussions.

Action of higher form symmetry operators

Choose a constant-time slice $\mathcal{M}_{D-1}$.

For each $\Sigma_{D-p-1} \subset \mathcal{M}_{D-1}$,

$W(C_p) \mapsto U_\alpha(\Sigma_{D-p-1}) W(C_p) U_\alpha^\dagger(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha q \oint_{C_p} \Gamma_\Sigma} W(C_p)$

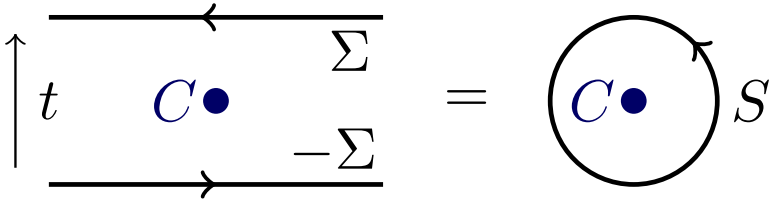
where Γ_Σ is the Poincaré dual of Σ_{D-p-1} in $\mathcal{M}_{D-1}$:

$\int_{\mathcal{M}_{D-1}} \eta^{D-p-1} \wedge \Gamma_\Sigma = \int_{\Sigma_{D-p-1}} \eta^{D-p-1}$, $\forall \eta_{D-p-1}$. ($d\Gamma_\Sigma = 0$ since $\partial\Sigma = 0$.)

- $U_\alpha(-\Sigma) = U_{-\alpha}(\Sigma) = U_\alpha^\dagger(\Sigma)$.
- Infinitesimal version:

$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i}q\#_{\mathcal{M}_{D-1}}(\Sigma, C) W(C) = \mathbf{i}q\ell_{\mathcal{M}_{D-1}}(S, C) W(C)$

- If we assume Lorentz symmetry:



$\mathcal{O}(C) \to U_\alpha(\Sigma) \mathcal{O}(C) U_\alpha^\dagger(\Sigma) = U_\alpha(S) \mathcal{O}(C)$

Consolidation.

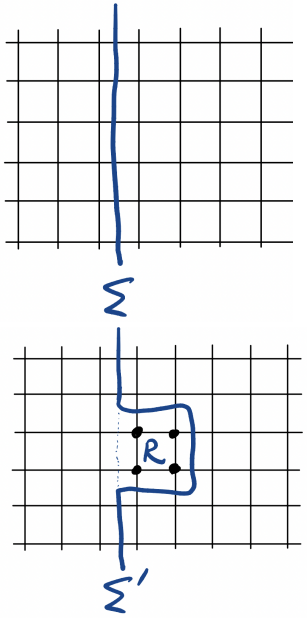
Noether's theorem relates continuous symmetries to topological defect operators $U_g(\Sigma)$.
Conservation $\implies$ topological.

Group law $\implies$ Fusion rule: $U_g(\Sigma)U_{g'}(\Sigma) = U_{gg'}(\Sigma)$.

Example: Ising model, Euclidean, any D:

$U_{-1}(\Sigma)$ is an instruction to flip the sign of J for any bond crossing Σ.
If $\Sigma' - \Sigma = \partial R$, $U_{-1}(\Sigma)$ and $U_{-1}(\Sigma')$ are related by redefining $\sigma_x \to -\sigma_x$ for $x \in R$.

$\implies$ σ_x is charged.



Useful reverse perspective:

Topological defect operators are a sufficient condition for symmetry.

- Continuous and discrete symmetries on equal footing.
- Noether symmetries and topological symmetries on equal footing.
- Allows generalizations!

Mean String Field Theory.

$$S_{\mathrm{LGW}}[\psi] = \int [dC] \left(V\left(|\psi[C]|^2\right) + \tfrac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi]$$

- Important disclaimer: Not at all UV complete, no gravity.
  We expect no connection to 'real' string field theory and are trying to do something much less difficult involving only effective strings.
- A *gauged* version of this model (without S_r) was studied [Soo-Jong Rey, 1989] as a description of 2-form Higgs mechanism, and [Franz 07, Beekman-Sadri-Zaanen 11] as a dual description of a $3+1$d superfluid.
- After posting our paper, we learned that a lattice version of this model is related by a Hubbard-Stratonovich transformation to Wilson lattice gauge theory [Banks 80, Yoneya 81].
- Still-difficult but well-posed Q: what does this model describe?
  Plausible goal: develop a crude picture of the phase diagram (and transitions) for systems with 1-form symmetries.

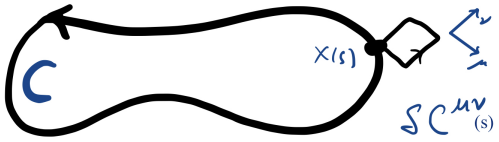
Classical mechanics of Mean String Field Theory.

Equations of motion: $0 = \frac{\delta S[\psi]}{\delta \psi^\star[C]}$

$$0 = -\frac{1}{2} e^{mL[C]} \oint ds \frac{\delta}{\delta C_{\mu\nu}(s)} \left(ds \frac{e^{-mL[C]}}{L[C]} \frac{\delta \psi[C]}{\delta C^{\mu\nu}(s)} \right) + \psi[C] V'(|\psi[C]|^2) + \frac{\delta S_r}{\delta \psi^\star[C]}$$

Requires a boundary condition at small loops.

This BC says: a small loop can shrink to nothing.

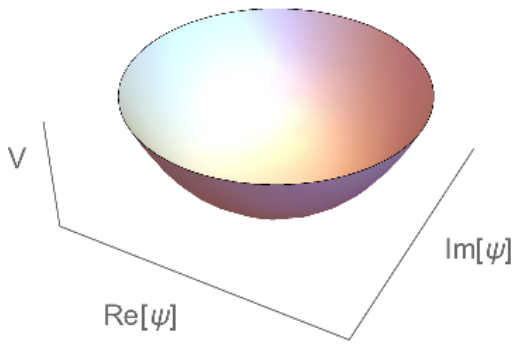


Setting ψ[small, contractible loop] $= g^{-2}$, some constant

- is consistent with the symmetries, since for a small, contractible loop, $C = \partial R$, $\psi[C] \to e^{\mathbf{i} \oint_C \Gamma} \psi[C] = e^{\mathbf{i} \int_R d\Gamma} \psi[C] = \psi[C]$ is neutral, and
- will match nicely to gauge theory in the broken phase.

Unbroken phase.

Let's ignore S_r for a moment, and take $r > 0$:



$$S_{\mathrm{LGW}}[\psi] = \int [dC] \left(r\psi[C]\psi^\star[C] + \frac{1}{2L[C]} \oint ds \frac{\delta\psi^\dagger[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right),$$

$r > 0 \implies \psi[C] \sim 0.$ ($\psi = 0$ is not consistent with B.C.)

Ansatz: $\psi[C] = e^{-s(A[C])}, \quad A[C] = \min_{\Sigma, \partial\Sigma = C} \mathrm{Area}(\Sigma)$

For large r, A, solution is self-consistently: $(s'(A))^2 = r + \mathcal{O}\left(A^{-1/2}\right)$

$$\implies \boxed{\psi[C] \simeq e^{-\sqrt{r}A[C]}}.$$

Area law. Confinement.

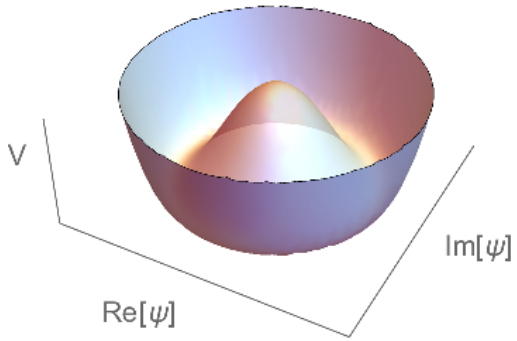
String tension $= \sqrt{r}$.

Broken phase.

Now consider $r < 0$:

$\psi[C] \sim v$ $(v = \sqrt{\frac{|r|}{2u}})$ "string condensed phase"

[Levin-Wen]



Fluctuations about groundstate:

$$\psi[C] = v \exp\left(\oint_C ds\,(\mathbf{i}t(x(s)) + \mathbf{i}a_\mu(x(s))\dot{x}^\mu(s) + \mathbf{i}h_{\mu\nu}(x(s))\dot{x}^\mu\dot{x}^\nu + \cdots)\right) \quad .$$

Plug back into action (worldline techniques, *e.g.* [Strassler's thesis]):

$$S[\psi] = \frac{v^2}{2}\int d^D x f_{\mu\nu}f^{\mu\nu} + \text{massive modes}, \quad (f \equiv da)$$

- Photon = Goldstone boson (slowly-varying 1-form symmetry transf).
- Gauge coupling is $g^2 = \frac{1}{2v^2}$, determined by stiffness.
- All other unprotected dofs massive.
- Perimeter-law factors $e^{-\oint_C \mathcal{L}} = e^{-mL[C]+\cdots}$ ambiguous by field redefinition of $\psi[C]$.
- $j^{\mu\nu} = \frac{\delta S}{\delta \mathcal{B}_{\mu\nu}} = v^2 f^{\mu\nu}$.

Topological defects in the broken phase.

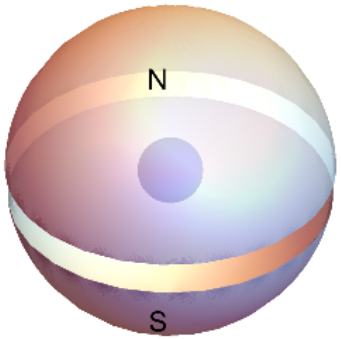
Another purpose of ordinary LG theory is to provide an understanding of topological defects of the broken phase [*e.g.* Mermin 1979].
Take $X \subset$ spacetime with $\psi \neq 0$ defines a map $LX \to \mathsf{U}(1)$, where LX is the free loop space, maps $S^1 \to X$.

Defects linked with X are then labelled by homotopy classes of such maps $[LX, \mathsf{U}(1)]$.

If $\pi_1(X) = 0$, then

$$[LX, S^1] = \pi_2(X).$$

For example, take $X = S^{q-1}$ surrounding a codim q locus. This predicts that the magnetic monopole is the *only* topological defect for $G = \mathsf{U}(1)$.



Discrete 1-form symmetries.

To break the $\mathsf{U}(1)$ 1-form symmetry down to a $\mathbb{Z}_p$ subgroup, add

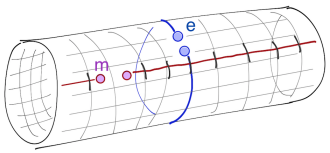
$$S_p = h \int [dC] \psi^p [C] + h.c.$$

In the broken phase, this is $S_p = 2hv^p \sum_C \cos\left(p \oint_C a\right)$.
For $h \gg 1$, minimizing S_p requires $\oint_C a = \frac{2\pi k}{p}, k = 0, \cdots p-1$ for all loops C, including nearby loops $\implies da = 0$.
Introducing a $D-2$-form Lagrange multiplier b to set $pda = 0$ gives

$$\int [d\psi] e^{-S_{\mathrm{LGW}}[\psi] - S_p[\psi]} U^k_{\mathcal{M}_{D-2}} \sim \int [dadb] e^{\mathbf{i} \frac{p}{2\pi} \int b \wedge da} e^{\mathbf{i} k \int_{\mathcal{M}_{D-2}} b}$$

where $U^k_{\mathcal{M}_{D-2}}$ is the 1-form symmetry operator.

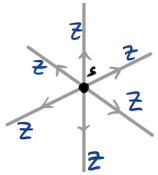
This is an EFT for $\mathbb{Z}_p$ gauge theory. ✓

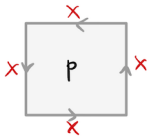


Regularization on the lattice.

A simple example of a system with 1-form symmetry:
$\mathbb{Z}_p$ gauge theory aka (perturbed) toric code. Cell complex, $\mathcal{H} = \otimes_{\text{links},\ \ell} \mathcal{H}_p$.

$$H_{\text{TC}} = -\infty \sum_{\text{sites},\ s} \quad - \Gamma \sum_{\text{plaquettes},\ p} \quad - g \sum_{\text{links},\ \ell} Z_\ell .$$





$g = 0$: $|\text{gs}\rangle = \sum_{\text{collections of closed loops, C}} |C\rangle$ (where $|$  $\rangle \equiv |Z_\ell = -1\rangle$).

$g \sim$ electric string tension.

'Product-state' ansatz: $|\psi\rangle =: e^{\sum_{c,\ \text{connected}} \psi[c] W[c]} : |0\rangle$

where $W[c]|0\rangle = |c\rangle$ creates the loop c. [Related ansatze: Levin-Wen 04, Vidal et al]

$$E[\psi] \equiv \langle \psi | H_{\text{TC}} | \psi \rangle$$

$$= \sum_c \left(- \sum_{\partial p \cap c \neq 0} \psi^\star[c] \psi[c + \partial p] + g L[c] \psi^\star[c] \psi[c] \right) - \underbrace{\sum_p \psi[\partial p]}_{\text{small-loop BC}} + \underbrace{H_r}_{\text{recombination}}$$

$0 = \frac{\delta E}{\delta \psi^\star}$ gives a lattice version of the MSFT EoM.

Comments and questions about MSFT.

- There is much more to understand about Mean String Field Theory. It is not quite under control yet, but likely can be understood.
- It gives an interesting new perspective on phase transitions of gauge theories. The presence of the cubic recombination term gives a reason that they are often first order.
- Can we find new RG fixed points this way?
- By adding topological and WZW terms, we can describe 1-form SPTs, and realize more general gauge theories as the broken phase.
- What is a gauge theory?
  Contrast with the work of Polyakov, Migdal, Makeenko and others reformulating a particular gauge theory as a field theory in loop space: Here, by writing a field theory in loop space, we arrive at some universal properties of gauge theory.