Lattice QCD input for the first row CKM unitarity tests

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Outline

- **Introduction**
  - Pion semi-leptonic decay - $\gamma W$-box diagram
  - Kaon semi-leptonic decay - $\chi$PT low energy constants
  - Conclusion and outlook
The unitarity relation for the first-row of the CKM matrix is:

$$\Delta_{\text{CKM}} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = 0.$$ 

According to PDG 2020, $\Delta_{\text{CKM}} = -0.0015(3)(4)$, where the first error is the uncertainty from $|V_{ud}|^2$ and the second error is the uncertainty from $|V_{us}|^2$.

- $|V_{ud}| = 0.97370(10)_{\text{exp.,nucl.}}(10)_{\text{RC}} [0.14 \text{ per mil}]$.
  From superallowed $0^+ \rightarrow 0^+$ nuclear $\beta$ decays, which are pure vector transitions at leading order. PDG 2022 include additional large (27) uncertainty for nuclear structure. Neutron and pion $\beta$ decay can determine as well, but currently have larger uncertainties.

- $|V_{us}| = 0.2245(4) [1.8 \text{ per mil}]$
  From combined analysis of $K \rightarrow l\nu$ and $K \rightarrow \pi l\nu$ decay processes. Lattice QCD inputs are essential and the current limiting factor.

- $|V_{ub}| = 0.00394(36)$. Very small, does not affect $\Delta_{\text{CKM}}$ much.

“That deviation could be due a problem with $|V_{ud}|$ theory (RC or NP), the lattice determination of $f_+(0)$ or new physics.”

– E. Blucher and W.J. Marciano (PDG 2020)
• In superallowed nuclear $\beta$ decay rates, one of the dominant uncertainties arises from the nucleus-independent electroweak radiative correction, $\Delta^V_R$, which is universal for both nuclear and free neutron beta decay. J.C. Hardy, I.S. Towner (2015)

• Sirlin established that only the axial $\gamma W$-box contribution is sensitive to hadronic scales. A. Sirlin (1978)

• Situation is similar for the charged pion $\beta$ decay.

\begin{align*}
H_{\mu\nu}^{VA}(t, \vec{x}) &= \langle H_f | T \left[ J_{\mu}^{em}(t, \vec{x}) J_{\nu}^{W,A}(0) \right] | H_i \rangle
\end{align*}
$K \rightarrow \ell \nu$, $K \rightarrow \pi \ell \nu$ and $|V_{us}|$

- Dashed line is the CKM matrix first row unitary constraint.
- All the bands and line should cross the same point. There are visible tensions in the plot.
- QED and strong isospin breaking corrections from ChPT calculations.
Dashed line is the CKM matrix first row unitary constraint.

All the bands and line should cross the same point. There are visible tensions in the plot.

QED and strong isospin breaking corrections for $f_{K^\pm}/f_{\pi^\pm}$ from lattice QCD.

$S_{E}^{\text{lat}} = - \sum_{\Box} \frac{6}{g^{2}} \text{Re} \text{ tr}_N(U_{\Box,\mu\nu}) - \sum_{q} \overline{q}(D_{\mu}^{\text{lat}} \gamma_{\mu} + am_{q})q$

**Wilson gauge action**  **Lattice fermion action**

Figure credit: Stephen R. Sharpe.
- Introduction

- **Pion semi-leptonic decay - $\gamma W$-box diagram**

- **Kaon semi-leptonic decay - $\chi$PT low energy constants**

- Conclusion and outlook
The $\gamma W$-box contribution for pion

\[ H_{\mu\nu}^{VA}(t, \vec{x}) = \langle \pi^0 | T \left[ J_{\mu}^{em}(t, \vec{x}) J_{\nu}^{W,A}(0) \right] | \pi^- \rangle \]

Hadronic function in lattice QCD calculations:

Gluons and quark loops not directly connected to the operators are automatically included in the numerical Monte Carlo integration over all possible QCD gauge field configurations.
\[
M_\pi(Q^2) = \frac{1}{6} \frac{\sqrt{Q^2}}{F_\pi m_\pi} \int d^4x \omega(t, \vec{x}) \epsilon_{\mu\nu\alpha\beta} x_\alpha H^{VA}_{\mu\nu}(t, \vec{x}),
\]

\[
\omega(t, \vec{x}) = \int_{-\pi/2}^{\pi/2} \frac{\cos^3 \theta d\theta}{\pi} j_1 \left( \sqrt{Q^2} |\vec{x}| \cos \theta \right) \frac{|\vec{x}|}{|\vec{x}|} \cos \left( \sqrt{Q^2} t \sin \theta \right)
\]

Results from the 64I ensemble (\(R\) is the upper limit of the Euclidean space-time range). Note that the weighting function \(\omega(t, \vec{x})\) is calculated in the infinite volume. The momentum exchange \(Q\) can vary continuously instead of being constrained by \(2\pi/L\).
The $\gamma W$-box contribution momentum integration

$$\square^V_{\gamma W}|_\pi = \frac{3\alpha_e}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_\pi(Q^2)$$

$$= 2.830(11)_{\text{stat}}(9)_{\text{PT}}(24)_{\text{a}}(3)_{\text{FV}} \times 10^{-3}$$

$$= 2.830(11)_{\text{stat}}(26)_{\text{sys}} \times 10^{-3}$$
Pion $\beta$ decay

$$
\Gamma_{\pi\ell 3} = \frac{G_F^2 |V_{ud}|^2 m_\pi^5 |f_+^\pi(0)|^2}{64\pi^3} (1 + \delta) l_\pi
$$

- Here, $m_\pi$ is the charged pion mass, $l_\pi$ is the kinematic factor.

- ChiPT A. Czarnecki, W. Marciano, A. Sirlin (2019)

$$
\delta = 0.0334(10)_{\text{LEC}}(3)_{\text{HO}}
$$

- Sirlin’s parametrization A. Sirlin (1978)

$$
\delta = \frac{\alpha_e}{2\pi} \left[ \tilde{g} + 3 \ln \frac{m_Z}{m_p} + \ln \frac{m_Z}{m_W} + \tilde{a}_g \right] + \delta_{\text{QED}}^{\text{QED}} + 2\Box_{\gamma W}^{\text{VA}} |_{\pi}
$$

$$
= 0.0332(1)_{\gamma W}(3)_{\text{HO}}
$$

where $\frac{\alpha_e}{2\pi} \tilde{g} = 1.051 \times 10^{-2}$, $\frac{\alpha_e}{2\pi} \tilde{a}_g \approx -9.6 \times 10^{-5}$, $\delta_{\text{HO}}^{\text{QED}} = 0.0010(3)$.

- Remaining theoretical uncertainty mainly comes from corrections of higher order in $\alpha_e$. 
Further theoretical improvement requires a complete electroweak two-loop analysis.

At present, with the PIBETA experiment, we have $|V_{ud}| = 0.9740(28)_{\text{exp}}^1(1)_{\text{th}}$ from pion $\beta$ decay. Recall PDG 2020 value is $|V_{ud}| = 0.9737(10)_{\text{exp}}(10)_{\text{nucl}}(10)_{\text{RC}}$.

Estimating the difference between pion and nucleon $\gamma W$-box by hadronic models, combined with the precise lattice results for pion, we obtain a new value for nucleon $\gamma W$-box. The tension persists.


This pion calculation provide experiences for the nucleon $\gamma W$-box lattice calculation. In particular, we expect the discretization error being similar.

Lattice calculation of the nucleon $\gamma W$-box diagram in progress.
Outline

- Introduction

- Pion semi-leptonic decay - $\gamma W$-box diagram
  

- Kaon semi-leptonic decay - $\chi$PT low energy constants
  
  

- Conclusion and outlook
• Goal is to obtain the $\chi$PT low energy constants relevant to the QED radiative corrections in kaon semi-leptonic decay, which is largely independent of quark mass.

• Calculate the kaon semi-leptonic decay process in the flavor SU(3) limit with the light meson masses, $m_\pi$, $m_K$, $m_\eta$, equal to the physical pion mass.

• Use the same set of contractions and propagators as the pion $\beta$ decay calculation, but adjust the charge factors appropriately.
\[ \square_{\gamma W}^{VA} \bigg|_{\kappa^0, SU(3)} = \frac{3\alpha_e}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_K(Q^2) \]

\[ = 2.437(20)_{\text{stat}}(15)_{\text{PT}}(36)_{\alpha}(1)_{FV} \times 10^{-3} \]

\[ = 2.437(20)_{\text{stat}}(39)_{\text{sys}} \times 10^{-3} \]
Using the lattice results of $\Box^V_{\gamma W}\mid_{K^0, SU(3)}$ evaluated in the flavor SU(3) limit, we can obtain the \(\chi\)PT low energy constants (LECs).

Previously, the LECs are obtained via the minimal resonance model


\[
X_1 = -3.7(3.7) \times 10^{-3} \rightarrow -2.2(4) \times 10^{-3}
\]
\[
\tilde{X}_6^{\text{phys}} = 10.4(10.4) \times 10^{-3} \rightarrow 13.9(7) \times 10^{-3}.
\]

In \(\chi\)PT, the radiative corrections (RCs) have two major sources of theoretical uncertainties:

- The input of the LECs at \(\mathcal{O}(e^2 p^2)\).
- The unknown \(\mathcal{O}(e^2 p^4)\) terms in the ChPT expansion.

In unit of percent:

\[
\delta_{K^0}^e = 0.99(19)e^2 p^4(11)_{\text{LEC}} \rightarrow 1.00(19)
\]
\[
\delta_{K^0}^\mu = 1.40(19)e^2 p^4(11)_{\text{LEC}} \rightarrow 1.41(19)
\]
\[
\delta_{K^\pm}^e = 0.10(19)e^2 p^4(16)_{\text{LEC}} \rightarrow -0.01(19)
\]
\[
\delta_{K^\pm}^\mu = 0.02(19)e^2 p^4(16)_{\text{LEC}} \rightarrow -0.09(19).
\]

More theory work done in this direction, as mentioned in the previous talk by C.Y. Seng.
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- Conclusion and outlook
Pion $\beta$ decay (the pion semi-leptonic decay $\pi\ell 3$)
\[
\Box_{\gamma W}^{VA}\big|_{\pi} = 2.830(11)_{\text{stat}}(26)_{\text{sys}} \times 10^{-3}
\]
\[
\delta = 0.0334(10)_{\text{LEC}(3)_{\text{HO}}} \rightarrow \delta = 0.0332(1)_{\gamma W(3)_{\text{HO}}}
\]

Kaon semi-leptonic decay in the flavor SU(3) limit (unphysical kaon mass):
\[
\Box_{\gamma W}^{VA}\big|_{K^0, SU(3)} = 2.437(20)_{\text{stat}}(39)_{\text{sys}} \times 10^{-3}
\]

$\chi$PT low energy constants relevant to kaon semi-leptonic decay:
\[
X_1 = -3.7(3.7) \times 10^{-3} \rightarrow -2.2(4) \times 10^{-3}, \quad \tilde{X}_6^{\text{phys}} = 10.4(10.4) \times 10^{-3} \rightarrow 13.9(7) \times 10^{-3}.
\]
Conclusion and outlook

- For better $|V_{ud}|$ from nucleus / neutron decay, nucleon $\gamma_{\gamma W}^{VA}$ lattice calculation underway.

- For better $|V_{ud}|$ from pion $\beta$ decay (also called pion semi-leptonic decay), more accurate experimental results is needed. The current theoretical determination is very accurate and clean.

- For better $|V_{us}|$ from meson leptonic decay: We are working towards a complete lattice calculation of QED correction to meson leptonic decay with infinite volume QED using the infinite volume reconstruction method. The theoretical framework for this calculation is in preparation: N. Christ, X. Feng, LJ, C.T. Sachrajda, T. Wang in preparation. First step, pion mass mass splitting calculation finished:

$$M_{\pi^\pm} - M_{\pi^0} = 4.534(42)_{\text{stat}}(43)_{\text{sys}} \text{MeV} \quad \text{[PDG 2020: 4.5936(5) MeV]}$$

PRL 128, 052003 (2022) X. Feng, LJ, M.J. Riberdy

Note that there are two other groups also working on this:

- Already finished calculation in $\text{QED}_L$ scheme by RM123 group:

- Another calculation from RBC-UKQCD also uses $\text{QED}_L$ scheme @ physical $m_\pi$:
- For better $|V_{us}|$ from meson semi-leptonic decay: a complete lattice calculation of QED correction to meson semi-leptonic decay is more difficult than leptonic decay, but should be feasible as well. Will be the next goal.

- For better $|V_{us}|$ from meson (semi-)leptonic decay: improving the precision leading order decay constant and form factors are also important.
For better $|V_{ud}|$ from pion leptonic decay: more accurate lattice determination of $f_{\pi^+}$ and the QED correction to the process are needed.

For better $|V_{ud}|$ from neutron decay: more accurate $g_A$ is needed. Currently experimental determination is more accurate. But the precision from lattice QCD is improving as well.
\( K \rightarrow \pi\pi \) and CP violation

- [PRD 102, 054509 (2020)] by the RBC-UKQCD collaborations. Chris Kelly (BNL).

- Final result \( \text{Re}(\epsilon' / \epsilon) = 2.17(26)_{\text{stat}}(62)_{\text{sys}}(50)_{\text{isospin}} \times 10^{-3} \)
  Or, include the ChPT evaluation of the QED and strong isospin breaking effects: \( 1.67 \times 10^{-3} \). Recall the experimental value is \( 1.66(23) \times 10^{-3} \).

- Good agreement at this precision. RBC-UKQCD efforts to reduce the error:
  - Repeat the calculation with finer lattices.
  - Non-perturbative 3- to 4-flavor operator matching. Masaaki Tomii.
  - Periodic boundary condition \( K \rightarrow \pi\pi \). Masaaki Tomii and Daniel Hoying.

- Developing method to study the QED and strong isospin breaking effects on the lattice
  - N. Christ and X. Feng, EPJ Web Conf. 175, 13016 (2018)

  Independent calculation with \( m_\pi = 260 \) MeV.
Rare kaon decays: $K \rightarrow \pi \nu \bar{\nu}$

- [PRD 100 (2019) 11, 114506] by the RBC-UKQCD collaborations Xu Feng (Peking University).

- The golden modes: an ideal process in which to search for signs of new physics.
  - NA62 at CERN: $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ (aim at 10% accuracy)
  - KOTO at J-PARC: $K_L \rightarrow \pi^0 \nu \bar{\nu}$ (long distance contributions negligible)

- The long distance ($O(1/m_c)$) contribution is estimated to be about 5% to 10% in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$.

- Two pioneer lattice calculations @ $m_\pi = 430$ MeV and $m_\pi = 170$ MeV and lighter charm quark mass due to coarse lattice spacing.
Rare kaon decays: $K \rightarrow \pi \ell^+ \ell^-$


- Similar to $K \rightarrow \pi \nu \bar{\nu}$:
  - NA62 at CERN: $K^+ \rightarrow \pi^+ \ell^+ \ell^-$
  - Prospective experiments planned at LHCb to study the: $K_S \rightarrow \pi^0 \ell^+ \ell^-$
  - Lattice calculations @ $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV!
  - Statistical error is currently pretty large: $V(z = 0.013(2)) = -0.87(4.44)$. Mostly come from the stochastically estimated quark loops.
Rare kaon decays: $K \rightarrow \mu^+\mu^-$


- Branching fraction is accurately measured: $\text{BR}(K_L \rightarrow \mu^+\mu^-) = 6.84 \pm 0.11 \times 10^{-9}$.

- Two mechanism of comparable sizes for the decay:
  - One-loop, second-order weak process, involving exchange of two weak bosons.
  - $O(\alpha^2_{EM} G_F)$ process shown below.

- First step calculation $\pi \rightarrow e^+e^-$ successful.

- Second step calculation $K_L \rightarrow \gamma\gamma$ in progress.

- Final goal: lattice calculation of $K_L \rightarrow \gamma^*\gamma^* \rightarrow \mu^+\mu^-$
Rare kaon decays: $K \rightarrow \ell \nu \ell' \pm \ell'^- \pm$


- Good test of the lattice calculation for the kaon form factors also needed for the QED corrections to the kaon leptonic decay (photon can be emitted from kaon).

- Techniques are developed to treat the four (non-interacting) particle final state.

- Unphysical pion mass $\sim 300$ MeV is used in both calculations.

- Experimental results from the E865 experiment at BNL. hep-ex/0204006, hep-ex/0505001

<table>
<thead>
<tr>
<th>Channel</th>
<th>RM123</th>
<th>Tuo et al.</th>
<th>ChPT</th>
<th>Experiment</th>
</tr>
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<tbody>
<tr>
<td>$Br[K \rightarrow \mu \nu_\mu e^+ e^-]$</td>
<td>8.26(13) $10^{-8}$</td>
<td>10.59(33) $10^{-8}$</td>
<td>8.25 $10^{-8}$</td>
<td>7.93(33) $10^{-8}$</td>
</tr>
<tr>
<td>$Br[K \rightarrow e \nu_e \mu^+ \mu^-]$</td>
<td>0.762(49) $10^{-8}$</td>
<td>0.72(5) $10^{-8}$</td>
<td>0.62 $10^{-8}$</td>
<td>1.72(45) $10^{-8}$</td>
</tr>
<tr>
<td>$Br[K \rightarrow e \nu_e e^+ e^-]$</td>
<td>1.95(11) $10^{-8}$</td>
<td>1.77(16) $10^{-8}$</td>
<td>1.75 $10^{-8}$</td>
<td>2.91(23) $10^{-8}$</td>
</tr>
<tr>
<td>$Br[K \rightarrow \mu \nu_\mu \mu^+ \mu^-]$</td>
<td>1.178(35) $10^{-8}$</td>
<td>1.45(6) $10^{-8}$</td>
<td>1.10 $10^{-8}$</td>
<td>$-$</td>
</tr>
</tbody>
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Thank You!